Frobenius k-characters, Fricke identities and Markov equation

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Dedication

Based on a joint work with Victor M. Buchstaber dedicated to the memory of our dear friend Boris Dubrovin

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Boris A. Dubrovin (1950-2019)

Markov, Frobenius and Fricke

Andrei A. Markov (1856-1922), Ferdinand G. Frobenius (1849-1917) and the grave of Robert Fricke (1861-1930)

Markov spectrum of real numbers

Markov constant $\mu(\alpha)$ of a real number α is the minimal possible c such that

$$|\alpha - \frac{p}{q}| < \frac{c}{q^2}$$

holds for infinitely many p, q. It is known that for $\alpha = [a_0, a_1, \ldots,]$ the Markov constant can be computed as

$$\mu(\alpha)^{-1} = \limsup_{N\to\infty}([a_{N+1}, a_{N+2}, \ldots] + [0; a_N, a_{N-1}, \ldots, a_1]).$$

The set of all possible values of $\mu(\alpha)$, $\alpha \in \mathbb{R}$ is the *Lagrange spectrum*, its part with $\mu > \frac{1}{3}$ is **discrete** and known as the *Markov spectrum*.

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α	$\mu(\alpha)$	
$\frac{1+\sqrt{5}}{2}$	$\frac{1}{\sqrt{5}}$	$=0.4472135\ldots$
$1+\sqrt{2}$	$\frac{1}{\sqrt{8}}$	$=0.3535533\ldots$
$\frac{9+\sqrt{221}}{10}$	$\frac{5}{\sqrt{221}}$	$=0.3363363\ldots$
$\frac{23+\sqrt{1517}}{26}$	$\frac{13}{\sqrt{1517}}$	$=0.3337725\ldots$
$\frac{5+\sqrt{7565}}{58}$	$\frac{29}{\sqrt{7565}}$	$=0.3334214\ldots$

Table: The top five most irrational numbers and their Markov constants

Markov triples and Markov Theorem

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All Markov triples can be obtained from $(1, 1, 1)$ by the Markov (Vieta) involution

$$\sigma : (x, y, z) \to (x, y, 3xy - z)$$

and permutations: $(1, 1, 1) \to (1, 1, 2) \to (1, 2, 1) \to (1, 2, 5) \to \ldots$

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Markov (1880): The Lagrange spectrum above $1/3$ is discrete and consists of

$$\mu = \frac{m}{\sqrt{9m^2 - 4}},$$

where m is one of the Markov numbers:

$$m = 1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985, \ldots$$

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Dubrovin (1996): Markov orbit describes quantum cohomology of $\mathbb{C}P^2$ (!!!)

Uniformisation problem and Fricke identity

The fundamental group of a one-holed torus is a free group with two generators, so the Fuchsian group $G \subset SL_2(\mathbb{R})$, solving the corresponding uniformisation problem, is freely generated by two matrices $A, B \in SL_2(\mathbb{R})$.

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Fricke (1896): The following identity holds for any $A, B \in SL_2(\mathbb{R})$, $C = AB$:

$$(tr\, A)^2 + (tr\, B)^2 + (tr\, C)^2 = tr\, A\, tr\, B\, tr\, C + tr\,(ABA^{-1}B^{-1}) + 2.$$

Denoting $x := tr\, A$, $y := tr\, B$, $z := tr\, C$ and $j_c = tr\, ABA^{-1}B^{-1}$, Fricke arrived at the real surface determined by the following cubic equation

$$x^2 + y^2 + z^2 - xyz - (j_c + 2) = 0,$$

describing what is now known as *Teichmüller space* of one-holed tori.

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The punctured tori correspond to the parabolic commutator $ABA^{-1}B^{-1}$ with $j_c = tr\, ABA^{-1}B^{-1} = -2$, so the Fricke relation becomes

$$x^2 + y^2 + z^2 - xyz = 0,$$

and, after rescaling by 3, the Markov equation (Keen (1977), Goldman (2003); Gorshkov (1953), Cohn (1955)).

Dedekind's group determinant

Let G be a finite group of order N and introduce the commuting variables x_g labelled by the elements $g_1 = e, g_2, \ldots, g_N \in G$.
The *group determinant* of G is the determinant of matrix X_G with $X_{i,j} = x_{g_i g_j^{-1}}$:

$$\Theta_G = \det X_G \in \mathbb{C}[x_e, x_{g_2}, \ldots, x_{g_N}].$$

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For example, for $G = \mathbb{Z}_3$

$$X = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_3 & x_1 & x_2 \\ x_2 & x_3 & x_1 \end{pmatrix}$$

$$\Theta_G = (x_1 + x_2 + x_3)(x_1 + \varepsilon x_2 + \varepsilon^2 x_3)(x_1 + \varepsilon^2 x_2 + \varepsilon x_3), \quad \varepsilon = e^{\frac{2\pi i}{3}}.$$

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Dedekind (1890s) noticed that the group determinants has a peculiar factorisation and made some conjecture about linear factors. The nature of the remaining factors was clarified by Frobenius in 1896, who essentially had to develop for this the representation theory for finite groups !

Frobenius's solution

Frobenius (1896): *The group determinant* $\Theta_G = \det X_G$ *has the factorisation of the form*

$$\Theta_G(x) = \prod_{i=1}^{r} \Psi_i(x)^{\deg \Psi_i},$$

where $\Psi_i(x)$, $i = 1, \ldots, r$ *are certain irreducible polynomials and* r *is the number of conjugacy classes in* G.

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The modern proof is easy (see e.g. Etingof et al, 2011). Consider the group algebra $\mathbb{C}[G]$, the regular representation $\rho : G \to End(V)$, $V = \mathbb{C}[G]$ and the $End(V)$-valued polynomial

$$L(x) = \sum_{g \in G} x_g \rho(g).$$

Its matrix in the basis $\{g_i\} \in V$, $i = 1, \ldots, N$ is precisely X_G since

$$L(x)h = \sum_{g \in G} x_g \rho(g) h = \sum_{g \in G} x_g gh = \sum_{g \in G} x_{gh^{-1}} g.$$

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Now the claim follows from the fact that the regular representation

$$V = \oplus_{i=1}^{r} (\dim V_i)\, V_i,$$

where V_i, $i = 1, \ldots, r$ are all non-equivalent irreducible representations of G, which is essentially what Frobenius had shown.

Frobenius k-characters

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Starting with any trace-like function $\chi : G \to \mathbb{C}$ Frobenius defines the polynomials $\Phi_k^\chi(x)$

$$\Phi_k^\chi = \frac{1}{k!} \sum_{h_1,\dots,h_k \in G} \chi_k(h_1,\dots,h_k) x_{h_1} \dots x_{h_k},$$

where the functions $\chi_k(h_1,\dots,h_k)$ are defined by the recurrence procedure

$$\chi_{k+1}(h_0,h_1,\dots,h_k) := \chi(h_0)\chi_k(h_1,h_2,\dots,h_k) - \sum_{j=1}^{k} \chi_k(h_1,h_2,\dots,h_0 h_j \dots,h_k),$$

with $\chi_1 = \chi$. In particular, $\Phi_1^\chi = \sum_{h \in G} \chi(h) x_h$ and

$$\chi_{k+1}(e,h_1,\dots,h_k) = (\chi(e) - k)\chi_k(h_1,\dots,h_k).$$

When $\chi = tr\,\rho(g)$ is the character of some representation $\rho : G \to GL(W)$ then χ_k is called *Frobenius k-character* of W.

Factors of group determinant as k-characters

Frobenius (1896): Let W be an irreducible representation of G of dimension n and χ be its character. Then the corresponding $\chi_j \equiv 0$ for all $j > n$ and $\Phi_n^\chi = \Psi_W$ is the corresponding irreducible factor of the group determinant.

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The corresponding polynomials $\Phi_k^\chi(x)$ appear as the coefficients in the following expansion

$$\Phi_n^\chi(x + u\epsilon) = u^n + \Phi_1^\chi(x)u^{n-1} + \Phi_2^\chi(x)u^{n-2} + \cdots + \Phi_n^\chi(x),$$

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where in Frobenius's notations $x + u\epsilon = (x_e + u, x_{g_2}, \ldots, x_{g_N})$.

Note that the condition that $\chi_{n+1} \equiv 0$ holds for the character χ of every n-dimensional representation of G, not necessarily irreducible and for every group G, not necessarily finite.

R. Taylor (1991): Conversely, if χ is a trace-like map from G to any algebraically closed field of characteristic zero with $\chi(e) = n \in \mathbb{N}$ and $\chi_{n+1} \equiv 0$, then χ is the character of an n-dimensional representation of G.

Diagonal k-characters and exterior powers of representations

Following Frobenius consider the restrictions of Frobenius characters on the diagonal:

$$\theta_k^\rho(g) = \frac{1}{k!}\chi_k(g, \ldots, g), \; g \in G.$$

Buchstaber, V. (2020): *The diagonal Frobenius k-characters of representation ρ are simply the usual characters of k-th exterior power of ρ:*

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To explain the meaning of the polynomials $\Phi_k^\chi(x)$ for any representation $\rho : G \to GL(W)$, $\chi(g) = tr\,\rho(g)$ consider the operator-valued polynomial

$$L^\rho(x) = \sum_{g\in G} x_g \rho(g) \in End(W)[x].$$

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Buchstaber, V. (2020): *Frobenius polynomial $\Phi_k^\chi(x)$ is the trace*

$$\Phi_k^\chi(x) = tr\,\Lambda^k(L^\rho(x)).$$

In particular, $\Phi_n^\chi(x) = det\,L^\rho(x)$, where $n = \dim W = \chi(e)$.

Frobenius relations and Fricke identity

In Frobenius notations the first two characters are

$$\chi_2(A,B) = \chi(A)\chi(B) - \chi(AB),$$

$$\chi_3(A,B,C) = \chi(A)\chi(B)\chi(C) - \chi(A)\chi(BC) - \chi(B)\chi(AC) - \chi(C)\chi(AB)$$
$$+\chi(ABC) + \chi(ACB), \quad A,B,C \in G.$$

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For the canonical 2-dimensional representation of Fuchsian group we have the relation $\chi_3(A,B,C) = 0$. Substituting here $A = ab = c$, $B = a^{-1}$, $C = b^{-1}$ we have

$$\chi(A)\chi(B)\chi(C) - \chi(A)\chi(BC) - \chi(B)\chi(AC) - \chi(C)\chi(AB) + \chi(ABC) + \chi(ACB)$$
$$= \chi(ab)\chi(a^{-1})\chi(b^{-1}) - \chi(ab)\chi(a^{-1}b^{-1}) - \chi(a^{-1})\chi(a) - \chi(b^{-1})\chi(aba^{-1})$$
$$+\chi(aba^{-1}b^{-1}) + \chi(e) = \chi(a)\chi(b)\chi(c) - \chi(a)^2 - \chi(b)^2 - \chi(c)^2 + \chi(aba^{-1}b^{-1}) + 2 = 0,$$

which coincides with the Fricke identity:

$$(tr\, a)^2 + (tr\, b)^2 + (tr\, c)^2 = tr\, a\, tr\, b\, tr\, c + tr\,(aba^{-1}b^{-1}) + 2.$$

Modern developments

The theory of k-characters of Frobenius appears to be largely forgotten until 1990s, when Hoenke and Johnson (1992) proved that k-characters with $k \leq 3$ determine the group. Formanek and Sibley (1991) proved that the group determinant also uniquely determines the finite group.

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Markov triples appeared unexpectedly in

- algebraic geometry as ranks of the exceptional bundles over $\mathbb{C}P^2$ in Rudakov (1988) and as weights in Hacking-Prokhorov (2010) classification of del Pezzo surfaces with quotient singularities
- theory of Frobenius manifolds and quantum cohomology of $\mathbb{C}P^2$ in Dubrovin (1996) following Kontsevich and Manin (1994).

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