Spectral stochastic finite elements for two nonlinear problems

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We are interested in a solution of

$$A(x,\xi)\,u^s(x,\xi) = \lambda^s(\xi)\,u^s(x,\xi)\,,$$

where $\xi = \{\xi_i\}_{i=1}^N$ is a set of i.i.d. random variables, and

$$A = \sum_{\ell=0}^{M'} A_\ell \psi_\ell(\xi)\,, \qquad \text{(SPD matrix operator).}$$

We search

$$\lambda^s = \sum_{k=0}^{M} \lambda_k^s \psi_k(\xi)\,, \qquad u^s = \sum_{k=0}^{M} u_k^s \psi_k(\xi)\,,$$

where $\{\psi_\ell(\xi)\}$ is a set of orthonormal polynomials, $\langle \psi_j \psi_k \rangle = 0$ or 1, which is referred to as the gPC basis. We will also use the notation

$$c_{\ell jk} = \langle \psi_\ell \psi_j \psi_k \rangle = \mathbb{E}\left[\psi_\ell \psi_j \psi_k\right].$$

Monte Carlo and stochastic collocation

Both approaches are based on sampling,

$$A\left(\xi^{(q)}\right) u^s\left(\xi^{(q)}\right) = \lambda^s\left(\xi^{(q)}\right) u^s\left(\xi^{(q)}\right), \qquad s = 1, \ldots, n_s.$$

Monte Carlo uses ensemble average

$$\lambda_k^s = \frac{1}{N_{MC}} \sum_{q=1}^{N_{MC}} \lambda^s\left(\xi^{(q)}\right) \psi_k\left(\xi^{(q)}\right), \quad u_k^s = \frac{1}{N_{MC}} \sum_{q=1}^{N_{MC}} u^s\left(\xi^{(q)}\right) \psi_k\left(\xi^{(q)}\right).$$

Stochastic collocation uses quadrature

$$\lambda_k^s = \sum_{q=1}^{N_q} \lambda^s\left(\xi^{(q)}\right) \psi_k\left(\xi^{(q)}\right) w^{(q)}, \quad u_k^s = \sum_{q=1}^{N_q} u^s\left(\xi^{(q)}\right) \psi_k\left(\xi^{(q)}\right) w^{(q)},$$

where $\xi^{(q)}$ are the quadrature points, and $w^{(q)}$ are quadrature weights.

Stochastic Galerkin method

The stochastic Galerkin method is based on the projection

$$\langle Au^s, \psi_k \rangle = \langle \lambda^s u^s, \psi_k \rangle, \qquad k = 0, \ldots, M, \quad s = 1, \ldots, n_s.$$

which yields a nonlinear algebraic system

$$\sum_{j=0}^{M} \sum_{\ell=0}^{M'} c_{\ell jk} A_\ell u_j^s = \sum_{j=0}^{M} \sum_{i=0}^{M} c_{ijk} \lambda_i^s u_j^s, \qquad k = 0, \ldots, M, \quad s = 1, \ldots, n_s.$$

In order to find interior eigenvalues we use deflation via

$$\widetilde{A}_0 = A_0 + \sum_{d=1}^{n_d} \left[C - \lambda_0^d \right] \left(u_0^d \right) \left(u_0^d \right)^T, \qquad \widetilde{A}_\ell = A_\ell, \quad \ell = 1, \ldots, M',$$

where $\left(\lambda_0^d, u_0^d\right)$, $d = 1, \ldots, n_d$, are the zeroth order coefficients, i.e., the mean value, of the eigenpairs to be deflated, and C is a constant.

Algorithm: Stochastic inverse subspace iteration

Algorithm

Find n_e eigenpairs of the deterministic mean value problem

$$A_0\,\overline{U} = \overline{U}\,\overline{\Lambda}, \qquad \overline{U} = \left[\ \overline{u}^1, \quad \ldots, \quad \overline{u}^{n_e}\ \right], \qquad \overline{\Lambda} = diag\left(\overline{\lambda}^1, \ldots, \overline{\lambda}^{n_e}\right),$$

1. *choose n_s eigenvalues of interest with indices $e = \{e_1, \ldots, e_{n_s}\}$*
2. *set up matrices $\mathcal{A}_\ell$, $\ell = 0, \ldots, M'$, either as $\mathcal{A}_\ell = A_\ell$, or $\mathcal{A}_\ell = \widetilde{A}_\ell$ using deflation*
3. *initialize*

$$u_0^{1,(0)} = \overline{u}^{e_1}, \quad u_0^{2,(0)} = \overline{u}^{e_2}, \quad \ldots \quad u_0^{n_e,(0)} = \overline{u}^{e_{n_s}},$$
$$u_i^{s,(0)} = 0, \qquad s = 1, \ldots, n_s, \quad i = 1, \ldots, M.$$

Algorithm: Stochastic inverse subspace iteration

Algorithm (cont'd)

for $it = 0, 1, 2, \ldots$

1. *Solve the stochastic Galerkin system for* $v_j^{s,(it)}$, $j = 0, \ldots, M$:

$$\sum_{j=0}^{M} \sum_{\ell=0}^{M'} c_{\ell jk} \mathcal{A}_\ell v_j^{s,(it)} = u_k^{s,(it)}, \qquad k = 0, \ldots, M, \quad s = 1, \ldots, n_s.$$

2. *If* $n_s = 1$, *normalize using the quadrature rule, or else if* $n_s > 1$ *orthogonalize using the stochastic modified Gram-Schmidt process.*
3. *Check convergence.*

end

Use the stoch. Rayleigh quotient to compute the eigenvalue expansions.

Matrix-vector multiplication

Matrix-vector product corresponds to evaluation of the projection

$$v_k = \langle v, \psi_k \rangle = \langle Au, \psi_k \rangle \,, \qquad k = 0, \ldots, M.$$

In more detail,

$$\left\langle \sum_{i=0}^{M} v_i \psi_i \left(\xi\right), \psi_k \right\rangle = \left\langle \left(\sum_{\ell=0}^{M'} A_\ell \psi_\ell \left(\xi\right) \right) \left(\sum_{j=0}^{M} u_j \psi_j \left(\xi\right) \right), \psi_k \right\rangle,$$

so the coefficients are

$$v_k = \sum_{j=0}^{M} \sum_{\ell=0}^{M'} c_{\ell jk} A_\ell u_j, \qquad k = 0, \ldots, M.$$

The use of this computation for the Rayleigh quotient is described next.

Stochastic Rayleigh Quotient

The stochastic Rayleigh quotient defines the coefficients of a stochastic expansion of the eigenvalue defined via a projection

$$\lambda_k = \langle \lambda, \psi_k \rangle = \left\langle u^T v, \psi_k \right\rangle, \qquad k = 0, \ldots, M,$$

where the coeff's of v are computed using the matrix-vector multiplication. In more detail,

$$\left\langle \sum_{i=0}^{M} \lambda_i \psi_i(\xi), \psi_k \right\rangle = \left\langle \left(\sum_{i=0}^{M} u_i \psi_i(\xi) \right)^T \left(\sum_{j=0}^{M} v_j \psi_j(\xi) \right), \psi_k \right\rangle,$$

so the coefficients are

$$\lambda_k = \sum_{j=0}^{M} \sum_{i=0}^{M} c_{ijk} \left\langle u_i^T v_j \right\rangle_{\mathbb{R}}, \qquad k = 0, \ldots, M,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ is the inner product on Euclidean N_{dof}-dimensional space.

Normalization and the Gram-Schmidt process

The coeff's of a normalized vector are computed from the coeff's v_k^1 as

$$u_k^1 = \sum_{q=1}^{N_q} \frac{v^1\left(\xi^{(q)}\right)}{\left\| v^1\left(\xi^{(q)}\right)\right\|_2} \psi_k\left(\xi^{(q)}\right) w^{(q)}, \qquad k = 0, \ldots, M.$$

The orthonormalization is achieved by a stochastic version of the modified Gram-Schmidt algorithm [Meidani and Ghanem (2014)], which is based on

$$u^s = v^s - \sum_{t=1}^{s-1} \frac{\langle v^s, u^t\rangle_{\mathbb{R}}}{\langle u^t, u^t\rangle_{\mathbb{R}}} u^t, \quad s = 2, \ldots, n_s,$$

noting that $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ can be again evaluated at the collocation points.

Error assessment

We do not have the coefficients of the expansion of the true residual

$$r_k^s = \langle Au^s - \lambda^s u^s, \psi_k \rangle, \qquad k = 0, \ldots, M, \quad s = 1, \ldots, n_s.$$

Instead, we can use a residual indicator

$$\widetilde{r}_k^{s,(it)} = \sum_{j=0}^{M} \sum_{\ell=0}^{M'} c_{\ell jk} A_\ell u_j^{s,(it)} - \sum_{j=0}^{M} \sum_{i=0}^{M} c_{ijk} \lambda_i^{s,(it)} u_j^{s,(it)}.$$

Error assessment (cont'd)

In computations, for each pair $s = 1, \ldots, n_s$, we can monitor:

1. the norms of the expected value of the indicator and its variance,

$$\varepsilon_0^{s,(it)} = \left\| \widetilde{r}_0^{s,(it)} \right\|_2, \qquad \varepsilon_{\sigma^2}^{s,(it)} = \left\| \sum_{k=1}^{M} \left(\widetilde{r}_k^{s,(it)} \right)^2 \right\|_2 .$$

2. the difference of the coefficients in two consecutive iterations.

$$u_\Delta^{s,(it)} = \left\| \begin{bmatrix} \mathrm{u}_0^{s,(it)} \\ \vdots \\ \mathrm{u}_M^{s,(it)} \end{bmatrix} - \begin{bmatrix} \mathrm{u}_0^{s,(it-1)} \\ \vdots \\ \mathrm{u}_M^{s,(it-1)} \end{bmatrix} \right\|_2 .$$

3. In addition, we can sample the true residual using Monte Carlo,

$$r^s\left(\xi^i\right) = A\left(\xi^i\right) u^s\left(\xi^i\right) - \lambda^s\left(\xi^i\right) u^s\left(\xi^i\right), \qquad i = 1, \ldots N_{MC}.$$

Error assessment (cont'd)

We also report:

1. pdf estimates of the normalized ℓ^2-norms of the true residual

$$\varepsilon_r^s\left(\xi^i\right) = \frac{\left\|r^s\left(\xi^i\right)\right\|_2}{\left\|A\left(\xi^i\right)\right\|_2}, \qquad i = 1, \ldots N_{MC}, \quad s = 1, \ldots, n_s.$$

2. pdf estimates of the ℓ^2-norm of the relative eigenvector approx. errors

$$\varepsilon_u^s\left(\xi^{(i)}\right) = \frac{\left\|u^s\left(\xi^{(i)}\right) - u_{MC}^s\left(\xi^{(i)}\right)\right\|_2}{\left\|u_{MC}^s\left(\xi^{(i)}\right)\right\|_2}, \quad i = 1, \ldots, N_{MC}, \; s = 1, \ldots, n_s,$$

where $u^s\left(\xi^{(i)}\right)$ are samples of eigenvectors obtained from either stochastic inverse subspace iteration or stochastic collocation.

Numerical experiments: Vibration of undamped structures

We considered Young's modulus

$$E(x,\xi) = \sum_{\ell=0}^{M'} E_\ell(x)\,\psi_\ell(\xi)$$

as a truncated lognormal process transformed from Gaussian r. process. Finite element spatial discretization leads to

$$\mathcal{K}(\xi)\,u = \lambda \mathcal{M} u,$$

where $\mathcal{K}(\xi) = \sum_{\ell=0}^{M'} K_\ell \psi_\ell(\xi)$ is the stochastic stiffness matrix, and $\mathcal{M}$ is the deterministic mass matrix. Using Cholesky $\mathcal{M} = LL^T$,

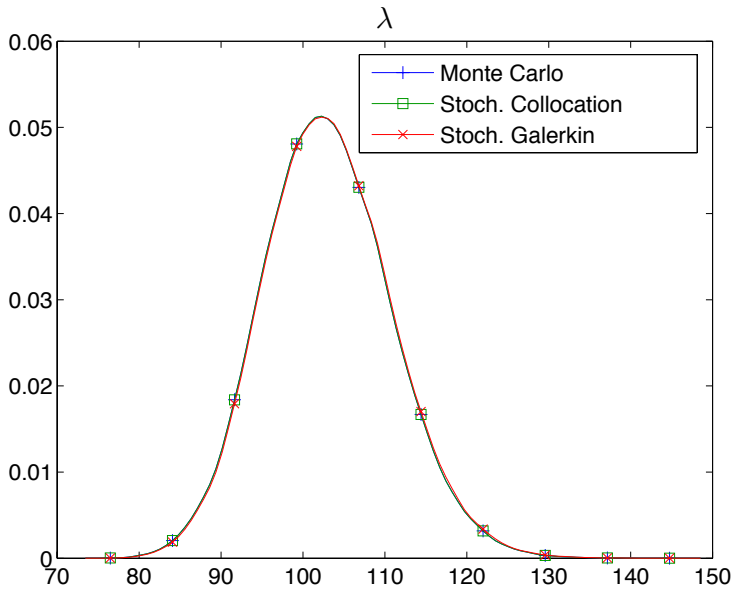
$$L^{-1}\mathcal{K}(\xi)\,L^{-T} w = \lambda w,$$

$$A = \sum_{\ell=0}^{M'} A_\ell \psi_\ell(\xi) = \sum_{\ell=0}^{M'} \left[L^{-1} K_\ell(\xi)\,L^{-T}\right] \psi_\ell(\xi)\,.$$

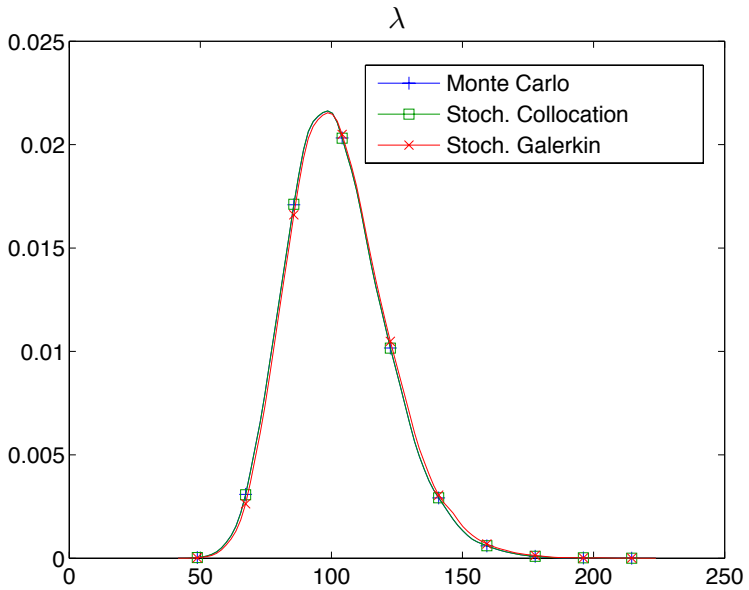
Numerical experiments: Timoshenko beam

Physical parameters: $E_0 = 10^8$, $\nu = 0.30$, with 20 finite elements (40 dof). Stochastic parameters: $N = 3$, $P = 3$, $P_{\log} = 6$, size of $c_{\ell jk}$: $20 \times 20 \times 84$, $N_{MC} = 5 \times 10^4$, Smolyak sparse grid (Gauss-Hermite) with 69 points.

pdf of λ_1 using only the stochastic Rayleigh quotient:



$CoV = 10\%$



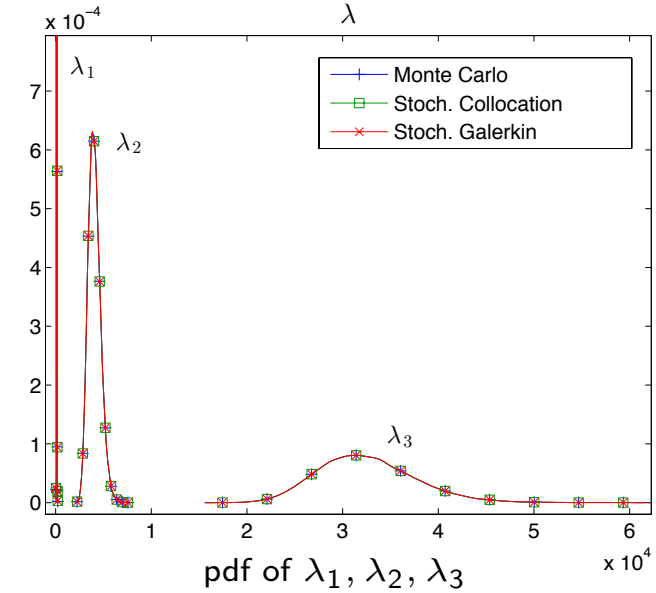
$CoV = 25\%$

Numerical experiments: Timoshenko beam ($CoV = 25\%$)

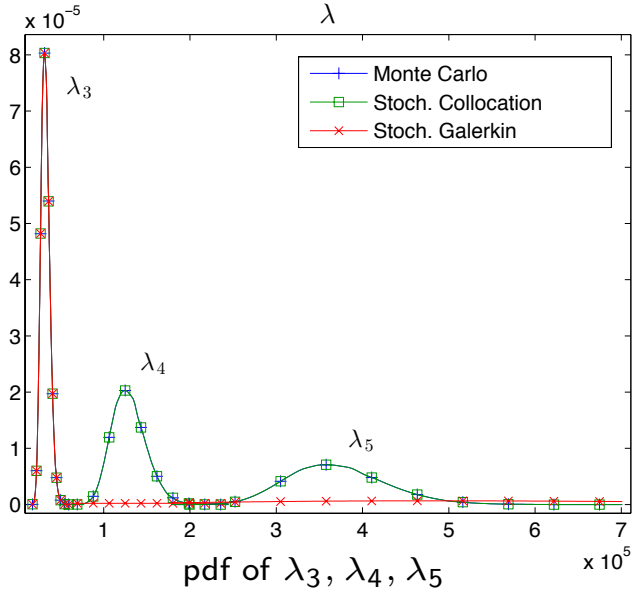
The first ten coefficients of the gPC expansion of the smallest eigenvalue using 0, 1 or 20 steps of stochastic inverse iteration, or stoch. collocation. Here d is the polynomial degree and k is the index of gPC basis function.

d	k	$\mathrm{RQ}^{(0)}$	$\mathrm{SII}^{(1)}$	$\mathrm{SII}^{(20)}$	SC
0	0	103.0823	102.1705	102.1670	102.1713
1	1	14.0453	13.9402	13.9402	13.9408
	2	-11.7568	-11.5862	-11.5859	-11.5848
	3	5.1830	5.0654	5.0651	5.0669
2	4	1.4284	1.2919	1.2918	1.2909
	5	-1.5368	-1.6766	-1.6767	-1.6764
	6	0.5090	0.9030	0.9032	0.9035
	7	1.1331	0.7533	0.7530	0.7529
	8	-0.8696	-0.2965	-0.2960	-0.2955
	9	0.5812	-0.2215	-0.2220	-0.2222

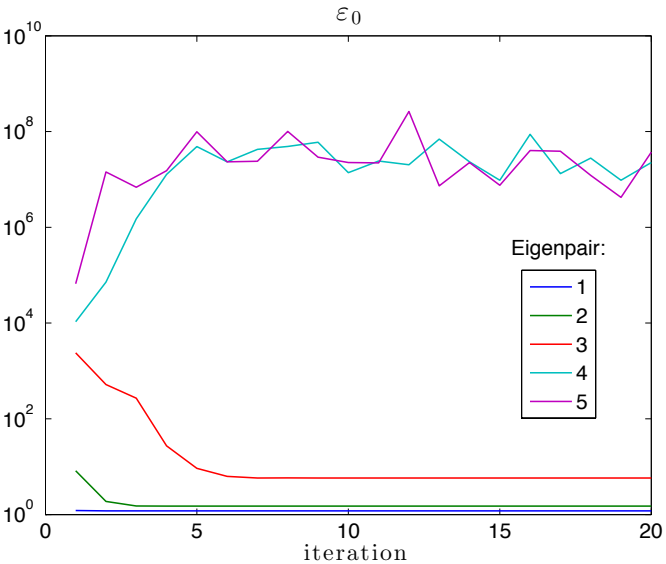
Numerical experiments: Timoshenko beam ($CoV = 25\%$)



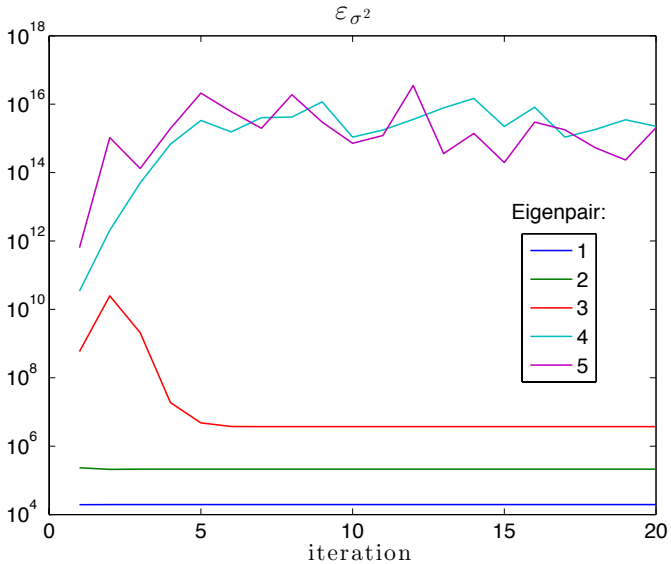
pdf of $\lambda_1, \lambda_2, \lambda_3$



pdf of $\lambda_3, \lambda_4, \lambda_5$

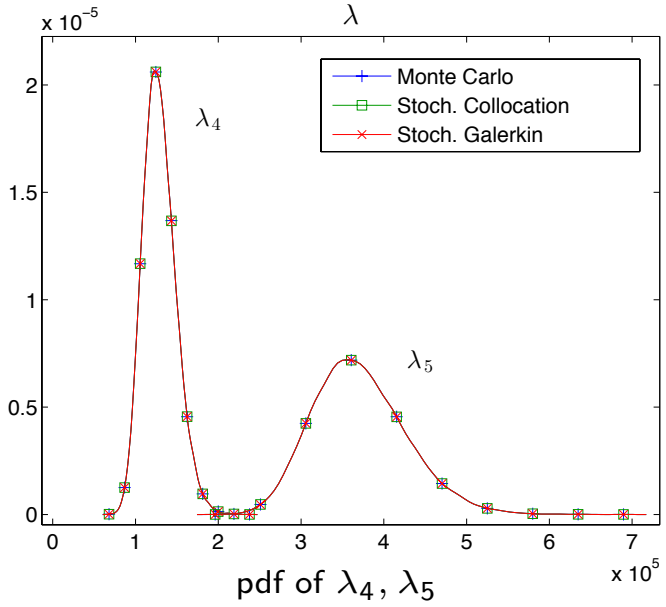


convergence history of ε_0

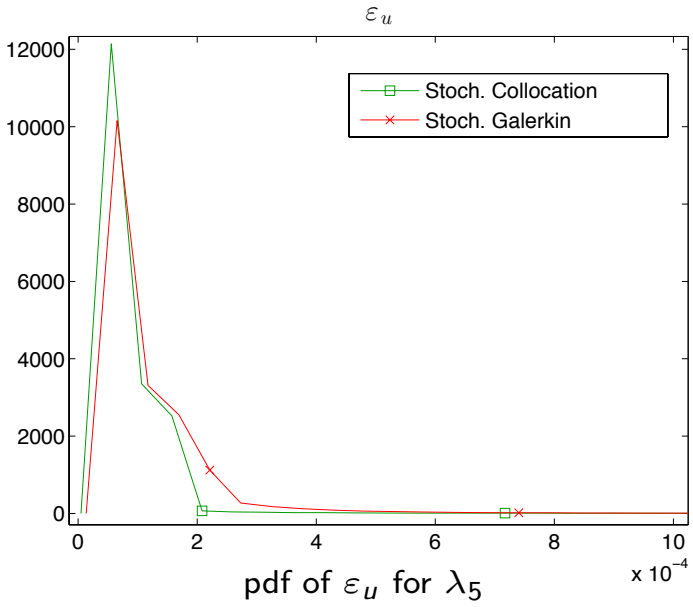


convergence history of ε_{σ^2}

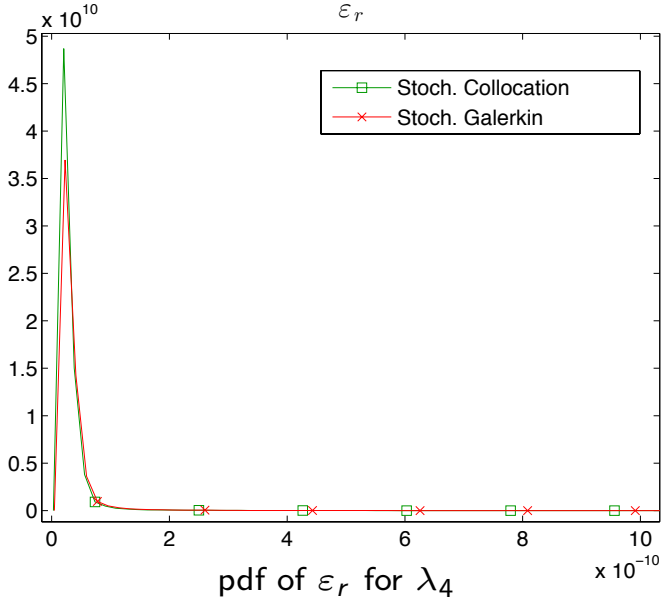
Numerical experiments: Timoshenko beam ($CoV = 25\%$)



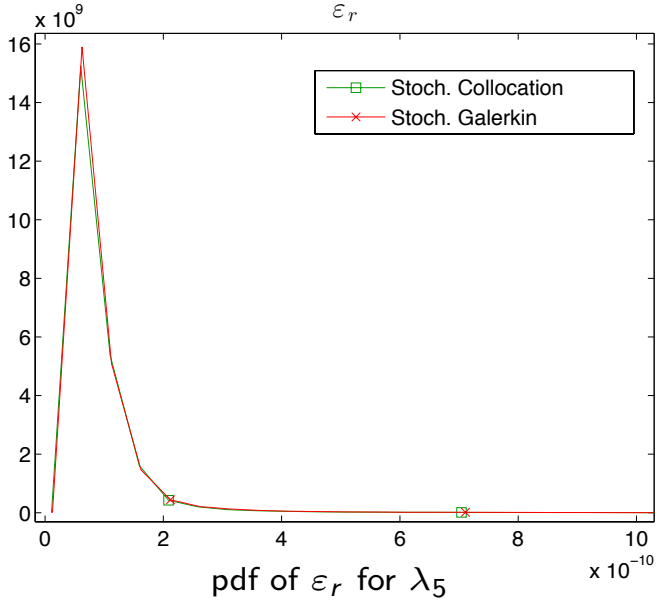
pdf of λ_4, λ_5



pdf of ε_u for λ_5

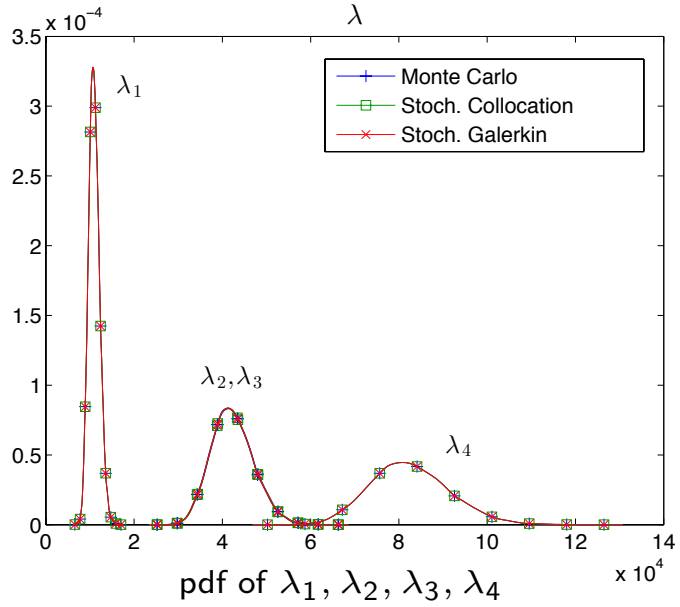


pdf of ε_r for λ_4

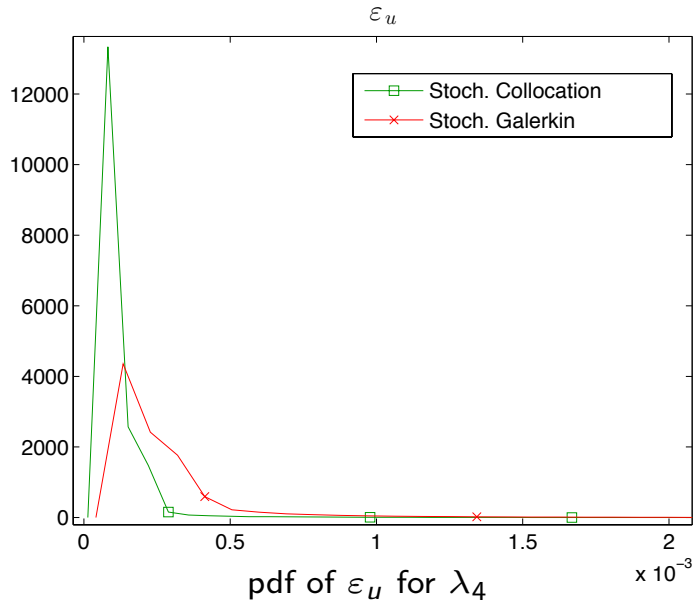


pdf of ε_r for λ_5

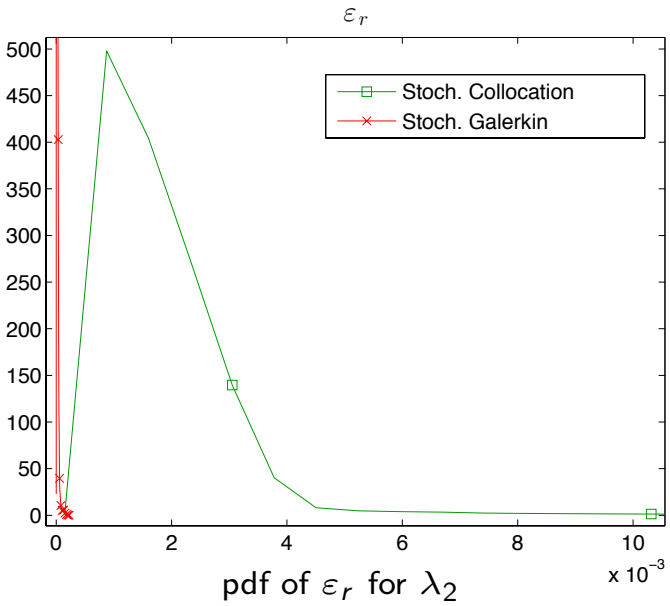
Numerical experiments: Mindlin plate ($CoV = 25\%$)



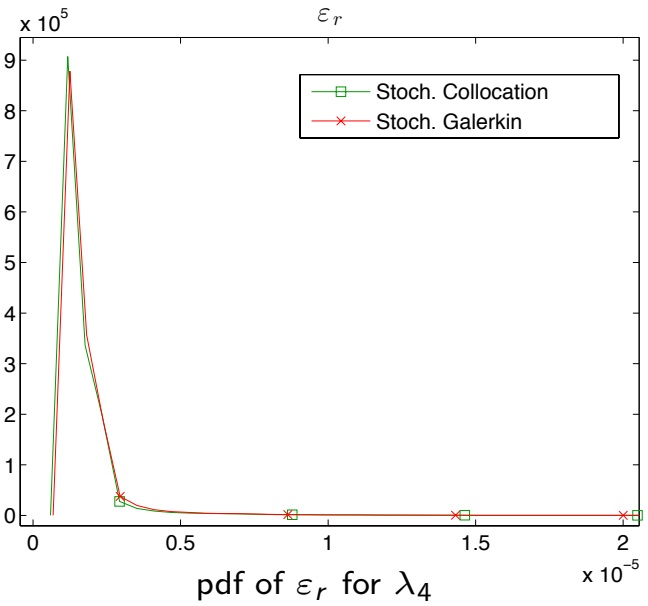
pdf of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$



pdf of ε_u for λ_4



pdf of ε_r for λ_2



pdf of ε_r for λ_4

The **viscosity** is given by a gPC expansion

$$\nu \equiv \nu(x,\xi) = \sum_{\ell=0}^{M_\nu - 1} \nu_\ell(x)\,\psi_\ell(\xi)$$

where $\{\nu_\ell(x)\}$ is a set of given deterministic spatial functions.

Possible applications:

1. the exact value of viscosity may not be known due to
   - measurement error
   - contaminants with uncertain concentrations
   - multiple phases with uncertain ratios
2. fluid properties might be influenced by an external field (MHD).

Equivalently, *stochastic Reynolds number* $\mathrm{Re}(\xi) = \frac{UL}{\nu(\xi)}$.

Stochastic Galerkin formulation

Idea: perform a Galerkin projection on the space T_Γ (expectation $\mathbb{E}$), and seek random fields for velocity $\vec{u} \in V_E \otimes T_\Gamma$, and pressure $p \in Q_D \otimes T_\Gamma$:

$$\mathbb{E}\left[\int_D \nu\, \nabla \vec{u} : \nabla \vec{v} + \int_D (\vec{u} \cdot \nabla \vec{u})\, \vec{v} - \int_D p\, (\nabla \cdot \vec{v})\right] = \mathbb{E}\left[\int_D \vec{f} \cdot \vec{v}\right] \quad \forall \vec{v} \in V_D \otimes T_\Gamma$$

$$\mathbb{E}\left[\int_D q\, (\nabla \cdot \vec{u})\right] = 0 \quad \forall q \in Q_D \otimes T_\Gamma$$

The *stochastic* counterpart of *Newton iteration* is

$$\mathbb{E}\left[\int_D \nu\, \nabla \delta \vec{u}^n : \nabla \vec{v} + c(\vec{u}^n; \delta \vec{u}^n, \vec{v}) + c(\delta \vec{u}^n; \vec{u}^n, \vec{v}) - \int_D \delta p^n\, (\nabla \cdot \vec{v})\right] = R^n$$

$$\mathbb{E}\left[\int_D q\, (\nabla \cdot \delta \vec{u}^n)\right] = r^n$$

where

$$R^n(\vec{v}) = \mathbb{E}\left[\int_D \vec{f} \cdot \vec{v} - \int_D \nu\, \nabla \vec{u}^n : \nabla \vec{v} - c(\vec{u}^n; \vec{u}^n, \vec{v}) + \int_D p^n\, (\nabla \cdot \vec{v})\right]$$

$$r^n(q) = -\mathbb{E}\left[\int_D q\, (\nabla \cdot \vec{u}^n)\right].$$

Stochastic Galerkin FEM: ordering of unknowns

We seek a discrete approximation of the solution of the form

$$\vec{u}(x,\xi) \approx \sum_{k=0}^{M-1} \sum_{i=1}^{N_u} u_{ik}\phi_i(x)\psi_k(\xi) = \sum_{k=0}^{M-1} \vec{u}_k(x)\psi_k(\xi)$$

$$p(x,\xi) \approx \sum_{k=0}^{M-1} \sum_{j=1}^{N_p} p_{jk}\varphi_j(x)\psi_k(\xi) = \sum_{k=0}^{M-1} \vec{p}_k(x)\psi_k(\xi)$$

Structure of operators depends on the ordering of the coeffs $\{u_{ik}\}$, $\{p_{jk}\}$. We will group velocity-pressure pairs for each k, the index of stochastic basis functions, giving the ordered list of coefficients

$$u_{1:N_u,0}, p_{1:N_p,0},\ u_{1:N_u,1}, p_{1:N_p,1},\ \ldots,\ u_{1:N_u,M-1}, p_{1:N_p,M-1}.$$

Stochastic Galerkin FEM: structure of Stokes operator

The discrete stochastic Stokes operator is built using matrices

$$\mathbf{A}_\ell = [a_{\ell,ab}], \quad a_{\ell,ab} = \left(\int_D \nu_\ell(x) \, \nabla\phi_b : \nabla\phi_a \right), \qquad \ell = 1, \ldots, M_\nu - 1$$

which are incorporated into the block matrices

$$\mathcal{S}_0 = \begin{bmatrix} \mathbf{A}_0 & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix}, \qquad \mathcal{S}_\ell = \begin{bmatrix} \mathbf{A}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \ell = 1, \ldots, M_\nu - 1.$$

These operators will be coupled with matrices arising from terms in T_P,

$$\mathbf{H}_\ell = [h_{\ell,jk}], \quad h_{\ell,jk} \equiv \mathbb{E}[\psi_\ell \psi_j \psi_k], \qquad \ell = 0, \ldots, M_\nu - 1, \quad j, k = 0, \ldots, M - 1.$$

The discrete stochastic (Galerkin) Stokes system is

$$\left(\sum_{\ell=0}^{M_\nu - 1} \mathbf{H}_\ell \otimes \mathcal{S}_\ell \right) \mathbf{v} = \mathbf{y}$$

where $\otimes$ corresponds to the matrix Kronecker product.

Stochastic Galerkin FEM: block saddle-point structure

The stochastic (Galerkin) Stokes matrix

$$\left(\sum_{\ell=0}^{M_\nu - 1} \mathbf{H}_\ell \otimes \mathcal{S}_\ell \right)$$

contains a set of M block 2×2 matrices of saddle-point structure

$$\mathcal{S}_0 + \sum_{\ell=1}^{M_\nu - 1} h_{\ell,jj} \mathcal{S}_\ell, \qquad j = 0, \ldots, M-1.$$

along its block diagonal $\implies$ **enables use of existing deterministic solvers for the individual diagonal blocks.**

Stochastic Galerkin FEM: linearized N-S operators

Step n of the stochastic nonlinear iteration entails

$$\left[\sum_{\ell=0}^{\widehat{M}-1} \mathbf{H}_\ell \otimes \mathcal{F}_\ell^n\right] \delta\mathbf{v}^n = \mathcal{R}^n \qquad \mathbf{v}^{n+1} = \mathbf{v}^n + \delta\mathbf{v}^n$$

where

$$\mathcal{R}^n = \mathbf{y} - \left[\sum_{\ell=0}^{\widehat{M}-1} \mathbf{H}_\ell \otimes \mathcal{P}_\ell^n\right] \mathbf{v}^n$$

and $\mathbf{v}^n$ and $\delta\mathbf{v}^n$ are current velocity and pressure coefficients and updates. Here

$$\mathcal{F}_0^n = \begin{bmatrix} \mathbf{F}_0^n & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \qquad \mathcal{F}_\ell^n = \begin{bmatrix} \mathbf{F}_\ell & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

$$\mathbf{F}_\ell^n = \mathbf{A}_\ell + \mathbf{N}_\ell^n + \mathbf{W}_\ell{}^n \qquad \text{for the stochastic Newton method,}$$

$$\mathbf{F}_\ell^n = \mathbf{A}_\ell + \mathbf{N}_\ell^n \qquad \text{for stochastic Picard iteration.}$$

Question: How are the matrices $\mathbf{A}_\ell$, $\mathbf{N}_\ell^n$, $\mathbf{W}_\ell{}^n$ computed?

Stochastic Galerkin FEM: components of N-S operators

At step n of the nonlinear iteration, let

$$\vec{u}_\ell^n(x) \quad \text{the } \ell\text{th term of the velocity iterate, } \ell = 1, \ldots, M$$

and let

$$\mathbf{N}_\ell^n = \left[n_{\ell,ab}^n\right] \qquad n_{\ell,ab}^n = \int_D (\vec{u}_\ell^n \cdot \nabla \phi_b) \cdot \phi_a$$

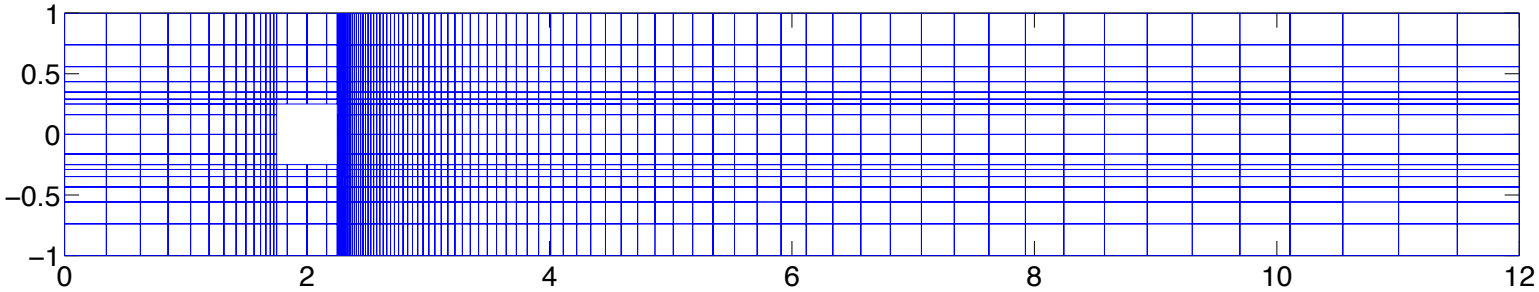
$$\mathbf{W}_\ell^n = \left[w_{\ell,ab}^n\right] \qquad w_{\ell,ab}^n = \int_D (\phi_b \cdot \nabla \vec{u}_\ell^n) \cdot \phi_a$$

and as before

$$\mathbf{A}_\ell = [a_{\ell,ab}], \quad a_{\ell,ab} = \left(\int_D \nu_\ell(x)\, \nabla \phi_b : \nabla \phi_a\right) \qquad \ell = 1, \ldots, M_\nu - 1$$

Model problem: Flow around an obstacle

Flow around an obstacle with 12 640 velocity and 1640 pressure dofs.



Stochastic Galerkin method implemented using the IFISS 3.3 package.[1]
Viscosity with mean $\nu_0 = 1/50$ or $1/150$ and lognormal distribution,

$$CoV = \frac{\sigma_\nu}{\nu_0}, \qquad \mathrm{Re}_0 = \frac{U\,L}{\nu_0} = 100 \text{ or } 300.$$

[1] http://www.cs.umd.edu/~elman/ifiss.html

Model problem: Random viscosity

Viscosity ν: a truncated lognormal process ($\nu_0 = 1/50$ or $1/150$), its representation computed from Gaussian random process (Ghanem, 1999) with the covariance function

$$C\left(X_1, X_2\right) = \sigma_g^2 \exp\left(-\frac{|x_2 - x_1|}{L_x} - \frac{|y_2 - y_1|}{L_y}\right)$$

L_x and L_y: correlation lengths of ξ_i in the x and y directions, respectively, (set to be equal to 25% of the domain size, i.e. $L_x = 3$ and $L_y = 0.5$

σ_g: the standard deviation of the Gaussian random field.

N: the stochastic dimension ($N = 2$)

P: the degree for the polynomial expansion of the solution ($P = 3$), the degree for the expansion of the lognormal process was $2P$

$CoV = \sigma_\nu / \nu_0$: coefficient of variation of the lognormal field (10% or 30%).

$\Longrightarrow M = 10$, $M_\nu = 28$ and $142,800$ dof in the global stochastic matrix.

Model problem: Nonlinear solvers

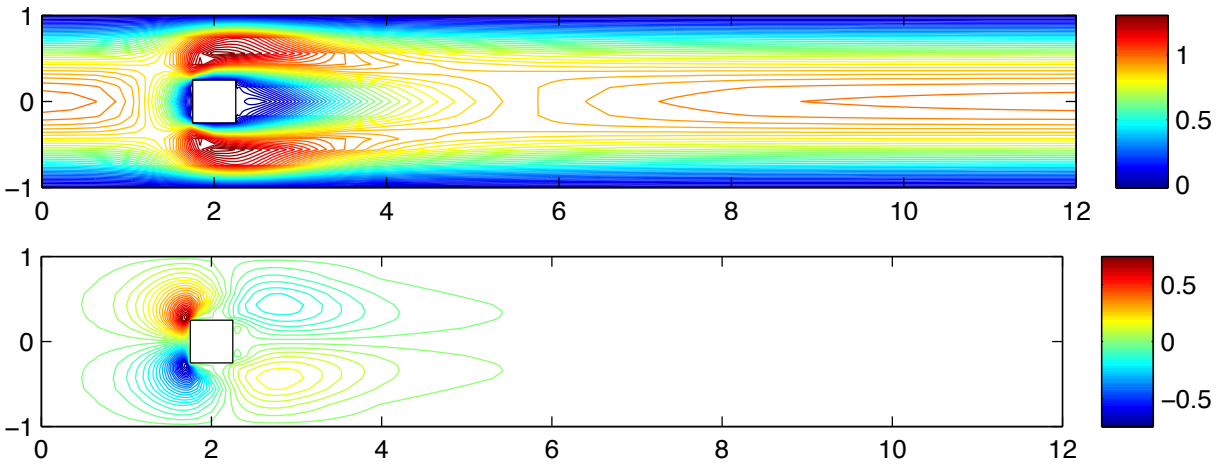
Hybrid scheme: Stokes $\rightarrow$ Picard $\rightarrow$ Newton
(direct solvers for the linear systems, rel. res. 10^{-8})

$Re = 100, \quad CoV = 10\%$:	6 Picard steps	1 Newton step(s)
$Re = 100, \quad CoV = 30\%$:	6	3
$Re = 300, \quad CoV = 10\%$:	20	1
$Re = 300, \quad CoV = 30\%$:	20	2

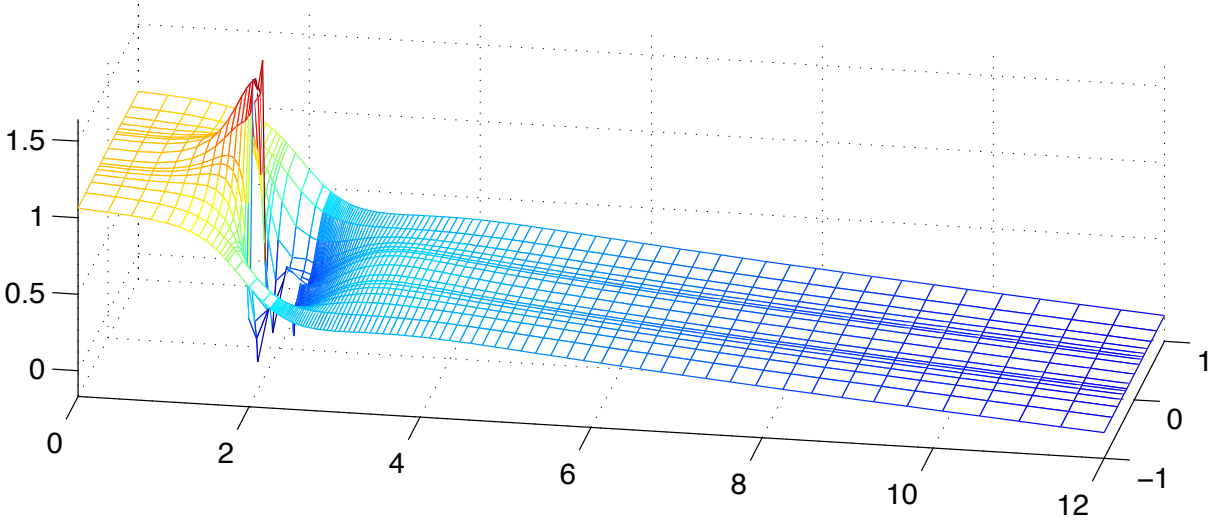
Variant with an *inexact* Picard iteration (block diagonal matrix $\mathbf{H}_0 \otimes \mathcal{F}_0^n$): for $CoV = 10\%$ at most one extra Newton step (both $Re = 100$ and 300), for $CoV = 30\%$ this inexact method failed to converge.

Model problem: Mean velocity and pressure (Re = 100 and $CoV = 10\%$)

Mean of horizontal and vertical velocity:

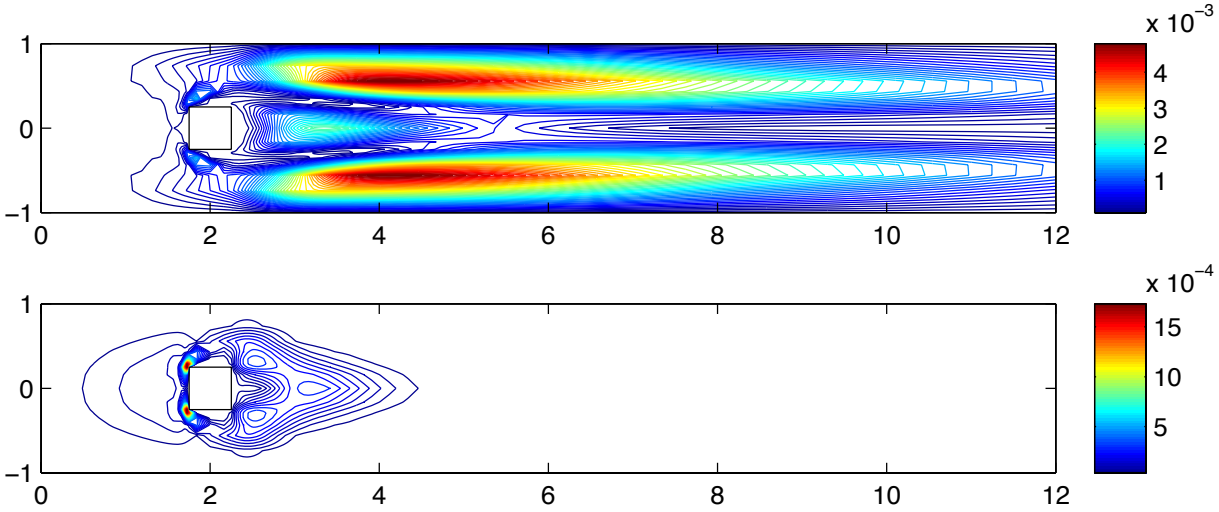


Mean of pressure:

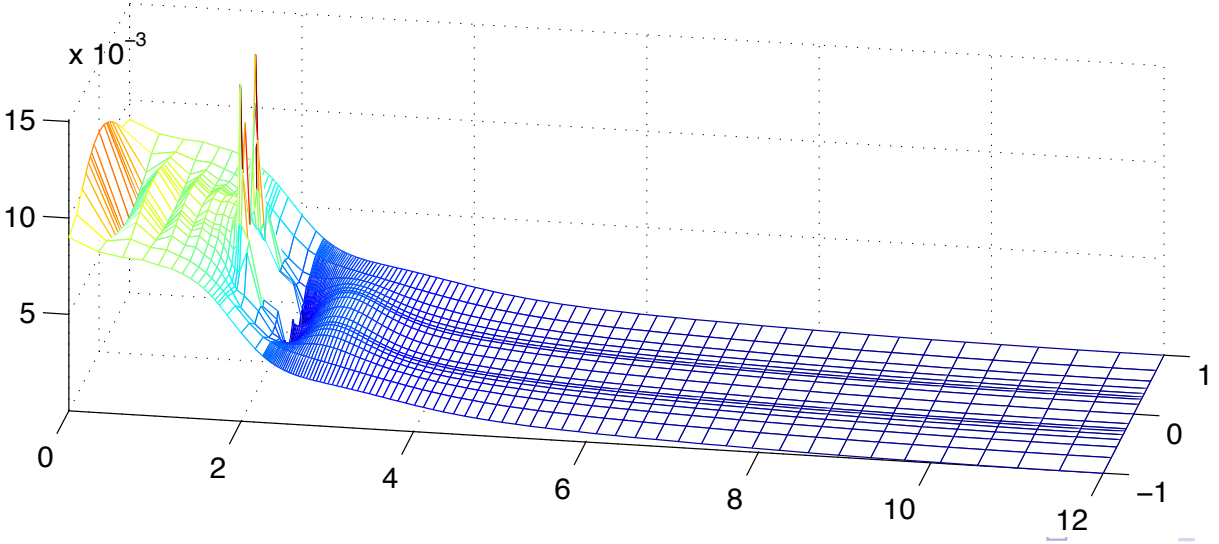


Model problem: Variance of velocity and pressure (Re = 100, $CoV = 10\%$)

Variance of horizontal and vertical velocity:

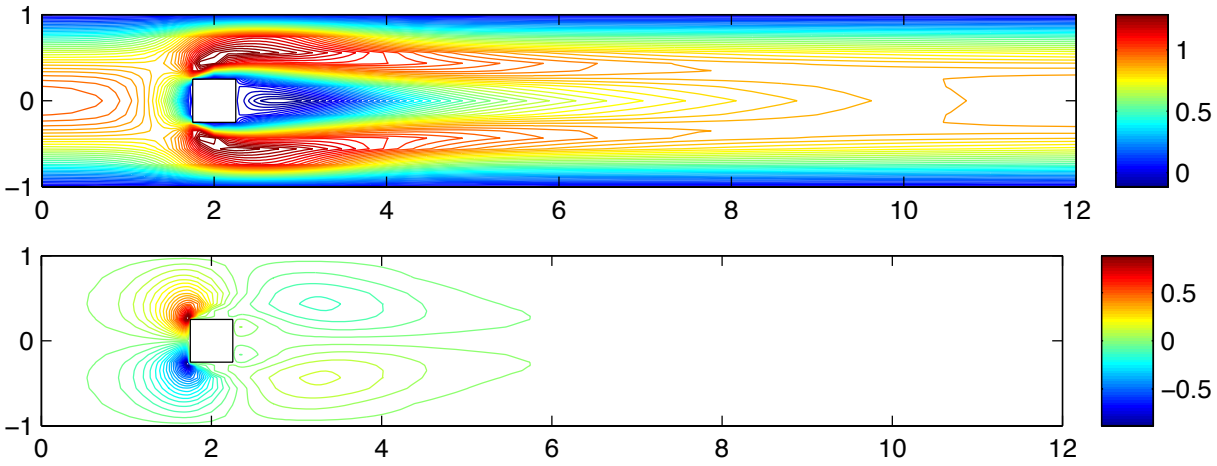


Variance of pressure:

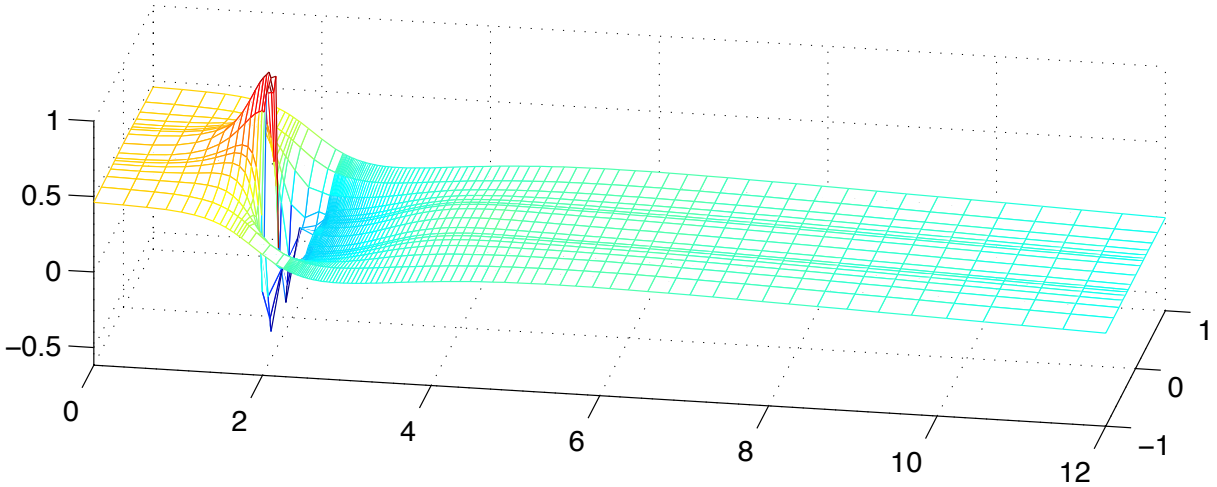


Model problem: Mean velocity and pressure (Re = 300 and $CoV = 10\%$)

Mean of horizontal and vertical velocity:

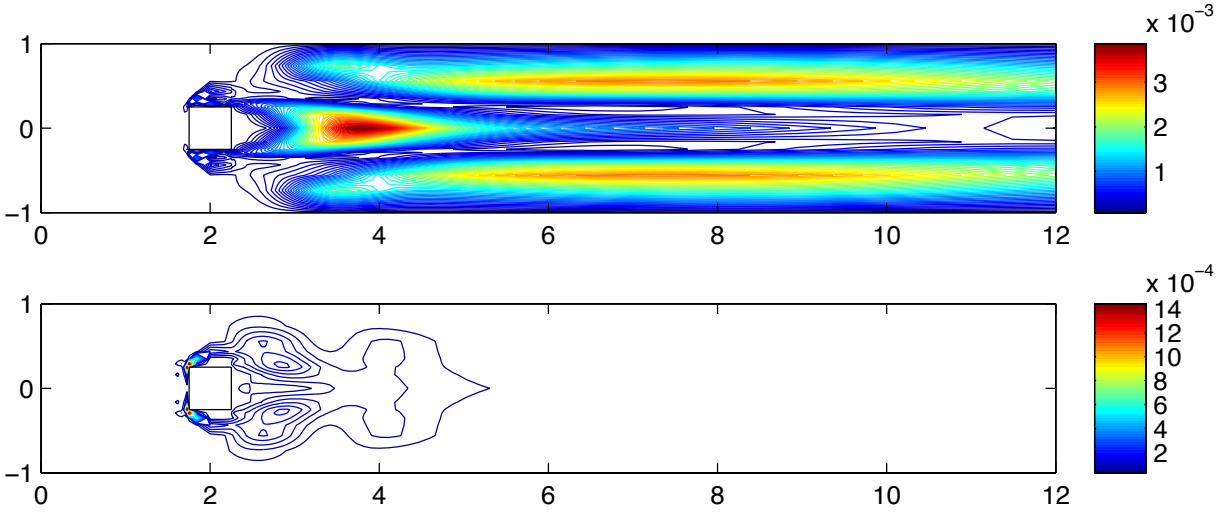


Mean of pressure:

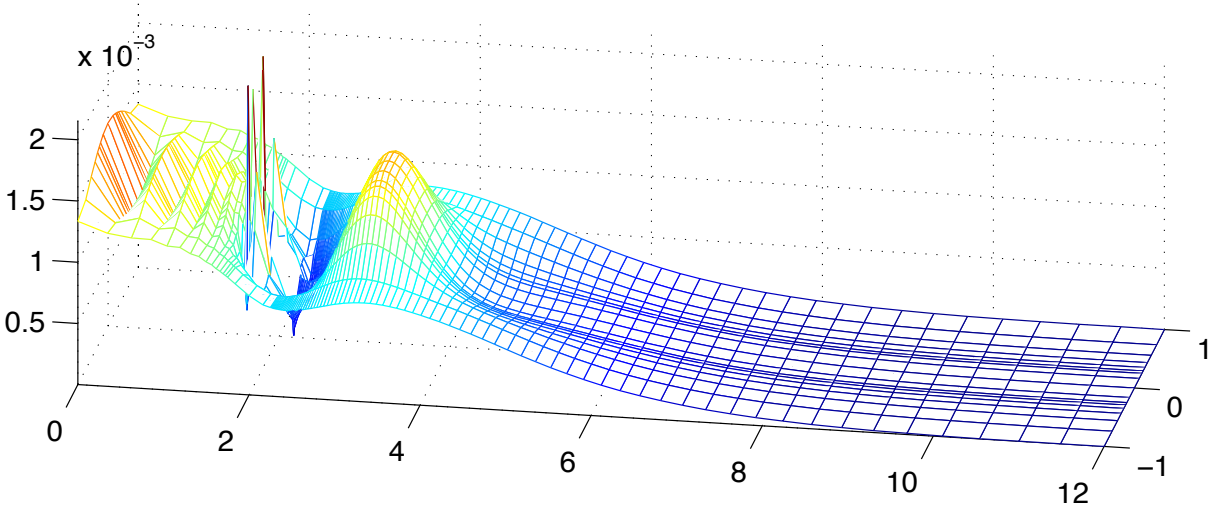


Model problem: Variance of velocity and pressure (Re = 300, $CoV = 10\%$)

Variance of horizontal and vertical velocity:

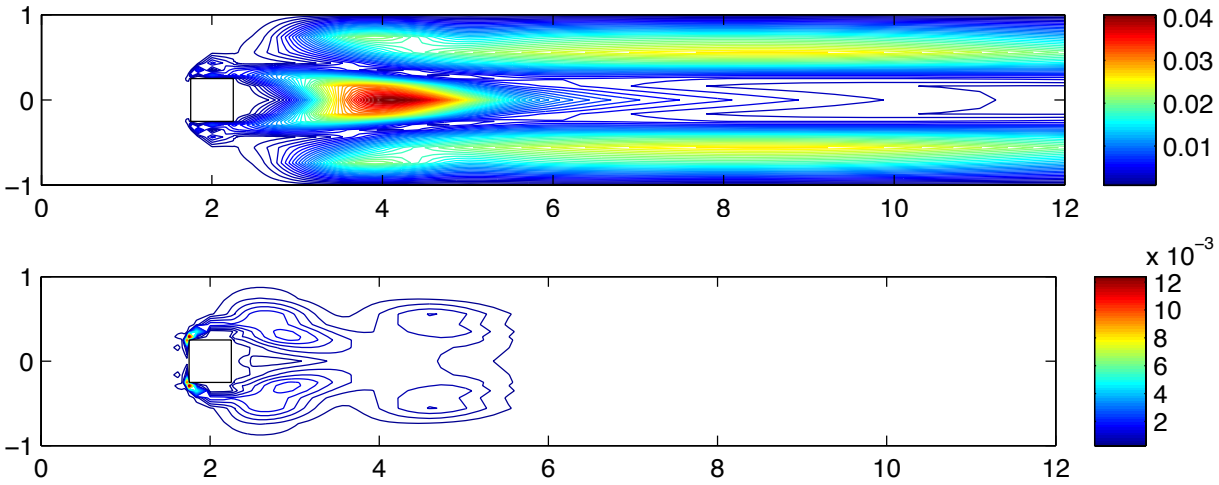


Variance of pressure:

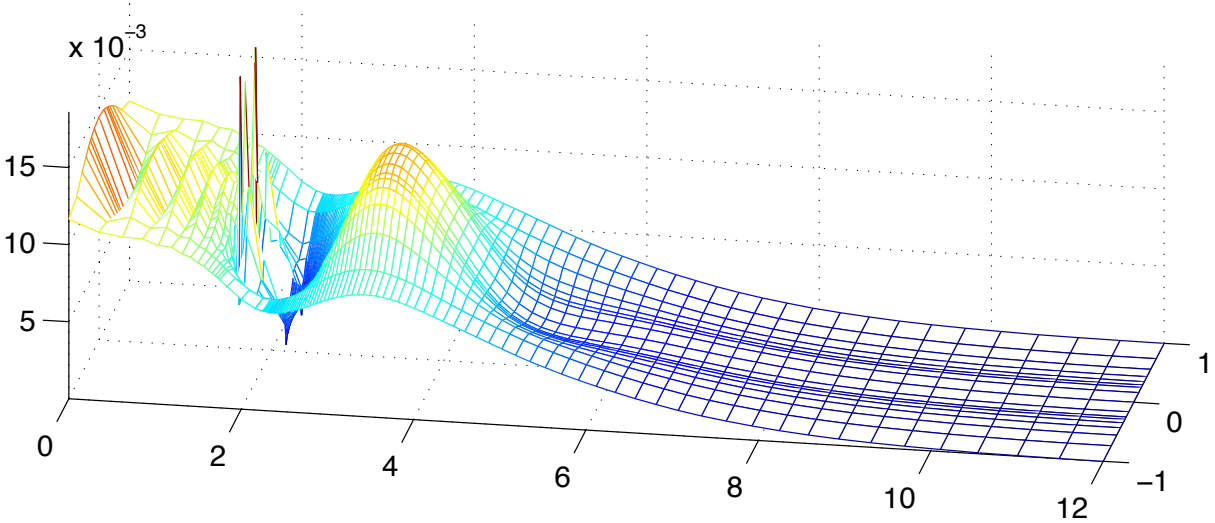


Model problem: Variance of velocity and pressure (Re = 300, $CoV = 30\%$)

Variance of horizontal and vertical velocity:



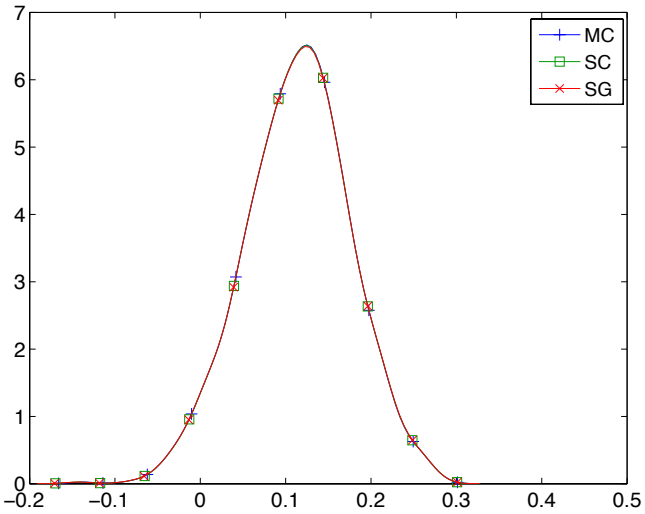
Variance of pressure:

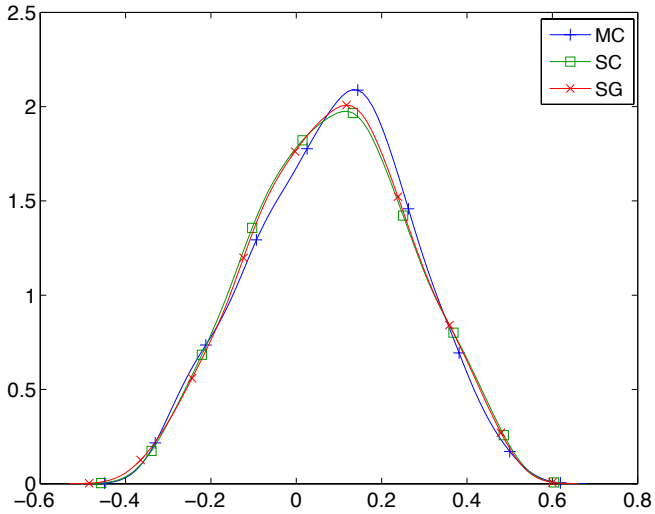


Model problem: pdf of horizontal velocity and pressure at point $(3.6436, 0)$

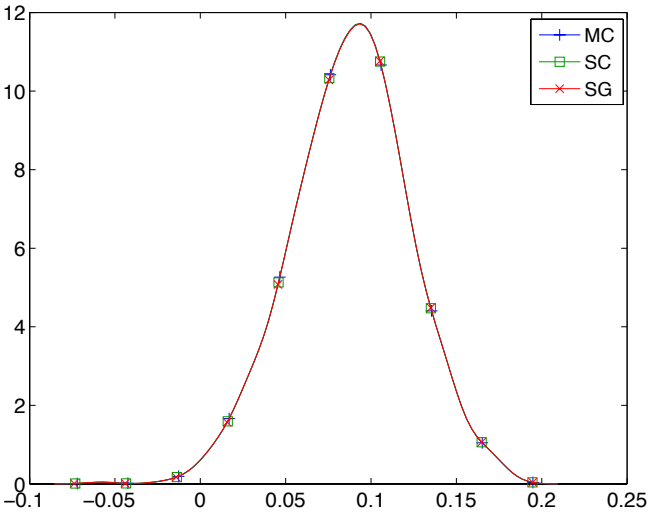
$CoV = 10\%$ $\qquad$ $CoV = 30\%$

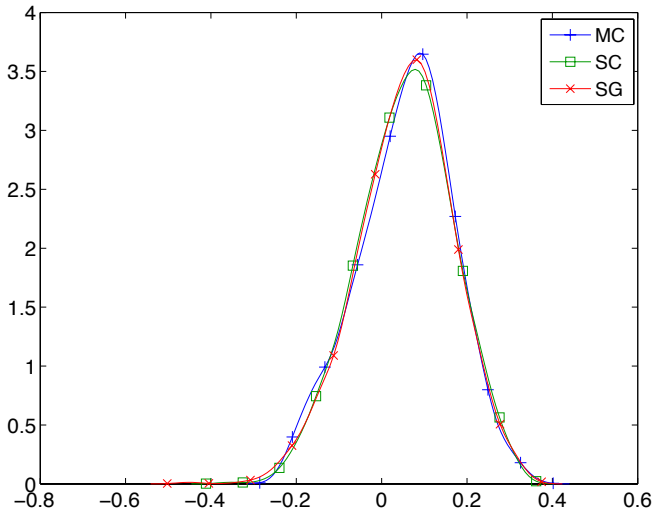
u_x





p





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