Counting curves: which, how and why

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Junior Math Days 2023

Counting points

Doing **enumerative geometry** means finding how many solutions a geometric problem has, without finding the solutions. Typically, the problem and the solutions will depend on parameters, but the number of solutions will not.

Examples

Two lines in a plane meet in **one** point.

Bezout's theorem: Two plane curves C of degree d and C' of degree d', have **exactly** dd' points in common.

Warning: we need complex, projective, and multiplicities!

Counting curves

- There is **one** line through two given points.
- Given four lines in space, there are **two** lines intersecting all of them.
- Any smooth cubic surface contains **27 lines**.
- Given 5 conics in the plane, there are **3264** conics tangent to all of them.

We count solutions when *expected dimension* is zero.
Exp. dim. $=$ $\#$ parameters - $\#$ equations

Which curves do we count

A **projective variety** X is a closed subset of $\mathbb{CP}^N$, zero locus of homogeneous polynomials.

- Chow: every compact complex submanifold of $\mathbb{CP}^N$ is a projective variety; we call it a smooth projective variety.
- Riemann: every compact Riemann surface is a projective curve C.

A map $f : C \to X$ has two invariants: the genus g of C, and the (homology/multidegree) class $\beta := f_*([C])$.

How we count, basis case

- The set of all lines in projective space is a smooth projective variety G of dimension 4.
- Degree three homogenous polynomials define a rank 4 vector bundle E on G; choosing a specific cubic defines a section s of E.

The number of points in G such that $s = 0$ is

$$\int_G c_4(E)$$

Moduli of maps

- **Good news** We fix X, g and β. The set $M(X; g, \beta)$ of all maps $f : C \to X$ of genus g and class β has a natural structure of moduli space.
  Conditions we are interested in can be expressed as vanishing of a section of a vector bundle E on $M(X; g, \beta)$.
- **Bad news** $M(X; g, \beta)$ is not compact/proper. Also, I didn't tell you what kind of structure.
- **Good news** There is a natural compactification $\overline{M}(X; g, \beta)$ (Gromov/Konstevich-Manin).
- **Bad news** $\overline{M}(X; g, \beta)$ has no well defined dimension (e.g. for $X = \mathbb{P}^2$, $g = 1$, $\beta = 3$).

$\overline{M}(X; g, \beta)$ has an *expected dimension* of is

$$d := (3 - \dim X)(g - 1) + c_1(T_X)\beta.$$

(index of a Fredholm operator/deformation theory).
Note special case $\dim X = 3$ and $c_1(T_X) = 0$ that is X is a Calabi-Yau threefold.
Problem Both spaces can locally be "made into" manifolds of dimension d. But to get numbers we need a global calculation, not a local one!

Why: string theory enter the picture

Based on string theory ideas, Edward Witten asked himself in 1991: what if we **pretend** $\overline{M}(X; g, \beta)$ is a smooth of the expected dimension d?

He then showed that, if this were the case, a lot of counting curves problems could in principle be solved: the solutions have been called **Gromov-Witten invariants**.

He combined the invariants in a partition function Z, and he and other physicist "derived" that Z must satisfy a series of partial differential equations, PDEs. In some cases such PDEs are enough to compute the invariants.

Interesting, promising results... based on an assumption that was *known to be wrong*.

Kontsevich and Manin produced an axiomatization of Witten's argument.

Kontsevich then showed that, if KM axioms could be made rigorous, the problem of counting plane curves of genus zero and any degree could be solved recursively.

Goal for mathematicians: replace Witten's "derivation" by PROOF(S).

Progress in symplectic geometry

Symplectic geometers solved the problem in a very large number of cases.

First of all, it was shown that in many cases one could make $\overline{M}(X; g, \beta)$ a compact manifold of dimension d by fiddling with the complex structure of X, to a generic almost complex structure.

Then, it was shown that in even more cases one could get a manifold by fiddling with the partial differential equation defining $\overline{M}(X; g, \beta)$ via a Fredholm operator.

Algebraic geometry shows up

Consider the curve C of equation $y + x^3 = 0$ in the plane $\{z = 0\}$, and let X be the cylinder over C (that is, the set of (x, y, z) such that $y + x^3 = 0$).

Which lines are contained in X? As a set: the vertical lines meeting $z = 0$ in a point of C. But what are the *equations*? It's called *algebraic* geometry, so let's do algebra.

Non horizontal lines L depend on 4 parameters (a, b, u, v) where $(a, b, 0) = L \cap \{z = 0\}$ and $(u, v, 1)$ is the direction of L.
We call such a line $L(a, b, u, v)$.

Parametrically, we can write the line $L = L(a, b, u, v)$ as

$$(a, b, 0) + t(u, v, 1) = (a + tu, b + tv, v).$$

Algebraically $L \subset X$ iff $(a + tu) + (b + tv)^3 = 0$: the equation of X restricts to zero on L.

We get four equations, the coefficients of $1, t, t^2, t^3$:

- $a^3 + b = 0$;
- $3a^2u + v = 0$;
- $3au^2 = 0$;
- $u^3 = 0$.

We can solve in v and b and get two free parameters a, u with conditions $3a^2u = a^3 = 0$. This does NOT mean $a = 0$!

Algebraic geometry progress

Kai Behrend and Yuri Manin recast the Kontsevich Manin axioms not in terms of numbers, but of a **virtual class**, a homology class on the moduli space of the expected dimension, over which to make integrals.

This wasn't a new idea in algebraic geometry, and in fact a very general formulation had been produced by Fulton and MacPherson; however such class existed only if there was a global presentation, not only a local one.

Problem solved!

In 1996, Jun Li and Gang Tian gave a construction of the virtual class, in full generality and proved all its properties.

The key ingredient is a so called tangent-obstruction complex, and the technique involves local choices, gluing, and showing that the result doesn't depend on the choices.

Why am I telling you this?

In 1997, Kai Behrend and I replaced Li and Tian's hard work by putting together more advanced, if still classical (before 1980), techniques:

1. algebraic stacks (Artin, 1974);
2. cotangent complex and deformations (Illusie, 1968);
3. derived categories and Picard stacks (Deligne, 1977).

Our paper is pretty unreadable! To everyone's relief, we also provided a very concrete criterion to apply our machinery in most cases where one can define an expected dimension at all.

Gromov Witten invariants: computational techniques

Tom Graber and Rahul Pandharipande used our construction to prove a virtual version of the classical localization formula in intersection theory, which became one of the key computational methods in the field.

The construction led naturally to proving relationships between enumerative invariants, without actually computing them. For instance, Kontsevich's original formula counts recursively the number N_d of curves of genus zero and degree d through $3d - 1$ generic points in the plane, using as seed just the number N_1.

Gromov Witten invariants are a source of examples of Frobenius manifolds, which brought fruitful interactions with the research in integrable systems.

Donaldson invariants

The original motivation for Li and Tian had been not just to give a definition of Gromov Witten invariants, but also to redefine Donaldson invariants, heavily based on analysis methods, in a purely algebraic geometry set-up.

This became indeed possible, but not in full generality because of a technical issue with strictly semistable sheaves.

Donaldson – Thomas invariants

In 1998, Donaldson proposed to his PhD student Richard Thomas to construct invariants for some well-behaved 6-dim manifolds.

Thomas tried symplectic techniques, but only succeeded by shifting to algebraic geometry and using our method on projective 3folds.

The special case where K_Y is trivial, so called *Calabi Yau threefold*s, became immediately popular among geometers and string theorists.

What next? Outside algebraic geometry

- Theoretical physics
- Integrable systems
- Symplectic geometry
- PDEs on formal power series
- Index of Fredholm operators among Banach orbibundles

What next? Gromov Witten

- Relative GW invariants, degeneration formulas
- GW for orbifolds
- open GW invariants
- GW invariants in characteristic p
- Other invariants (non-contracting maps, quasimaps...)

What next? Donaldson-Thomas

- Invariants in CY threefolds, including noncompact ones
- Wall crossing methods
- Invariants for Quot schemes
- Degeneration formulas
- Behrend function

Theoretical advances, mild

- Virtual pushforward and pullback
- Virtual structure sheaf
- Symmetric obstruction theories
- Use of master spaces
- Virtual Grothendieck Riemann Roch

Infinity category advances

- Quasismooth derived schemes
- Shifted symplectic structures
- Categorification of Behrend function
- Trimester at IHP 2023!

Thank you!