Geometry and Control

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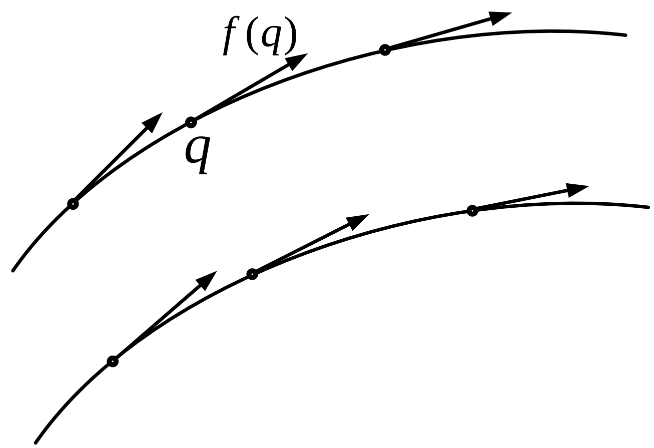
Smooth dynamical system:

$$\dot{q}(t) = f(q(t)), \quad q \in M, \ t \in \mathbb{R},$$



generates a flow

$$P^t : M \to M, \quad P^t : q(0) \mapsto q(t), \quad t \in \mathbb{R}.$$

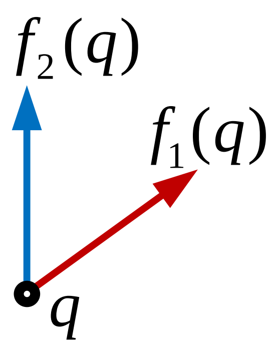
Control system:

$$\dot{q} = f_u(q), \quad u \in U.$$

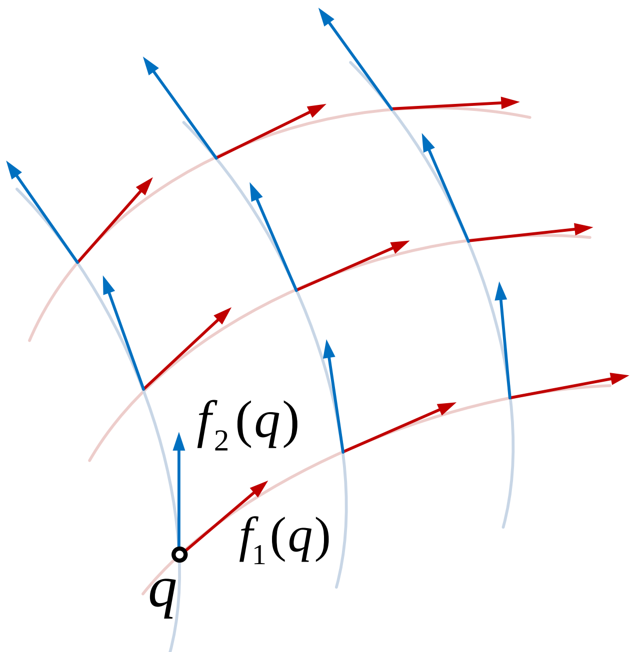
Control: $t \mapsto u(t),\ t \geq 0$.

Trajectory: $t \mapsto q(t)$, where $\dot{q}(t) = f_{u(t)}(q(t))$.

Special case: $U = \{1, 2\}$:

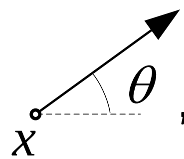


Trajectories:

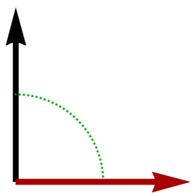
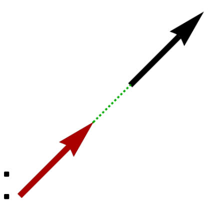


Example. Unicycle:

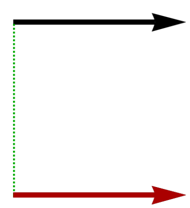
$$q = (x, \theta) :$$



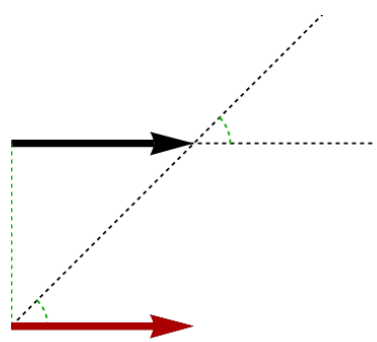
$$, \; M = \mathbb{R}^2 \times S^1.$$

P_1^t :  , P_2^t : 

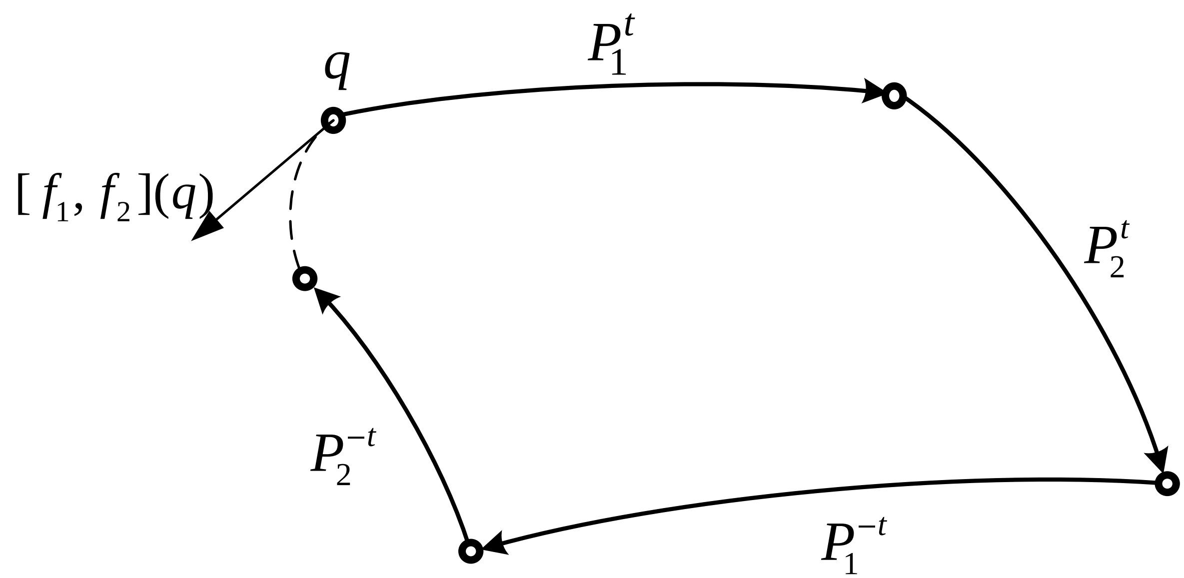
Third direction is the "parallel parking":



We realize it using only $P_1^{\pm t}$ and $P_2^{\pm t}$:



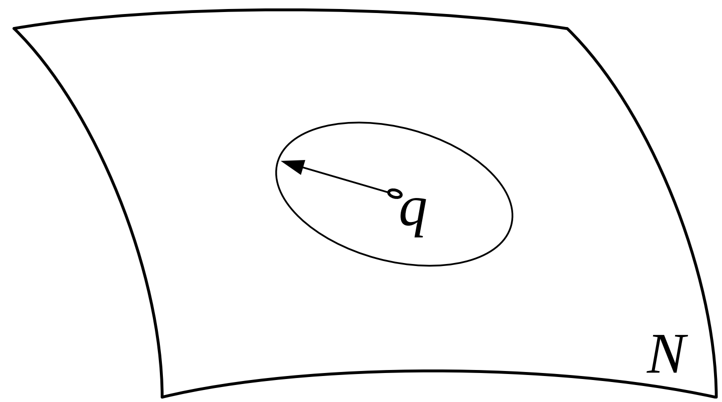
The commutator of flows and vector fields:



$$P_2^{-t} \circ P_1^{-t} \circ P_2^t \circ P_1^t(q) = q + t^2[f_1, f_2](q) + O(t^3).$$

In the example, the field $g = [f_1, f_2]$ generates the parallel parking.

Similar problem on a non flat surface:



M is the "spherical bundle" of all tangent vectors of the length 1 to the surface N.

The flow P_1^t is as before, P_2^t is the geodesic flow:

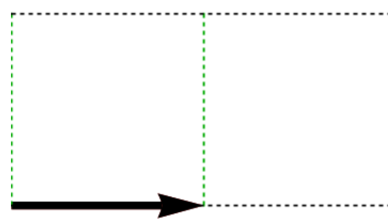
$$P_2^t : (x(0), \dot{x}(0)) \mapsto (x(t), \dot{x}(t)),$$

where $\ddot{x}(t) \perp N$.

Then $g = [f_1, f_2]$ generates the "parallel parking".

The difference between the flat and non flat cases:

"translation" f_2 and "parallel parking" g obviously commute in the flat case.



In general, $[g, f_2] = \kappa f_1$, where κ is the curvature!

In higher dimensions, the field $[g, f_2]$ may be linearly independent on $f_1, f_2, [f_1, f_2] = g$.

General case:

$$\text{Lie} f_U = span\{[f_{u_1}, [\cdots, f_{u_k}] \ldots] : u_i \in U,\ k = 1, 2, \ldots\},$$

Theorem (Rashevskij–Chow). *If* $Lie f_U|_q = T_q M, \ \forall q \in M$, *then* $\forall\, q_0, q_1 \in M \ \exists\, u_i, \in U, t_i \in \mathbb{R}, \ i = 1, \ldots, k,$ *such that*

$$q_1 = P_{u_k}^{t_k} \circ \cdots \circ P_{u_1}^{t_1}(q_0).$$

We set

$$\text{Gr} f_U = \{P_{u_k}^{t_k} \circ \cdots \circ P_{u_1}^{t_1} : u_i, \in U,\ t_i \in \mathbb{R},\ i = 1, \ldots, k,\ k > 0\} \subset \text{Diff} M.$$

Corollary. *Let* $\dim M > 1$ *and* $\text{Lie} f_U$ *is everywhere dense in* $\text{Vec}(M)$ *in the* C_0*-topology. Then for any finite families of points* $x_\alpha, y_\alpha \in M, \ \alpha \in \mathcal{A}, \ \#\mathcal{A} < \infty$, *there exists* $P \in \text{Gr} f_U$ *such that* $P(x_\alpha) = y_\alpha, \ \forall\, \alpha \in \mathcal{A}$.

Let $\ell > 0$, $K \Subset M$; we set:

$$\mathrm{Lie}^{\ell}_{K} f_U = \left\{ g \in \mathrm{Lie} f_U : \sup_{x \in K} (|g(x)| + \|\nabla_x g\|) < \ell \right\}.$$

Definition. *We say that f_U has property (A) if for any smooth vector field X and any $K \Subset M$ there exists $\ell > 0$ such that*

$$\inf \left\{ \sup_{x \in K} |g(x) - X(x)| : g \in \mathrm{Lie}^{\ell}_{K} f_U \right\} = 0.$$

Theorem. *If f_U has property (A), then for any isotopic to the identity diffeomorphism $\Phi : M \to M$, $K \Subset M$, and $\varepsilon > 0$, there exists $P \in \mathrm{Gr} f_U$ such that $\sup_{x \in K} \delta(P(x), \Phi(x)) < \varepsilon$, where $\delta(\cdot, \cdot)$ is the Riemannian distance in M.*

Examples:

1. $M = \mathbb{R}^n$; the family of vector fields:

$$\frac{\partial}{\partial x_i}, \quad e^{-|x|^2}\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n,$$

has property (A). The iterated commutators of these vector fields produce Hermit polynomials.

2. $M = \mathbb{T}^n = \{(\theta_1, \ldots, \theta_n) : \theta_i \in \mathbb{R}/2\pi\mathbb{Z}\}$. The family of vector fields:

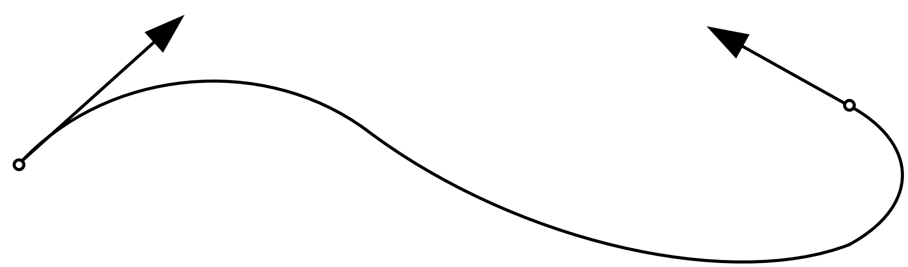
$$\frac{\partial}{\partial \theta_i}, \quad \sin(\theta_i)\frac{\partial}{\partial \theta_i}, \quad \sin(2\theta_i)\frac{\partial}{\partial \theta_i}, \quad \sum_{j=1}^{n} \sin(\theta_j)\frac{\partial}{\partial \theta_i}, \quad i = 1, \ldots, n,$$

has property (A).

Optimal control

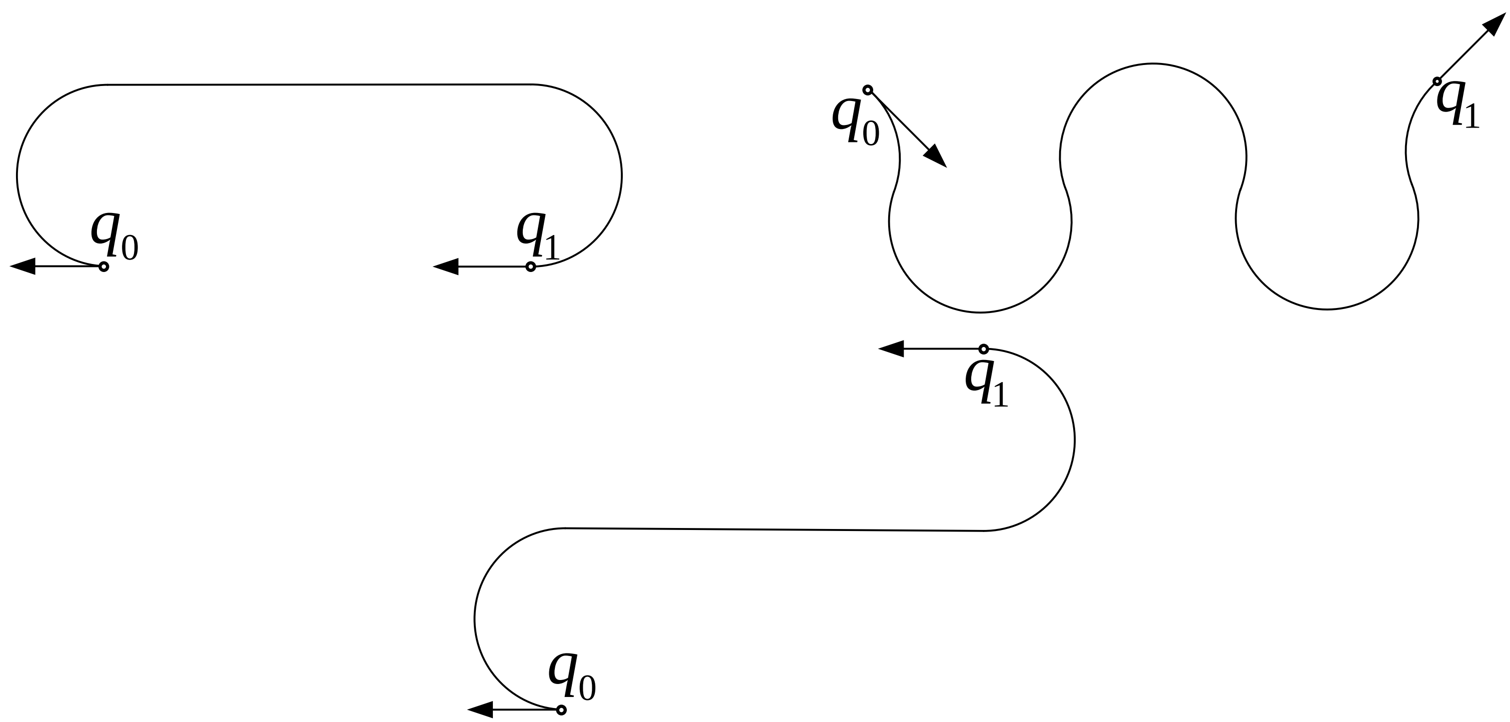
Example. $M = \mathbb{R}^2 \times S^1$, $\dot{q} = uf_1(q) + f_2(q)$.

Admissible trajectories $t \mapsto q(t) = (x(t), \theta(t))$, where $\dot{x} = \binom{\cos\theta}{\sin\theta}$ and $\dot{\theta} = u$ is the curvature of the curve $x(\cdot)$.



I) Markov–Dubins problem: $|u| \le c$, minimise the length of $x(\cdot)$ (= time t_1).

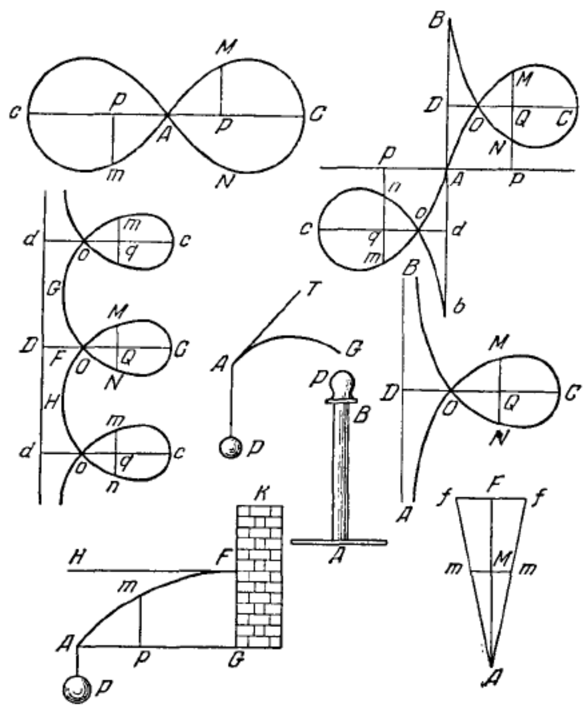
“Geodesics” are special concatenations of circle and linear segments:



Optimal piece has no more than 3 switchings.

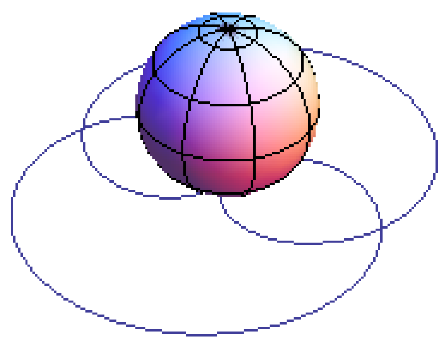
II) Euler elastic problem: $u \in \mathbb{R}$, minimise $\int\limits_0^{t_1} u^2(t)\,dt$.

"Geodesics" are *elasticae*: the curves whose curvature is a linear function of coordinates, $u(t) = \langle a, x(t) \rangle + \alpha$.



Both problems are translation and rotation invariant, where $S^1 = \mathsf{SO}(2)$ is the group of rotations.

Example (rolling without slipping or twisting).
Here $M = \mathbb{R}^2 \times \mathsf{SO}(3)$, $q = (x, X)$, where $x \in \mathbb{R}^2$, $X \in \mathsf{SO}(3)$.



$$u = \binom{u_1}{u_2}, \quad \dot{x} = u, \quad \dot{q} = u_1 F_1(q) + u_2 F_2(q).$$

Minimize $\int_0^{t_1} |u(t)|\, dt$, the length of $x(\cdot)$.

Minimal length:

$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)|\, dt : q(0) = q_0, q(t_1) = q_1 \right\}$$

is a metric on M:

$$\delta(q_0, q_1) = \delta(q_1, q_0), \quad \delta(q_0, q_2) \leq \delta(q_0, q_1) + \delta(q_1, q_2).$$

"Rolling geodesics" are again elasticae.

General sub-Riemannian problem:

$$\dot{q} = \sum_{i=1}^{k} u_i F_i(q), \quad q \in M, \quad u = (u_1, \ldots, u_k)^T \in \mathbb{R}^k,$$

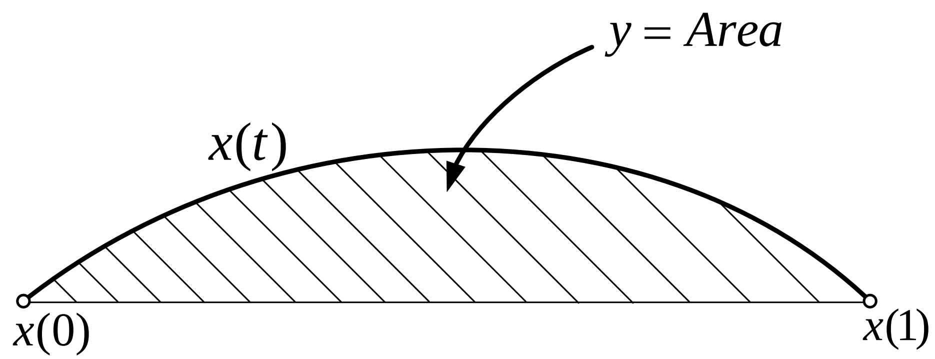
defines a "Carnot–Caratheodory metric" on M:

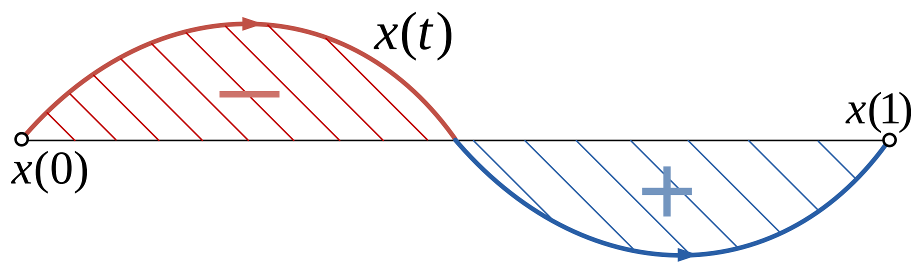
$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)| \, dt : q(0) = q_0, q(t_1) = q_1 \right\}$$

3-dim examples:

I) Dido problem. $q = (x, y),\ x \in \mathbb{R}^2,\ y \in \mathbb{R}$,

$$\begin{cases} \dot{x} = u \\ \dot{y} = \frac{1}{2} x \wedge u \end{cases}$$



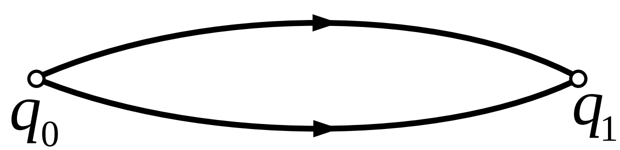

$x(t)$
$-$
$+$
$x(0)$
$x(1)$

II) Interpolation problem.

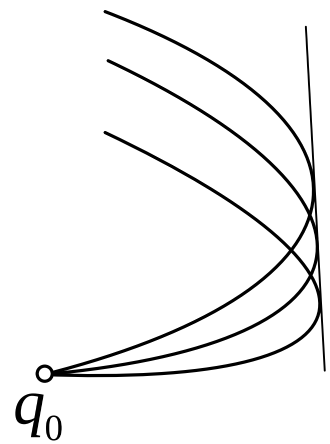
$$M = \mathbb{R}^2 \times S^1, \quad \dot{q} = u_1 f_1 + u_2 f_2.$$

Where to “cut” geodesics?

Maxwell points: at least two geodesics of equal length connect the same points.



Conjugate points: the envelope of the family of geodesics starting from q_0.

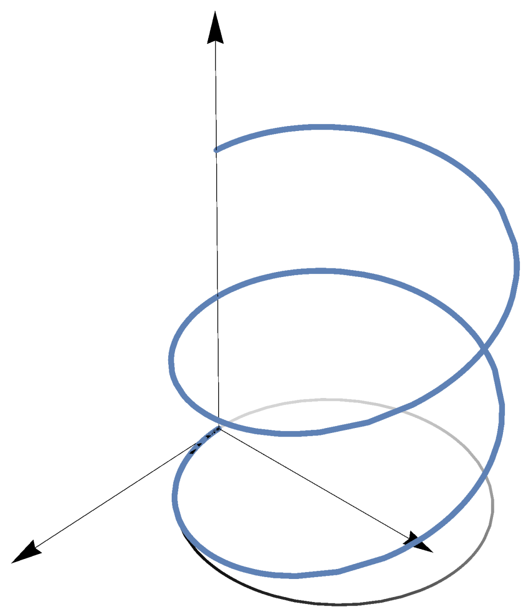


The wave front $W_{q_0}(r)$ is the set of endpoints of the length r geodesics starting from q_0.

The sphere $S_{q_0}(r)$ is a part of the wave front.

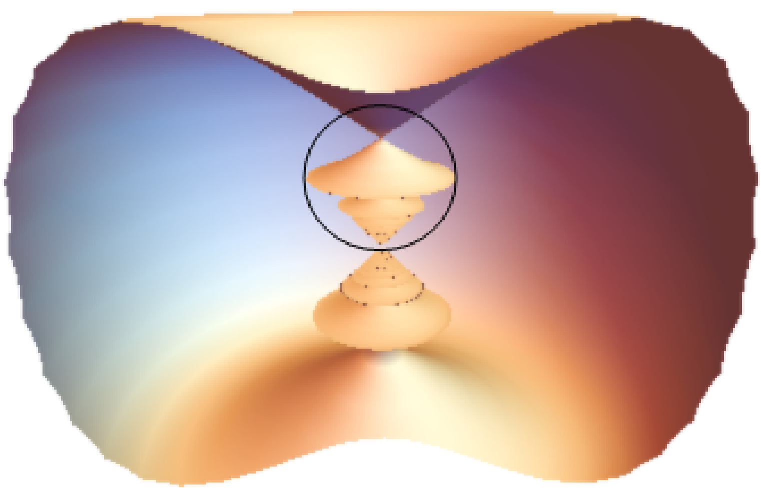
Dido problem.

Geodesics:

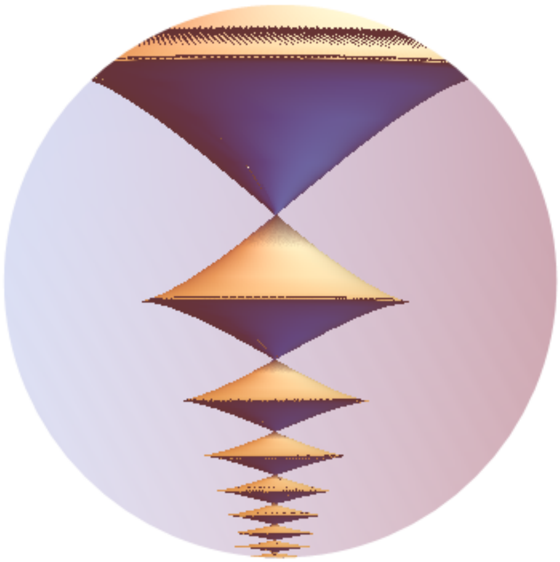


They are characterized by the initial velocity and acceleration: greater the acceleration, more tough is the spiral.

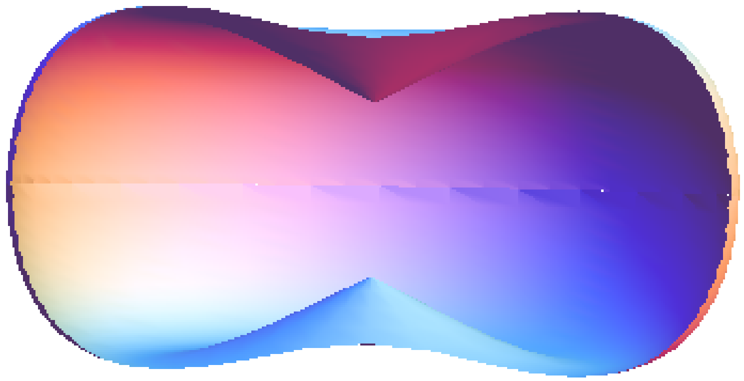
The wave front:



$\Rightarrow$



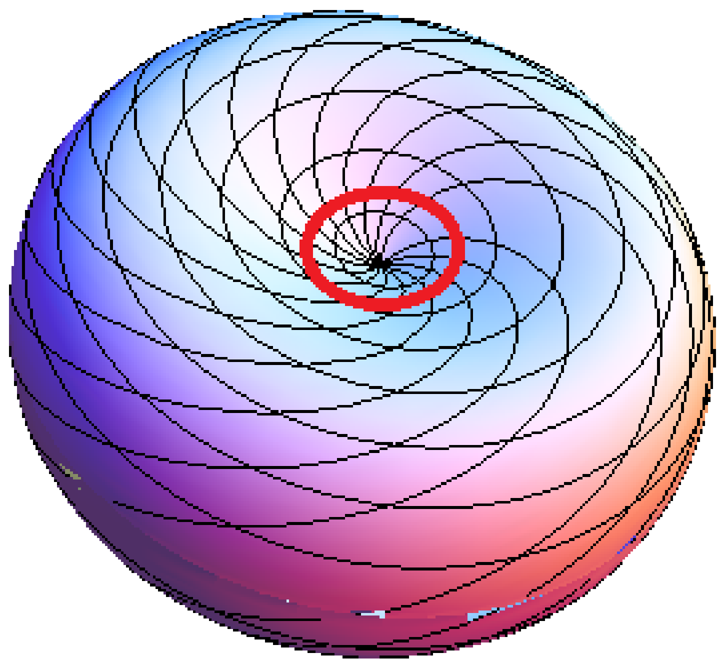
The sphere:



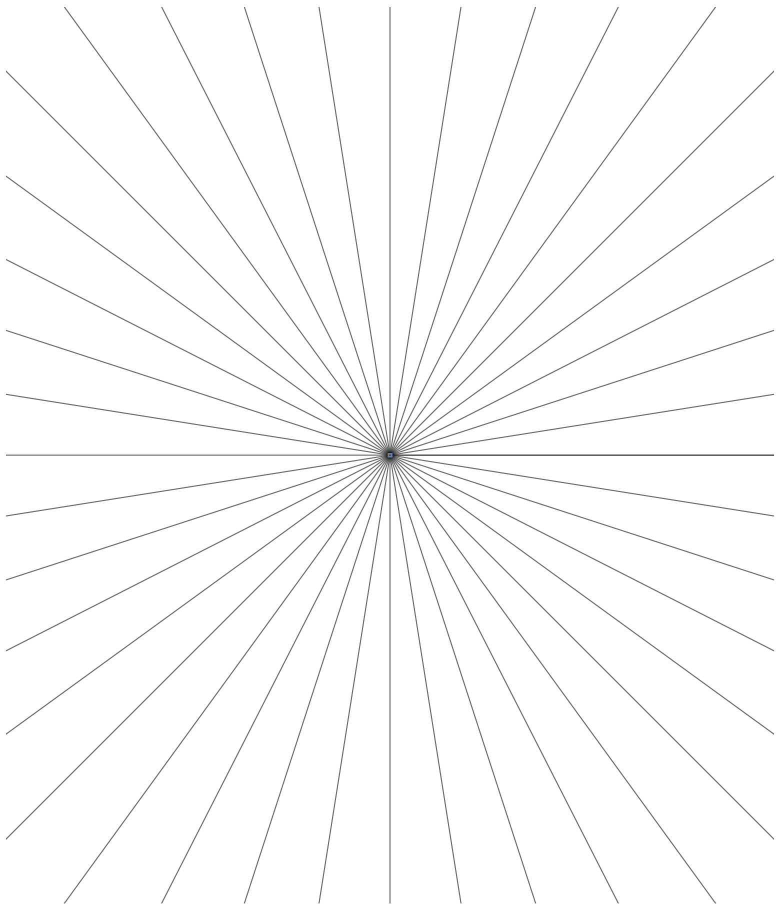
First Maxwell and conjugate points coincide and belong to the vertical line due to the rotational symmetry.

Breaking the symmetry.

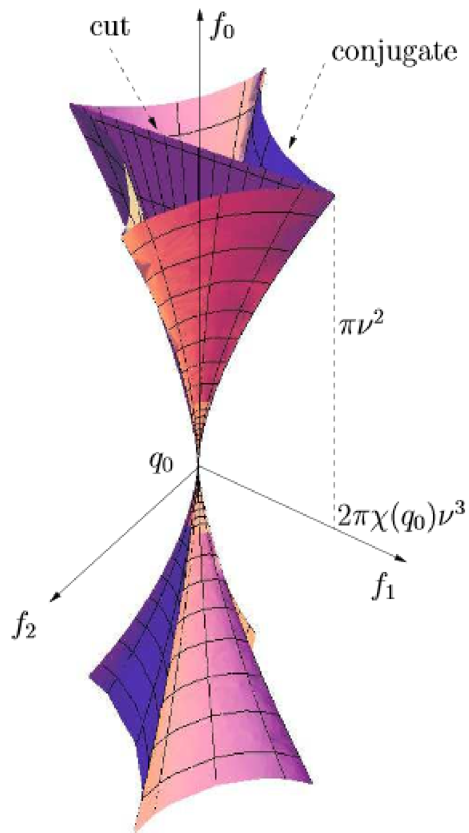
Look under the microscope:



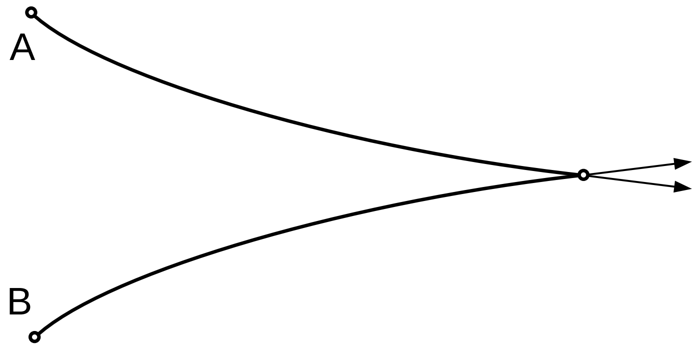
Symmetric $\Rightarrow$ Generic



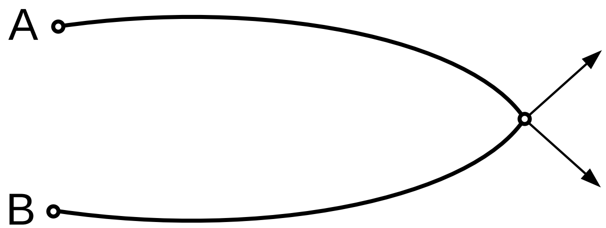
First conjugate and cut loci:



Curvature, basic idea:



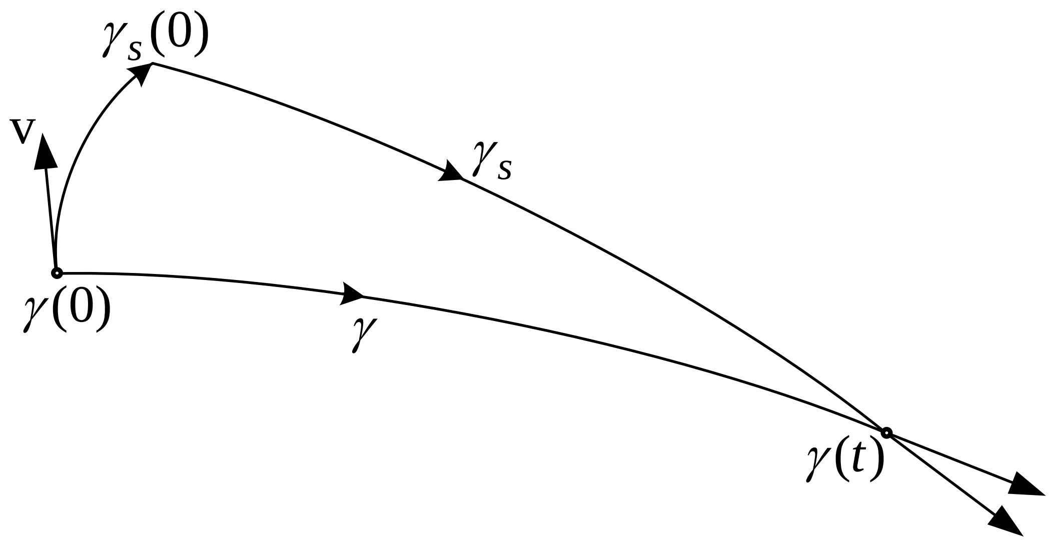
negative curvature



positive curvature

Bigger the curvature – bigger the difference between velocities in the intersection point.

Construction. Given a geodesic γ, horizontal tangent vector $\mathsf{v} \perp \dot\gamma(0)$ and $t > 0$, consider a family of geodesics:



The curvature:

$$\frac{4}{3}R_\gamma(\mathsf{v}) = \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial s^2}\left(t^2|\dot\gamma_s(t) - \dot\gamma(t)|^2\right)\Big|_{t=s=0}.$$

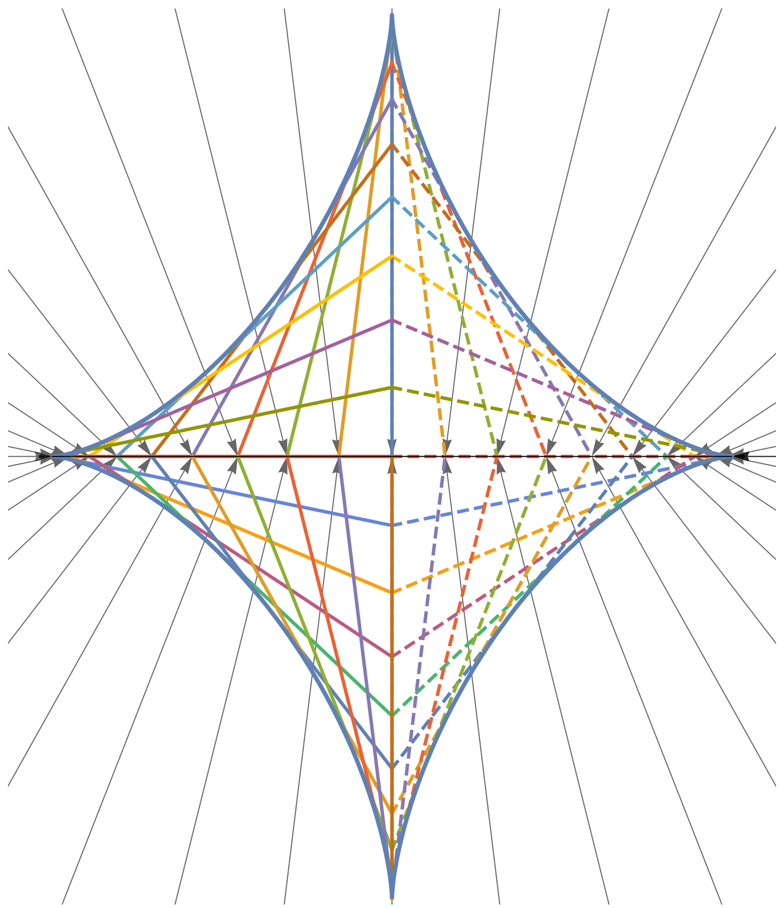
R_γ is a quadratic form of $(k-1)$ variables where k is the dimension of the horizontal subspace. If k equals 2, then $R_\gamma \in \mathbb{R}$.

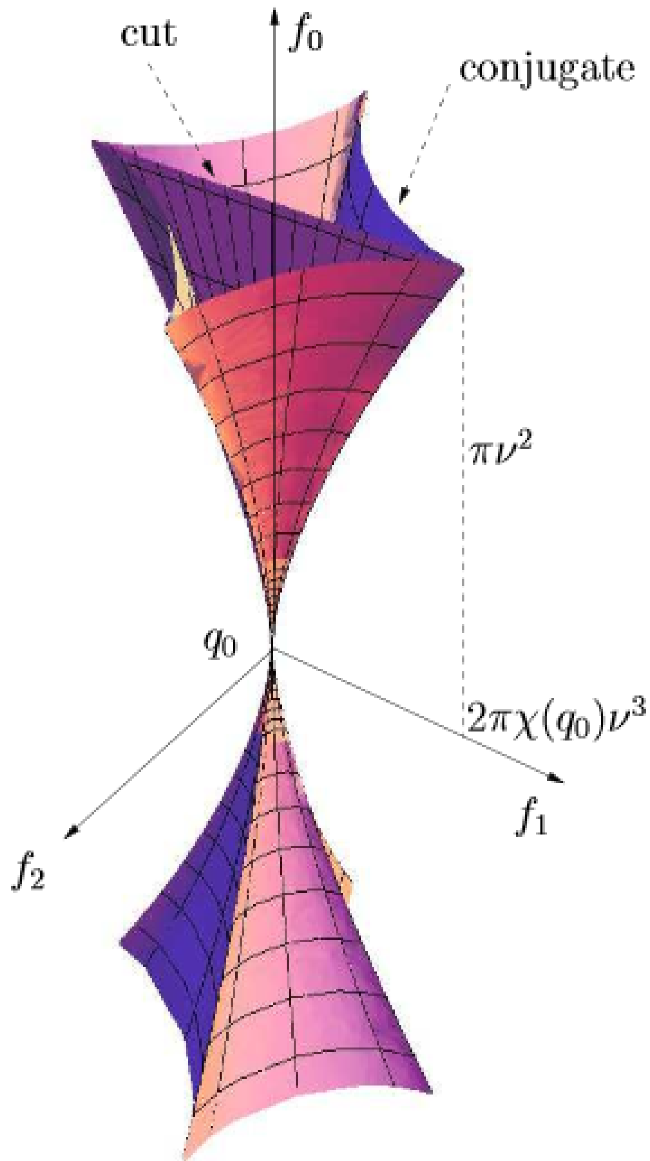
3-dim problems. Let ν be the initial acceleration of γ; then

$$R_\gamma = \nu^2 + Q(\dot{\gamma}(0)),$$

where Q is a quadratic form. $Q = 0 \Leftrightarrow$ we are locally isometric to the Dido problem.

Maxwell segments are eigenvectors of Q; the visible on the sphere first Maxwell segment corresponds to the bigger eigenvalue.




cut
f_0
conjugate
$\pi\nu^2$
q_0
$2\pi\chi(q_0)\nu^3$
f_1
f_2

Number $\chi(q_0)$ is the difference between the eigenvalues of Q.

Fast-oscillating control is a way to control the state of a very big dimension by few control parameters. Let

$$f_u = \sum_{i=1}^{k} u^i f_i, \quad u = (u^1, \ldots, u^k) \in U,$$

where U is a neighborhood of $0 \in \mathbb{R}^k$.

Take a sample vector-function $t \mapsto v(t)$, $supp\{v(\cdot)\} \subset [0,1]$ and let

$$\dot{q}_\varepsilon(t) = f_{v\left(\frac{t}{\varepsilon}\right)}(q_\varepsilon), \quad q_\varepsilon(0) = q_0.$$

Then, for any "observable" $a : M \mapsto \mathbb{R}$, we have:

$$a(q_\varepsilon(t)) \approx a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int\int_{\Delta_i} f_{v(t_i)} \circ \cdots \circ f_{v(t_1)} a(q_0)\, dt_1 \ldots dt_i$$

$$= a(q_0) + \sum_{i=1}^{\infty} \varepsilon^i \int\int_{\Delta_i} p_i(v(t_i)), \ldots, v(t_1))\, dt_1 \ldots dt_i,$$

where $\Delta_i = \{(t_1, \ldots, t_i) : 0 \leq t_i \leq \cdots \leq t_1 \leq 1\}$ and p_i is a i-linear form, $p_i(v_i, \ldots, v_1) = \langle \omega_i, v_i \otimes \cdots \otimes v_1 \rangle$.

We set $\gamma(t) = \int_0^t v(\tau)\, d\tau$, the i-th order term takes the form:

$$\varepsilon^i \left\langle \omega_i, \int\int_{\Delta_i} d\gamma(t_i) \otimes \cdots \otimes d\gamma(t_1) \right\rangle.$$

We set:

$$D^n(\gamma) = \int\int_{\Delta_n} d\gamma(t_n) \otimes \cdots \otimes d\gamma(t_1),$$

where γ is a Lipschitz curve in $\mathbb{R}^k$, $\gamma(0) = 0$.

In particular, $D^1(\gamma) = \gamma(1)$. If γ is a closed curve then principal term is $D^2(\gamma)$. Moreover,

$$D^2(\gamma) = \int_0^1 \dot{\gamma}(t) \wedge \gamma(t)\, dt + \frac{1}{2}\gamma(1) \otimes \gamma(1).$$

Let $\Omega_n = \{\gamma : D^1(\gamma) = \cdots = D^{n-1}(\gamma) = 0\}$.

We know that $D^n(\Omega_n) = Lie^n(\mathbb{R}^k) \subset (\mathbb{R}^k)^{\otimes n}$.

If $\gamma \in \Omega_n$ and $D^n(\gamma) = \pi(e_1, \ldots, e_n)$, where π is a "Lie polynomial", then

$$q_\varepsilon(t) = q_0 + \varepsilon^n \pi(f_1, \ldots, f_n)(q_0) + O(\varepsilon^{n+1}).$$

We are looking for symmetries of $D^n\Big|_{\Omega_n}$ in order to better understand the structure of Ω_n.

Let Σ_n be the symmetric group and $\overline{\Sigma}_n = \{\sum\limits_i c_i\sigma_i : \sigma_i \in \Sigma_n\}$ its group algebra. We set:

$$D^n_\sigma(\gamma) = \int\int\limits_{\Delta_n} d\gamma(t_{\sigma(n)}) \otimes \cdots \otimes d\gamma(t_{\sigma(1)}), \quad D^n_{\sum c_i\sigma_i} = \sum c_i D^n_{\sigma_i}.$$

Let $\sigma \in \Sigma_n$, the *monotonicity type* of σ is a word $w_\sigma = s_1 \ldots s_{n-1}$ in the alphabet $\{\alpha, \beta\}$,

$$s_i = \begin{cases} \alpha, & \sigma(i) < \sigma(i+1); \\ \beta, & \sigma(i) > \sigma(i+1). \end{cases}$$

Given a word w, we set $\overline{w} = \sum\limits_{\{w_\sigma = w\}} \sigma$. The *descent subalgebra* of $\overline{\Sigma}_n$:

$$\mathfrak{M}_n = span\left\{\overline{w} : w = s_1 \cdots s_{n-1},\ s_i \in \{\alpha, \beta\}\right\}.$$

It admits a homomorphism:

$$r : \mathfrak{M}_n \to \mathbb{Z}, \quad r(\overline{s_1 \ldots s_{n-1}}) = (-1)^{\#\{i : s_i = \beta\}}.$$

Example:

$$\overline{\alpha\cdots\alpha} = 1, \quad \overline{\beta\cdots\beta} = \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix}, \quad r(\overline{\beta\cdots\beta}) = (-1)^{n-1}.$$

Theorem. *A curve γ belongs to Ω_n if and only if*

$$D^n_{\mathfrak{m}}(\gamma) = r(\mathfrak{m}) D^n(\gamma), \quad \forall\, \mathfrak{m} \in \mathfrak{M}.$$

Affine in control system:

$$\dot{q} = h(q) + \sum_i u^i f_i(q), \quad u = (u^1, \ldots, u^k) \in \mathbb{R}^k. \qquad (*)$$

If $h \in Lie\{f_1, \ldots, f_k\}$, then we can neutralize the drift h, but this inclusion is violated for many important apparently controllable systems. Examples:

1. *Acceleration control:* $\ddot{x} = \sum_i u^i g_i(x)$. We rewrite:

$$\dot{x} = y, \ \dot{y} = \sum_i u^i g_i(x); \qquad q = (x, y),$$

$$\dot{q} = h(q) + \sum_i u^i f_i(q), \quad [f_i, f_j] = 0, \ i, j = 1, \ldots, k.$$

2. *"Fluid dynamics:"* $\dot{y} = Ay + B(y, y) + \sum_i u^i g_i$.

Theorem. *Assume that* $[f_i, f_j] = 0,\ i, j = 1, \ldots, k$. ***If***

$$conv \left\{ \sum_{i,j} u^i u^j [f_i, [f_j, h]] : u^i, u^j \in \mathbb{R} \right\}$$

is a subspace, then system

$$\dot{q} = h(q) + \sum_{\iota} u^{\iota} f_{\iota}(q) + \sum_{i,j} u^{ij} [f_i, [f_j, h]](q)$$

has "the same control properties" as system (*).

If the fields $[f_i, [f_j, h]], f_\iota$ are all commuting then we iterate the theorem etc.

Hint: Use a fast-oscillating control variation:

$$u_\varepsilon^i(t) = \frac{1}{\varepsilon}\sin\left(\frac{t}{\varepsilon^2}\right), \quad u_\varepsilon^j(t) = \frac{1}{\varepsilon}\cos\left(\frac{t}{\varepsilon^2}\right)$$

to single out the desired bracket.

Indeed:

$$\int_0^1 u_\varepsilon^\iota \, dt = O(\varepsilon), \quad \iint_{\Delta_2} u_\varepsilon^i(t_1) u_\varepsilon^j(t_2)\, dt_1 dt_2 = O(1),$$

as $\varepsilon \to 0$.