Integrable Systems: A bird's-eye view

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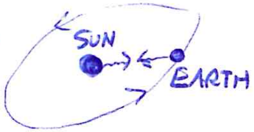
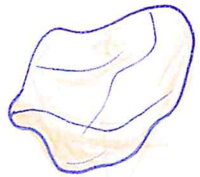
* Review of Classical integrable systems - finite dimensional

Beginning of XIX century: some problems were analytically solved:

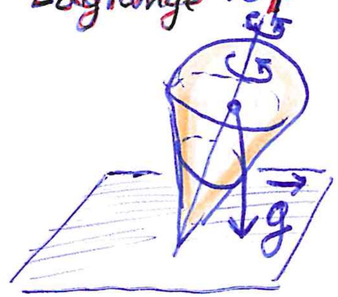
Kepler problem



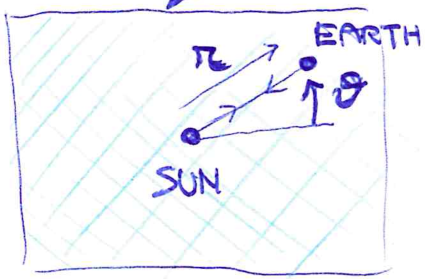
Euler top



Lagrange top



EXAMPLE:



Motion in a plane.

$$\mathcal{L} = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r)$$

$$V(r) = -\frac{K}{r}$$

$\Downarrow$ Lagrange eqs, integrated once:

$$\begin{cases} \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r) = E & \text{Energy} \\ m r^2 \dot{\theta} = L & \text{Angular momentum} \end{cases}$$

constant!

$$\Downarrow$$

$$t - t_0 = \sqrt{\frac{m}{2}} \int_{r_0}^{r} \frac{dr'}{\sqrt{E - \left(\frac{L^2}{2 m r'^2} - \frac{K}{r'} \right)}} \qquad \square$$

The solution reduces to an integration

$\Downarrow$

"Integrable system"

Hamiltonian Formulation (Hamilton, Jacobi, Liouville XIX century)

- $(q,p) = (q_1, \dots, q_n, p_1, \dots, p_n) \in M \simeq \mathbb{R}^{2n}$ phase space

  Motion: $\dfrac{d}{dt} q_j = \dfrac{\partial H}{\partial p_j}$, $\dfrac{d}{dt} p_j = -\dfrac{\partial H}{\partial q_j}$, $j = 1, \dots, n$

  $H = H(q,p,t)$ Hamiltonian

- $(M, \{\cdot,\cdot\})$

  $$\{f,g\} := \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$ Poisson brackets

  $$\boxed{\frac{dq_i}{dt} = \{q_i, H\} \quad , \quad \frac{dp_i}{dt} = \{p_i, H\}}$$

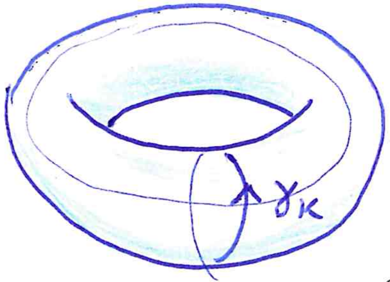
- Integrable system
  - $\exists\ f_1(q,p) = H(q,p),\ f_2(q,p), \dots, f_n(q,p)$ independent
  - $\dfrac{d f_j}{dt} = 0$ "First integrals"
  - $\{f_i, f_j\} = 0 \quad \forall\, 1 \le i \neq j \le n$

How ?

Invent: $\begin{cases} f_1(q,p) = A_1 \\ \vdots \\ f_n(q,p) = A_n \end{cases}$ constant $\Longrightarrow$ $p = p(q, A)$

$\{(q,p) \in M \mid f(q,p) = A\}$

is a torus $\mathbb{T}^n$ [Arnol'd]



compute integrals *

$$\begin{cases} J_K = \dfrac{1}{2\pi} \oint_{\gamma_K} \sum_{j=1}^{n} p_j(q,A)\, dq_j \\ \phi_K = \dfrac{\partial}{\partial J_K} \int^{q} \sum_{j=1}^{n} p_j(q, A(J))\, dq^j \end{cases}$$

ANGLE-ACTION

$\begin{cases} (q,p) \longmapsto (\phi, J) \text{ is canonical.} \\ H = H(J) \end{cases}$ $\Longrightarrow$ $\begin{cases} \phi_K(t) = \dfrac{\partial H(J)}{\partial J_K} \cdot t + \phi_K(0) \\ J_K \text{ constant} \end{cases}$ $K = 1, \dots, n.$

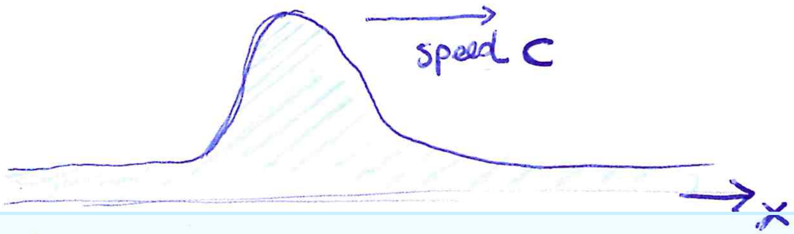
* The system is "integrable".

Integrability implies "regularity" : motion is quasi-periodic on $\mathbb{T}^n$

It does not mean that we can write explicitly the solutions ...

• There are ∞-dimensional integrable systems.

- In 1834 J. S. Russel observed a solitary wave in Union canal (Edinburgh $\rightsquigarrow$ Glasgow)



- Korteweg and de Vries in 1895 :

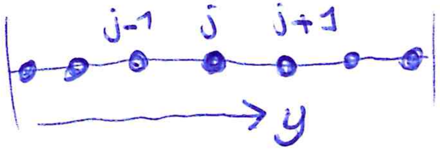
$$\boxed{u_t + 6u \cdot u_x + u_{xxx} = 0 \quad (\mathrm{KdV})}$$

If you look for a solution $f(x-ct)$, you find

$$u(x,t) = \frac{c/2}{\mathrm{Ch}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)}$$

- Around 1965, Zabusky and Kruskal

Fermi-Pasta-Ulam



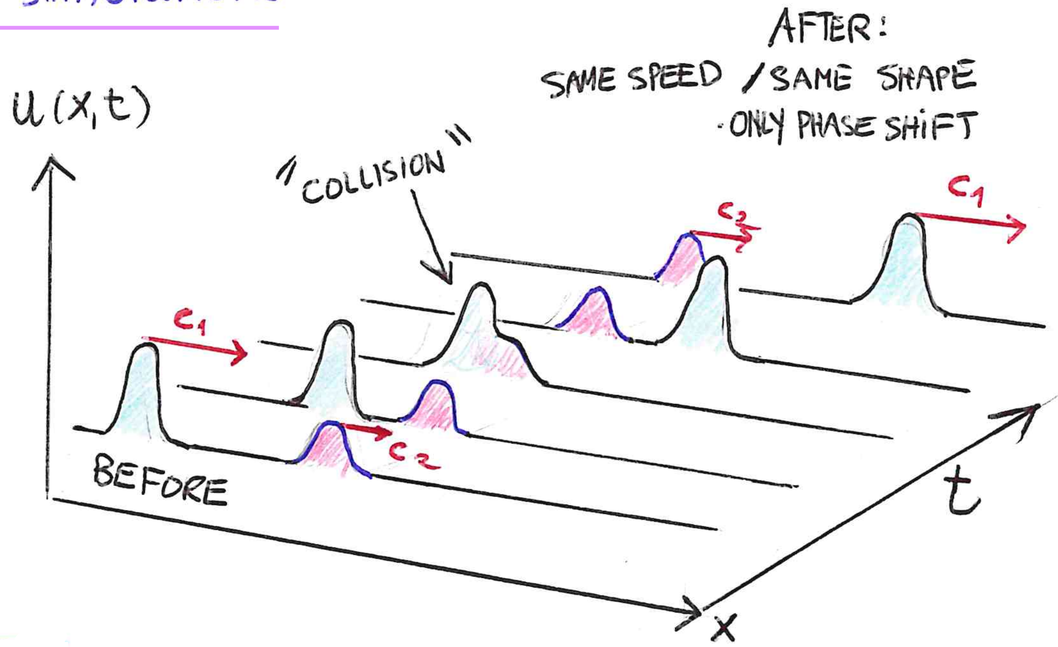
$$m\frac{d^2y_j}{dt} = f(y_{j+1}-y_j) - f(y_j - y_{j-1})$$

$$f(x) = x + \alpha x^2$$

$\xrightarrow{\text{continuous limit}}$ (KdV).

Zabuski - Kruskal did numerical simulations

They found solutions of (KdV) with "elastic collision" of solitary waves
↳ called solitons



"Integration" of (KdV)

- Put $u = \frac{\psi_{xx}}{\psi} + \lambda, \quad \lambda \in \mathbb{C}.$ $\rightsquigarrow$ $-\psi_{xx} + u\psi = \lambda\psi$

Naive idea: write an equation for ψ, equivalent to (KdV)

- Lax (1968): $\boxed{\begin{cases} L\psi = \lambda\psi \\ \psi_t = M\psi \end{cases}}$

where:

$$L\psi := -\psi_{xx} + u(x,t)\psi$$

$$M\psi := -4\psi_{xxx} + 6u\psi_x + 3u_x\psi$$

LAX PAIR

$$\boxed{\begin{cases} L\psi = \lambda\psi \\ \psi_t = M\psi \end{cases}}$$

has solution $\Longleftrightarrow$ compatibility $\partial_t L = L\partial_t$

$$\boxed{\partial_t L = [M, L]}$$

with $\partial_t \lambda = 0$

$\Updownarrow$

$\boxed{u \text{ satisfies (KdV)}}$

- How to use this to solve (KdV)?

Gardner, Green, Kruskal, Miura (1967)

$\hookrightarrow$ INVERSE SCATTERING TRANSFORM (IST)

$u \mapsto -u$ We have changed sign...

$$u_t - 6uu_x + u_{xxx} = 0$$

$$\boxed{-\psi_{xx} + u(x)\psi = \lambda\psi}$$ Shrödinger eq.

- DISCRETE SPECTRUM : $\lambda_m = -K_m^2$, $m = 1, \dots, N$ $\rightsquigarrow$ $\psi_m(x)$ eigenfunctions
- CONTINUUM : $\lambda = K^2$, $K \in \mathbb{R}$ $\rightsquigarrow$ $\psi(x, K)$ generalized eigenfunction

u rapidly decreasing at $x \to \pm\infty$.

- $$\psi(x,k) \sim \begin{cases} a_+(k)e^{ikx} + a_-(k)e^{-ikx}, & x \to -\infty \\ b_+(k)e^{ikx}, & x \to +\infty \end{cases} \Longleftarrow \begin{cases} u(x) \to 0. \\ -\psi_{xx} \approx \lambda\psi \end{cases} \text{ for } x \to \infty$$

- Write $$\psi(x,t) = b_+(x)e^{ikx} + \int_x^{+\infty} \mathbb{K}(x,z)e^{ikz}\,dz.$$

- Imposing that it is a solution, we find:

$$\begin{cases} u(x) = -2\left(\frac{\partial \mathbb{K}(x,z)}{\partial x} + \frac{\partial \mathbb{K}(x,z)}{\partial z}\right)\Big|_{z=x}, \\ \mathbb{K}(x,z) + F(x+z) + \int_x^{+\infty} \mathbb{K}(x,y)\,F(y+z)\,dy = 0, \quad x > z > -\infty \end{cases}$$

(1951)
GEL'FAND - LEVITAN
MARCHENKO (1955)
EQUATION

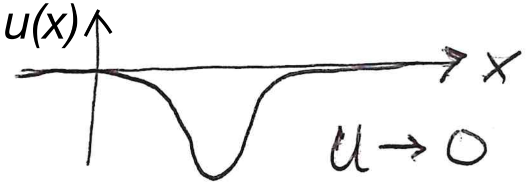
Including also

$$\psi_n(x) \approx \begin{cases} e^{k_n x}, & x \to -\infty \\ c_n e^{-k_n x}, & x \to \infty \end{cases} \quad \text{in } L^2(-\infty,\infty)$$

$$F(\xi) := \sum_{n=1}^{N} c_n^2 e^{-k_n \xi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{a_-(k)}{a_+(k)} e^{ik\xi}\,dk.$$

Solve initial value problem for (KdV):

$u(x)$, $t=0$



$u \to 0$, $x \to \pm\infty$.

a) $\boxed{-\psi_{xx} + u(x)\psi = \lambda\psi}$

$\underline{t=0}$

$$\psi(x,k) \sim \begin{cases} a_+(k)e^{ikx} + a_-(k)e^{-ikx}, & x \to -\infty \\ b_+(k)\, e^{ikx}, & x \to +\infty \end{cases}$$

$$\psi_m(x) \sim \begin{cases} e^{k_m x}, & x \to -\infty \\ c_m e^{-k_m x}, & x \to +\infty \end{cases}$$

b) $\underline{\psi \text{ evolves}}$: use Lax pair!

$$\boxed{\psi_t} = M\psi = -4\psi_{xxx} + 6u(x,t)\psi_x + 3u(x,t)\psi$$

$$\boxed{\underset{x\to\infty}{\sim} -4\psi_{xxx}}$$ linear, without $u(x,t)$

Substitute

$$\frac{da_\pm}{dt} = \pm 4ik^3 a_\pm$$

$$\frac{db_+}{dt} = 4ik^3 b_+$$

$$\frac{dc_m}{dt} = 4k_m^3 c_m$$

$$\Longrightarrow \begin{cases} a_\pm(k,t) = a_\pm(k)\, e^{\pm 4ik^3 t} \\ b_+(k,t) = b_+(k)\, e^{4ik^3 t} \\ c_m(t) = c_m \cdot e^{4k_m t} \end{cases}$$

Compute $\boxed{K(x,z,t)}$ from:
at time t

$$\begin{cases} \mathbb{K}(x,z,t)+F(x+z,t)+\displaystyle\int_x^{\infty}\mathbb{K}(x,y,t)F(y+z,t)\,dy=0, \\ \text{where } \quad F(\xi)=\displaystyle\sum_{m=1}^{N}c_m^2(t)e^{-\kappa_m\xi}+\int_{-\infty}^{\infty}\frac{a_-(k,t)}{a_+(k,t)}e^{ik\xi}\,\frac{dk}{2\pi}. \end{cases}$$

Then, we receive:

$$\boxed{u(x,t)=-2\left(\frac{\partial\mathbb{K}(x,z,t)}{\partial x}+\frac{\partial\mathbb{K}(x,z,t)}{\partial z}\right)\Bigg|_{z=x}}$$ solution of i.v.p. for (KdV).

Solve a linear Shrödinger eq.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{cases} u(x) \\ t=0 \end{cases} \arrow[r, mapsto] \arrow[d, dashed, "(\mathrm{KdV})"'] & \left\{\kappa_m,\, c_m,\, a_-(k)/a_+(k)\right\} \arrow[d, "\text{Linear time evolution}"] \\
u(x,t) & \left\{\kappa_m,\, c_m(t),\, a_-(k,t)/a_+(k,t)\right\} \arrow[l, mapsto]
\end{tikzcd}$$

← Direct scattering

← "Inverse" scattering

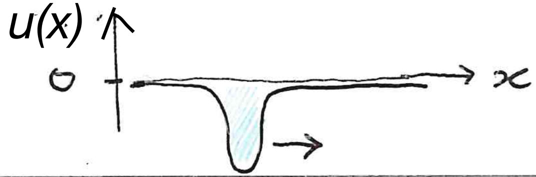
Solve a linear integral equation

A non-linear PDE solved by two linear problems.

This has been applied.

EXAMPLE: * <u>N = 1</u> Only one bound state K_1 $\to$

$a_- = 0$



$$u(x,t) = \frac{-2K_1^2}{ch^2(K_1(x-4K_1^2t)+\delta)} .$$

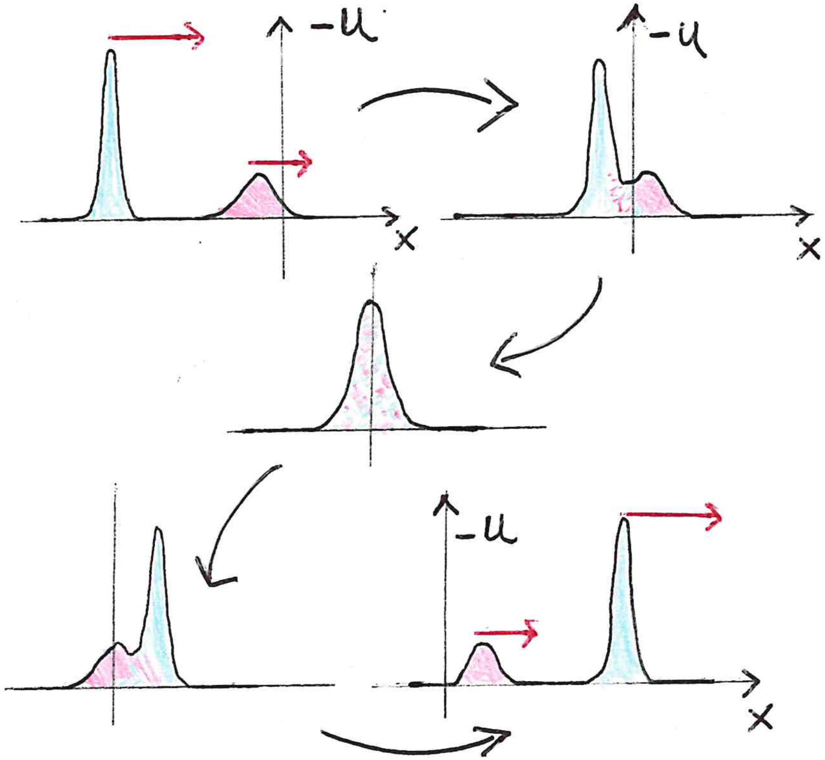
1-soliton.

* <u>N=2</u>, two bound states K_1, K_2

$$\rightsquigarrow u(x,t) = -2(K_2^2-K_1^2)\frac{\dfrac{K_2^2}{sh^2\eta_2} + \dfrac{K_1^2}{ch^2\eta_1}}{(K_2\coth\eta_2 - K_1\tanh\eta_1)^2} .$$

$$\eta_j = K_j x - 4K_j^3 t + \delta_j .$$

2-solitons.



* <u>For any N:</u>

We get a N-solitons solution.

Another point of view of integrability

(m)		(∞)
$(q_1,\dots,q_m,p_1,\dots,p_m)$	$\longrightarrow$	$u(x,t)$
$\sum_{j=1}^{m}(\dots)$	$\longrightarrow$	$\int dx(\dots)$
$f(q,p)$	$\longrightarrow$	$F[u]=\int dx\, f(x,u,u_x,u_{xx},\dots)$ Functional.
$\frac{\partial f}{\partial q},\frac{\partial f}{\partial p}$	$\longrightarrow$	$\frac{\delta F[u]}{\delta u(x)} := \sum(-1)^k \frac{\partial^k}{\partial x^k}\frac{\partial f}{\partial u^{(k)}} = \frac{\partial f}{\partial u}-\frac{\partial}{\partial x}\frac{\partial f}{\partial u_x}+\dots$
$\{f,g\}$		

$$\{F,G\}_0=\int\frac{\delta F}{\delta u(x)}\,\partial_x\,\frac{\delta G}{\delta u(x)}\,dx$$

$$\{F,G\}_1=\int\frac{\delta F}{\delta u(x)}\left(-\partial_x^3+4u\,\partial_x+2u_x\right)\frac{\delta G}{\delta u(x)}\,dx$$

(KdV) is a bi-Hamiltonian system:

$$u_t=6uu_x-u_{xxx}$$

$$u_t=\{u(x),H_1[u]\}_0,$$

$$u_t=\{u(x),H_0[u]\}_1,$$

$$H_1[u]:=\int\left(\frac{u_x^2}{2}+u^3\right)dx.$$

$$H_0[u]:=\int\left(u_{xx}-u^2\right)dx.$$

Observe that $\boxed{\{u,H_0\}_1 = \{u,H_1\}_0 \Rightarrow \{F,H_0\}_1 = \{F,H_1\}_0 \quad \forall F[u]}$

$\Downarrow$

We can define Hamiltonians $\boxed{H_2, H_3, \dots, H_n, \dots}$ by

$$\boxed{\begin{aligned} H_2 &: \quad \{F,H_2\}_0 = \{F,H_1\}_1 \\ &\ \vdots \\ H_n &: \quad \{F,H_n\}_0 = \{F,H_{n-1}\}_1 \\ &\qquad\qquad\qquad \text{etc...} \end{aligned}}$$

$\leftarrow$ The explicit construction is due to Magri. (1978)

They are independent and commute

$$\underline{\{H_n, H_m\}_{0,1} = 0}$$

Solution of (KdV) satisfies a hierarchy of <u>commuting flows</u>

$$\boxed{u_{t_n} = \{u, H_n\}_0 = \frac{\partial}{\partial x}\frac{\delta H_n}{\delta u}}$$

A remark: Zakharov and Faddeev (1971) proved that the inverse scattering method allows to construct (explicitly) a canonical transformation

$$u \longmapsto \begin{pmatrix} P = P(k), & p_1, \dots, p_N \\ Q = Q(k), & q_1, \dots, q_N \end{pmatrix}$$

~~$\{P,P\}_0 = \{Q,Q\} \neq 0$, $\{P,Q\} =$~~

s.t.

$$H[u] = 8 \int_{-\infty}^{\infty} k^3 P(k)\, dk - \frac{32}{5} \sum_{\ell=1}^{N} p_\ell^{5/2}$$

(KdV) is a COMPLETELY INTEGRABLE HAMILTONIAN SYSTEM.

$$\Rightarrow \begin{cases} \dfrac{dP}{dt} = \dfrac{dp_\ell}{dt} = 0 \\ \dfrac{dQ}{dt} = 8k^3, \quad \dfrac{dq_\ell}{dt} = -8k_\ell^3 \end{cases}$$

Again on Lax Pairs

For (KdV): $\boxed{\begin{cases} L\psi = \lambda\psi \\ \psi_t = M\psi \end{cases}}$

It is: $\begin{cases} \psi_{xx} = (u(x,t) - \lambda)\psi \\ \psi_t = -4\psi_{xxx} + 3u\psi_x + 6u_x\psi \end{cases}$

Define $\boxed{\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \vec{\phi} \quad \text{by} \quad \begin{cases} \phi_1 = \psi \\ \phi_2 = \psi_x \end{cases}}$ $\rightsquigarrow$ $\boxed{\begin{cases} \dfrac{\partial}{\partial x}\vec{\phi} = \begin{pmatrix} 0 & 1 \\ u-\lambda & 0 \end{pmatrix}\vec{\phi} \\ \dfrac{\partial}{\partial t}\vec{\phi} = \begin{pmatrix} * & * \\ * & * \end{pmatrix}\vec{\phi} \end{cases}}$

This is in general the "structure" of integrability: there is a pair:

$$\boxed{\begin{cases} \partial_x \Phi = A(x,t,u,\lambda)\,\Phi \\ \partial_t \Phi = B(x,t,u,\lambda)\,\Phi \end{cases}}$$

Lax Pair

Matrices

More general

$$\begin{cases} \dfrac{\partial \Phi}{\partial x} = A(x,\vec{t},u_1,\dots,u_k)\,\Phi \\[2ex] \dfrac{\partial \Phi}{\partial t_j} = B_j(x,\vec{t},u_1,\dots,u_k)\,\Phi \end{cases}$$

A, B are $m\times m$ matrices

$j=1,\dots,p$, $\vec{t}=(t_1,\dots,t_p)$

Compatible system ($\exists$ a fundamental matrix solution $\Phi(x,t)$)

invertible, $m\times m$ matrix.

$\Updownarrow$

$j=1,\dots,m$ $$\boxed{\frac{\partial A}{\partial t_j} - \frac{\partial B_j}{\partial x} + [A,B_j] = 0}$$

When written explicitly, they are non-linear PDEs (or ODEs)

for $u_1,\dots,u_K$.

"integrable"

EXAMPLE: Preliminary: - a linear ODE has branch points determined by the equation

$$\frac{d}{dt} y - \frac{1}{2(t-a)} y = 0 \longrightarrow y(t) = c \cdot \sqrt{t-a}$$

- a non-linear ODE, in general, has branch point not fixed by the equation

$$y \frac{dy}{dt} = \frac{1}{2} \Longrightarrow y(t) = \sqrt{t-c}$$

$t = c$ "movable" branch point

- Thus: Linear ODE allow to define special functions

- The 2^{nd}-order non-linear ODEs with branch points determined by equation are classified $\rightsquigarrow$ Painlevé equations [beginning of XX century]

there are 6 of them:

(1) $y'' = 6y^2 + t$

(2) $y'' = 2y^3 + ty + \alpha \,, \quad \alpha \in \mathbb{C}$

$\vdots$

rational

(6) $y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right) y'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right) y' + Q(y,t)$

- Painlevé equations define non-linear special functions

- They appear in applications

Ex: (KdV) $u_t - 6uu_x + u_{xxx} = 0$

Find $u(x,t) = y(\underbrace{x+3ct^2}_{z}) + ct$

substitute: $$\frac{d^3y}{dz^3} = 6y\frac{dy}{dz} - c$$

Integrate: $$\frac{d^2y}{dz^2} = 3y^2 - cz + \alpha$$

Painlevé (1).

- They are "integrable": there is a Lax Pair for each of them:

$$\boxed{\begin{cases} \dfrac{\partial \Phi}{\partial x} = A(x,t,y)\,\Phi \\[2ex] \dfrac{\partial \Phi}{\partial t} = B(x,t,y)\,\Phi \end{cases}}$$

A, B 2x2 matrices

$\rightarrow$ compatibility / solvability $\Longleftrightarrow$ $\boxed{\partial_t A - \partial_x B + [A,B] = 0}$

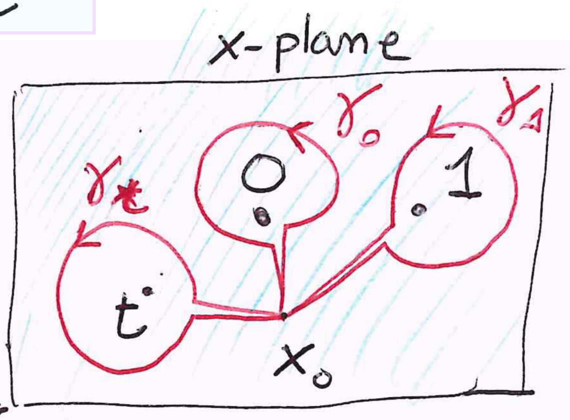
This reduces to a Painlevé eq. for y(t).

Example : Equation (6).

$$\boxed{A(x,t,y) = \frac{A_0(t)}{x} + \frac{A_1(t)}{x-1} + \frac{A_t(t)}{x-t}.}$$

Matrix solution

$$\Phi(x,t) \xrightarrow[\gamma_0]{} \Phi M_0$$
$$\xrightarrow[\gamma_1]{} \Phi M_1$$
$$\xrightarrow[\gamma_t]{} \Phi M_t$$



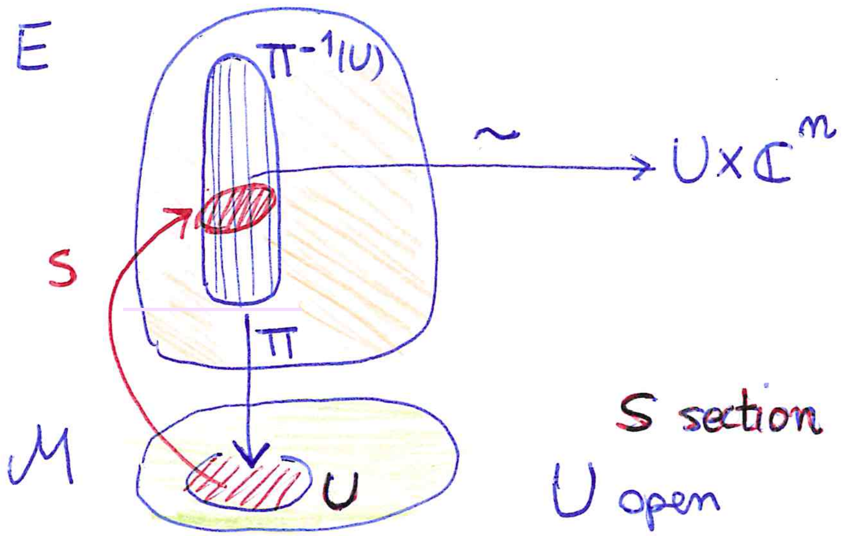
Lax Pair implies that the monodromy matrices M_0, M_1, M_t are constant (of the motion for the Painlevé eq.)

Tecniques to find analytic properties of $y(t) = y(t, \underbrace{M_0, M_1, M_t}_{\text{the "constants of integration"}})$

Method of monodromy preserving deformations

or isomonodromy deformations

Last remark: Our Lax Pairs are a particular istance of <u>Flat connections on vector bundles</u>



s section

U open

Vector bundle $(E, \mathcal{M}, \pi)$

∇ connection

X, Y vector fields on $\mathcal{M}$.

$$\begin{cases} \nabla_{fX+gY}\, s = f \nabla_X s + g \nabla_Y s, & f, g \in C^\infty. \\ \nabla_X (a s_1 + b s_2) = a \nabla_X s_1 + b \nabla_X s_2, & a, b \in \mathbb{C}. \\ \nabla_X (f s) = \underbrace{X(f)}_{df(X)}\, s + f \cdot \nabla_X s, & \end{cases}$$

- Independent vector fields: $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m}$

- Independent sections: $e_1, \dots, e_n$ $\Rightarrow s = \sum_{j=1}^{n} \Phi^j e_j$

$$\Longrightarrow \nabla_{\partial/\partial x^i} \Big(\sum_j \Phi^j e_j \Big) = \sum_j \frac{\partial \Phi^j}{\partial x^i} e_j + \Phi^j \underbrace{\nabla_{\partial/\partial x^i} e_j}_{\Gamma_{ij}^k e_k}$$

$$= \left(\frac{\partial \Phi^k}{\partial x^i} + \Gamma_{ij}^k \Phi^j \right) e_k$$

Equation $\boxed{\nabla s = 0}$ is: $\boxed{\dfrac{\partial \Phi}{\partial x^i} + \Gamma_i \Phi = 0}$, $i = 1, \ldots, m$

$$\Gamma_i := \left(\Gamma_{ij}^{K} \right)_{j,K=1}^{n}$$

$$\Phi = \begin{pmatrix} \Phi^1 \\ \vdots \\ \Phi^n \end{pmatrix}$$

Let $\begin{cases} d\Phi = \displaystyle\sum_{i=1}^{m} \frac{\partial \Phi}{\partial x^i} dx^i \\ \Gamma := \displaystyle\sum_{i=1}^{m} \Gamma_i \, dx^i \end{cases} \longrightarrow$

It is $\boxed{d\Phi + \Gamma \Phi = 0}$ $(*)$

If a solution exists, then $d^2 \Phi = 0 \Rightarrow \boxed{d\Gamma + \Gamma \wedge \Gamma = 0}$ $(**)$

- **Theorem:** $(*)$ has n independent solutions $\Longleftrightarrow$ $(**)$

EXAMPLE: $(x^1, \ldots, x^m) = (x, t)$, $\Gamma_1 = A$, $\Gamma_2 = B$

Then: $\begin{cases} (*) \text{ is } \boxed{\dfrac{\partial \Phi}{\partial x} = A\Phi, \ \dfrac{\partial \Phi}{\partial t} = B\Phi} \\ (**) \text{ is } \boxed{\dfrac{\partial A}{\partial t} - \dfrac{\partial B}{\partial x} + [A, B] = 0} \end{cases}$