Full Bargmann gas solutions of the mKdV equation

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EMORY UNIVERSITY

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The mighty "wave of translation"

- *KdV equation*
  [Scott-Russell, 1834; Korteweg, de Vries, 1895]

$$q_t - 6qq_x + q_{xxx} = 0$$

Shallow-water waves, internal waves in the ocean, ion acoustic waves in plasma, acoustic waves on a crystal lattice, ...

- *Modified KdV equation*
  [Zabusky, '67; Miura, '68]

$$q_t + 6q^2 q_x + q_{xxx} = 0$$

Electric circuits, multi-component plasmas, ...

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The Spirited Horse, the Engineer, and the Mathematician: Water Waves in Nineteenth-Century Hydrodynamics

OLIVIER DARRIGOL

Communicated by J.Z. BUCHWALD

Once upon a time, a spirited horse was dragging a boat along a peaceful canal of Northern Scotland. Suddenly, a ferocious dog charged from the bushes. The horse took fright and ran off, drawing the boat after him. To the horse's surprise, the boat offered almost no resistance at this high speed. A clever naval engineer, who chanced to witness the scene, pursued the matter through numerous experiments and confirmed the horse's discovery. Not knowing his limits, the engineer ventured to propose a mechanical explanation of this paradox based on the existence of a wonderful, solitary wave that carried the boat with it. A renowned mathematician, who excelled in the learned calculus of water motion, mocked this amateurish attempt. There was nothing in the engineer's equations to justify his reasoning, and much to condemn it. Yet for a few years Scottish canal travelers enjoyed the commercial exploitation of the paradox. Half a century later, the aged mathematician resumed his calculations. Thanks to his long experience, he now saw new meanings in his old symbols. From this enriched analysis the solitary wave and the possibility of vanishing ship resistance emerged as if by magic. At last, the wise man rejoiced, mathematics could do as well as a galloping horse.

This fable is an imaginary simplification of a real story of which the engineer John Scott Russell, and the mathematicians (in a broad sense) George Biddell Airy, George Gabriel Stokes, Joseph Boussinesq, and Lord Kelvin were the main actors (besides the spirited horse). It is intended to indicate a major nineteenth-century transformation of the mathematical physicist's tool kit through which practically important solutions of long-known equations became much more easily accessible. The main purpose of the present article is to analyze the nature and the water-wave circumstances of this transformation.

Waves on the surface of water were an obvious field of application of the new hydrodynamics of Jean le Rond d'Alembert, Leonhard Euler, and Joseph Louis Lagrange. The latter mathematician himself wrote the basic equations of water waves, and solved them in the simplest case of small waves on shallow water. Most of what is today known on water waves was found in the nineteenth century: the celerity of small, plane, monochromatic waves on water of constant depth, the pattern of waves created by a local action on the water surface, the shape of oscillatory or solitary waves of finite size, the effect of friction, wind, and a variable bottom on the size and shape of the waves. There is, however, a puzzling contrast between the conciseness and ease of the modern treatment of these topics, and the long, difficult struggles of nineteenth century physicists

Classical solutions

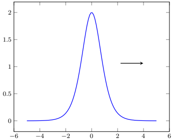
Travelling wave ansatz:

$$q(x,t) = \varphi\left(x - vt\right)$$

the PDE becomes an ODE in the variable $\xi = x - vt$: $-v\varphi' - 6\varphi\varphi' + \varphi''' = 0$.

1. rapidly decreasing, localized travelling wave (*soliton*):

$$q_{\text{sol}}(x,t) = \pm\kappa\,\text{sech}\left(2\kappa(x - 4\kappa^2 t - x_0)\right)$$



with $\kappa > 0$.

2. *periodic travelling wave* solutions: $q_{\text{ell}}(x,t) =$

$$-\beta_1 - \beta_2 - \beta_3 + 2\frac{(\beta_2+\beta_3)(\beta_1+\beta_3)}{\beta_2+\beta_3-(\beta_2-\beta_1)\,\text{cn}^2\left(\sqrt{\beta_3^2-\beta_1^2}(x - 2(\beta_1^2+\beta_2^2+\beta_3^2)t) + x_0 \mid m\right)}$$

with wave parameters $\{\beta_j\}$ and $\text{cn}\,(z \mid m)$ Jacobi elliptic function of modulus $m \in (0,1)$.

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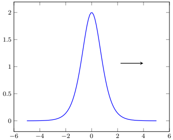

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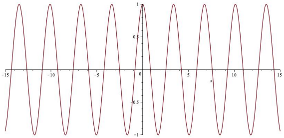
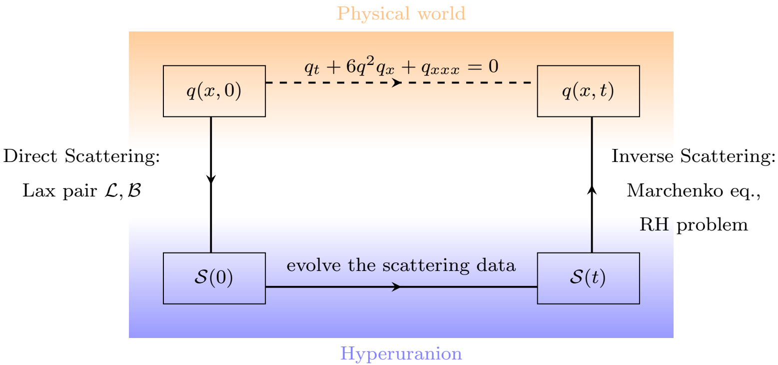


The general solution

Recipe (for fast decaying or step-like IC):



Integrable PDEs

The procedure applies to a vast class of PDEs (KdV, Boussinesq, mKdV, cubic NLS, etc.) that are *integrable*:

the equation arises as the compatibility condition between two linear differential operators – **Lax pair** [Tanaka, '72; Wadati, '73]:

$$\dot{\mathcal{L}} = [\mathcal{L}, \mathcal{B}]$$

$$\mathcal{L} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D - \mathrm{i} \begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix} \qquad D := \partial_x$$

$$\mathcal{B} = -4 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} D^3 + 3D \begin{bmatrix} -q^2 & \mathrm{i}q_x \\ \mathrm{i}q_x & -q^2 \end{bmatrix} + 3 \begin{bmatrix} -q^2 & \mathrm{i}q_x \\ \mathrm{i}q_x & -q^2 \end{bmatrix} D \ .$$

Equivalently, the compatibility condition can be presented as the existence of simultaneous solutions to

$$\mathcal{L}\psi = \lambda\psi, \qquad \psi_t = \mathcal{B}\psi \ .$$

The Direct Scattering

Consider the Lax pair with initial profile $q(x, t = 0)$.

From the equation

$$\mathcal{L}\psi = \lambda\psi$$

solving the eigenvalue-eigenfunction problem means to find the

scattering data,

which describe the solution space of the Schrödinger/Dirac operator:

$$\begin{aligned}\mathcal{S}_0 = \big\{ -\kappa_1^2, \ldots, -\kappa_n^2 &\text{ eigenvalues (discrete spectrum)},\\ \chi_1, \ldots, \chi_n &\text{ norming constants of the eigenfunctions},\\ \rho(k) &\text{ reflection coefficient (continuum spectrum) } \big\}\end{aligned}$$

Turning on time

If $q(x,t)$ depends on a parameter t, one expects $\mathcal{S}_t$ to vary with t as well.

If the t dependence of $q(x,t)$ is given in terms of the mKdV equation:

$$q_t = -6q^2 q_x - q_{xxx} \ ,$$

- the discrete eigenvalues are constant of motion: $-\kappa_j^2$;
- the norming constants satisfies: $\chi_j(t) = \chi_j(0)e^{A\kappa_j^3 t}$, $A \in \mathbb{R}$;
- the reflection coefficient satisfies $\rho(k;t) = \rho(k;0)e^{\mathrm{i}Bk^3 t}$, $B \in \mathbb{R}$.

Finally, there exist standard spectral theory techniques to reconstruct the potential $q(x,t)$ from $\mathcal{S}_t$ (*we'll be using a Riemann–Hilbert problem*).

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Where is the soliton?

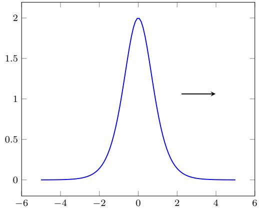
If $q(x,0)$ yields only one eigenvalue $\lambda = -\kappa^2$ and no reflection coefficient $\rho(k) \equiv 0$ for the Schrödinger equation.

The solution $q(x,t)$ is a 1-soliton solution:

$$q_{\text{sol}}(x,t) = \pm\kappa \,\text{sech}\left(2\kappa(x - 4\kappa^2 t - x_0)\right)$$

with phase shift

$$x_0 = \frac{1}{2\kappa} \log \frac{2\kappa}{|\chi|} .$$



In general,

1. Multi-soliton solutions correspond to the discrete eigenvalues $\{-\kappa_j^2\}$ of the operator $\mathcal{L}$.
2. The reflection coefficient $\rho(k)$ corresponds to a radiative part that propagates to the left and decays at rate t^{-1}.

What's so special about solitons?

- Solitons = localized travelling wave solutions.
- Solitons = discrete eigenvalues of the $\mathcal{L}$ operator; *they arise in the long-time behaviour.*
- Soliton interaction is elastic: they "survive" collisions [Zabusky-Kruskal, '65]

$$q(x,t) = \sum_{j=1}^{N} q_{\mathrm{sol,j}}(x + \delta_j^{\pm}, t) + o(1) \qquad \text{as } t \to \pm\infty$$

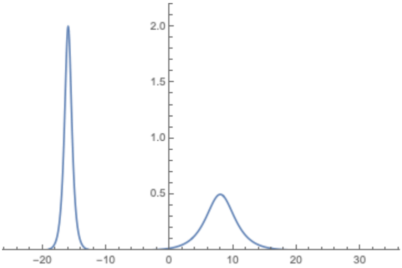


Figure: VIDEO

pairwise interaction yields logarithmic phase shifts $\delta_j^+ - \delta_j^- \sim \log\left|\frac{\kappa_j + \kappa_i}{\kappa_j - \kappa_i}\right|$

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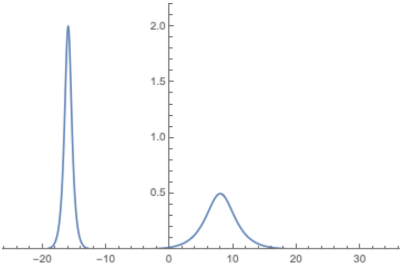

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What is a soliton gas?

Definition (informal)

A **soliton gas/ensemble** is a -random- configuration of a large number of solitons.

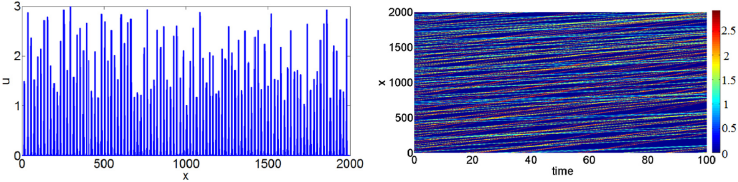
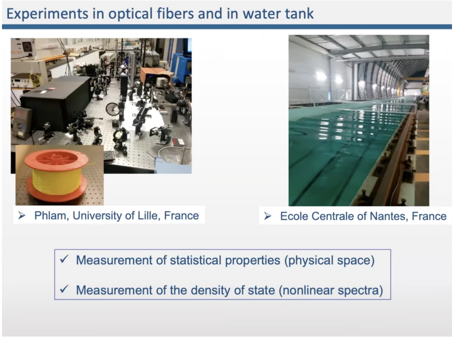


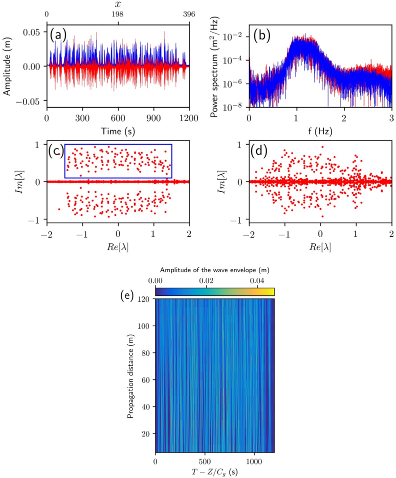
Figure: Initial distribution of the soliton gas (left). (t, x)-diagram of soliton field (right). From [Shurgalina, Pelinovski, '16].

Several numerical experiments: [Gelash, Agafontsev, '18]. [Bonnemain, Congy, El, Roberti *et al.*, '18–'24], [Pelinovsky, Didenkulova *et al.*, '16–'24], etc.

Soliton gasses have been observed *experimentally*

in water waves [Costa *et al.*, '14; Redor *et al.*, '19; Suret *et al.*, '20; ...] and in optics [Marcucci *et al.*, '19; ...]





From Suret's talk at INI in Summer 2022, and [Suret *et al.*, '20].

A soltion gas:

1. **Continuum limit of solitons:** gas solutions belong to the closure of the set of multisoliton potentials
[Marchenko '88–'91], [Zaitsev, Whitham, '83], [Boyd, '84], [Gesztesy, Karwowski, Zhao, '92]

2. **Kinetic theory:** soliton gas as a special large genus (thermodynamic) limit of a finite gap (N-phase nonlinear wave) solution to the PDE

and a trial soliton velocity satisfies kinetic+continuity equations (*nonlinear dispersive relations*)

$$v(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| (v(\kappa) - v(s))\, \varrho(s; x, t)\, \mathrm{d}s\,, \quad \varrho_t + (v\varrho)_x = 0$$

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- These two approaches are equivalent [Jenkins, '24+].
- We focus on the former approach.

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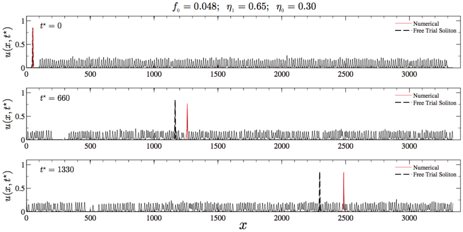

From [Carbone, Dutyk, El, '16].

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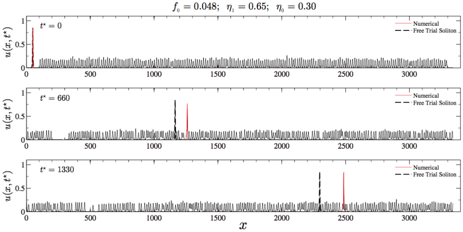
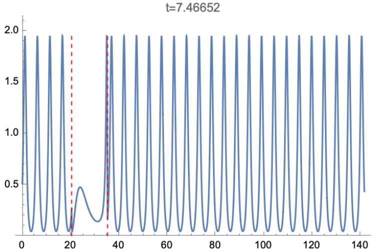


From [Carbone, Dutyk, El, '16].

Results

1. construct a limiting $2N$-soliton solution to the (m)KdV equation as $N \to \infty$, via RH problem

2. analyze its asymptotic profile at $t = 0$ for $x \to \pm\infty$ and at $t \gg 1$ for $x \in \mathbb{R}$



$$q(x,t) = q_{\text{lead}}(x,t) + \{\text{error}\}$$

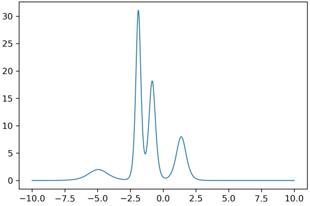
Multi-soliton potential

Given the scattering data for $q_t + 6q^2 q_x + q_{xxx} = 0$

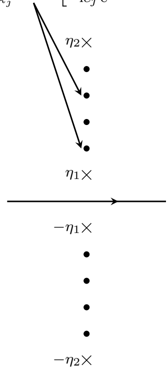
$$\mathcal{S} = \left\{ \{\mathrm{i}\kappa_j\}_{j=1}^{2N}, \{c_j\}_{j=1}^{2N}, \rho(k) \right\}$$

the solution [Wadati, '72; Deift-Zhou, '93]

$$q_{2N}(x,t) = \lim_{k\to\infty} 2\mathrm{i}k\boldsymbol{M}_{12}(k;x,t)$$



$$\operatorname*{res}_{k=\mathrm{i}\kappa_j} \boldsymbol{M} = \lim_{k\to\mathrm{i}\kappa_j} \boldsymbol{M}(k) \begin{bmatrix} 0 & 0 \\ -\mathrm{i}c_j\, \mathrm{e}^{2\mathrm{i}\theta(k;x,t)} & 0 \end{bmatrix}$$



$$\boxed{\theta = 4tk^3 + xk}$$

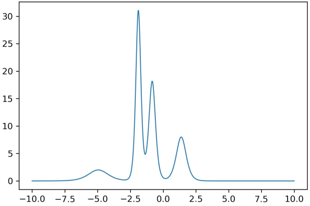
Blaschke factor trick

Define:

$$\widetilde{\boldsymbol{M}}(k) := \boldsymbol{M}(k)\left(\prod_{j=1}^{N}\frac{k-\mathrm{i}\mu_j}{k+\mathrm{i}\mu_j}\right)^{\boldsymbol{\sigma}_3}$$

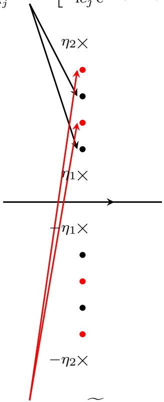
the *same* $2N$-solitons solution:

$$q_{2N}(x,t) = \lim_{k\to\infty} 2\mathrm{i}k\widetilde{\boldsymbol{M}}_{12}(k;x,t)$$



Intuition: at $t=0$, the λ_j-bumps are located "on the right" and the μ_j-bumps are located "on the left".

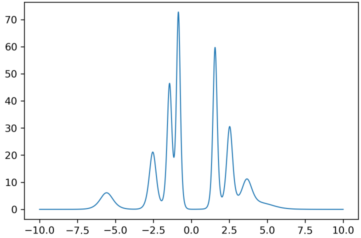
$$\operatorname*{res}_{k=\mathrm{i}\lambda_j}\widetilde{\boldsymbol{M}} = \lim_{k\to\mathrm{i}\lambda_j}\widetilde{\boldsymbol{M}}(k)\begin{bmatrix}0 & 0\\ -\mathrm{i}c_j\,\mathrm{e}^{2\mathrm{i}\theta(k;x,t)} & 0\end{bmatrix}$$



$$\operatorname*{res}_{k=\mathrm{i}\mu_j}\widetilde{\boldsymbol{M}} = \lim_{k\to\mathrm{i}\mu_j}\widetilde{\boldsymbol{M}}(k)\begin{bmatrix}0 & -\mathrm{i}\chi_j\,\mathrm{e}^{-2\mathrm{i}\theta(k;x,t)}\\ 0 & 0\end{bmatrix}$$

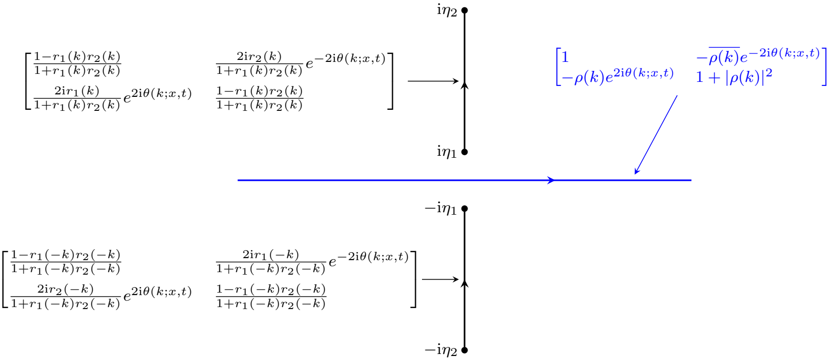
Assume the eigenvalues are accumulating uniformly within a bounded interval $\Sigma := [\mathrm{i}\eta_1, \mathrm{i}\eta_2]$:

- Rescale
$$c_j \mapsto \frac{c_j}{N^\gamma} \qquad \text{and} \qquad \chi_j \mapsto \frac{\chi_j}{N^\gamma} ,$$
where $\gamma = 1$ in the bulk and $\gamma = \frac{3}{2}$ at the edges.
- The norming constants c_j, χ_j are discretization of smooth functions $r_1(k), r_2(k)$ (positive-valued, sufficiently regular, $r_1(k)r_2(k) < 1$ on Σ).
- Take the limit $N \to +\infty$ ...



This $2N$-soliton solution has *very broad* support. For $N \gg 1$, it decays only for $|x| > CN$ for some $C \in \mathbb{R}_+$.

The limit RH problem



$$\boxed{\theta(k;x,t) = 4tk^3 + xk}$$

Theorem

The limiting RH problem has a unique solution and the corresponding mKdV (classical) solution

$$q(x,t) := \lim_{k\to\infty} 2\mathrm{i}k\boldsymbol{X}_{12}(k;x,t)$$

is the uniform limit of the $2N$*-soliton solution* $q_{2N}(x,t)$ *for* $(x,t) \in \mathbb{R}\times\mathbb{R}_+$.

Sketch of the proof:

- Existence and uniqueness of the solution to the limiting RHP follows from the Vanishing Lemma [Zhou, '89].
- Uniform limit follows from standard Stirling formula expansion.

Properties:

- this gas is an instance of a *condensate* gas [El, Tovbis, '20];
- this gas is *smooth* and *dense* (due to the scaling of the norming constants);
- and *deterministic*.

The dispersionless gas

In the case where the reflection coefficient $\rho \equiv 0$, we recover a *primitive/Bargmann potential*.
[Dyachenko, Nabelek, Zakharov, Zakharov, '16–'20].

These potentials are constructed via *formal* limiting procedure from a scalar $\bar{\partial}$-problem with dressing operator [Zakharov, Manakov, '85] and they are encoded in the solution of a scalar *nonlocal* RH problem.

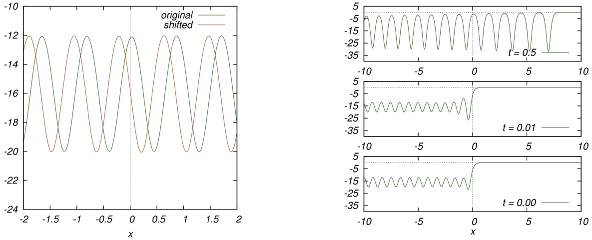


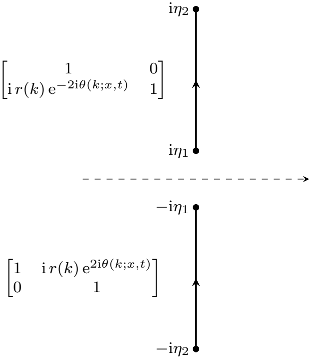
Figure: (left) periodic potential $r_1 = r_2 = \frac{1}{\pi}$; (right) one-sided dressing $r_2 = 0$.

Conjecture

All primitive/Bargmann potentials are regular, dense soliton gasses.

The half-gas

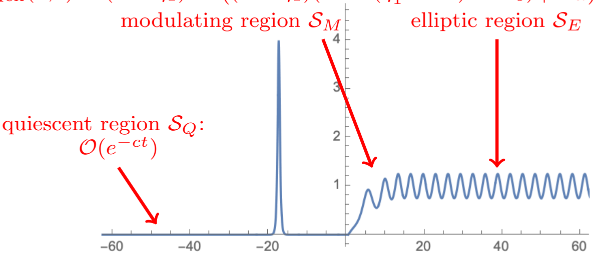
The case where $\rho \equiv 0$ and one of the limiting norming constant $r_2 \equiv 0$ has already been analyzed in [Girotti, Grava, Jenkins, McLaughlin, Minakov, '23].



$$\boxed{\theta = 4tk^3 + xk}$$

1. Long-time asymptotic analysis:

$$q_{\text{ell}}(x,t) = (\alpha+\eta_1)\,\text{dn}\left((\alpha+\eta_1)(x-2(\eta_1^2+\alpha^2)t-x_0)\mid m_\alpha\right)+\mathcal{O}(t^{-1})$$



2. gas-soliton interaction and derivation of the kinetic equations:

$$\dot{x}_{\text{peak}} = -\frac{2\varphi_t(\mathrm{i}\kappa_0)-\partial_t \ln\Psi(x,t;\kappa_0,\eta_1)}{2\varphi_x(\mathrm{i}\kappa_0)-\partial_x \ln\Psi(x,t;\kappa_0,\eta_1)}\Big|_{x=x_{\text{peak}}(t)}+\mathcal{O}\left(t^{-1}\right)$$

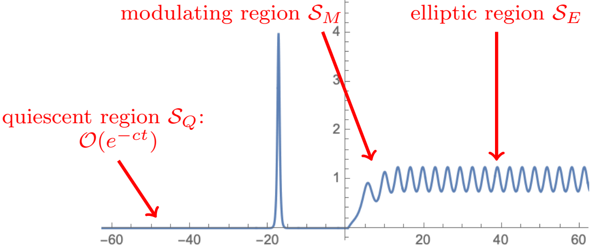
and

$$\bar{v}_{\text{sol}}(\kappa_0) = 4\kappa_0^2+\frac{1}{\kappa_0}\int_{\eta_1}^{\alpha}\ln\left|\frac{\kappa_0-s}{\kappa_0+s}\right|\left(v_{\text{group}}(s)-\bar{v}_{\text{sol}}(\kappa_0)\right)\partial_x\varrho(\mathrm{i}s)\,\mathrm{d}s\ .$$

where $\bar{v}_{\text{sol}}$ is the *average velocity* of the soliton peak over one period of the background.

1. Long-time asymptotic analysis:

$$q_{\text{ell}}(x,t) = (\alpha+\eta_1)\,\text{dn}\left((\alpha+\eta_1)(x-2(\eta_1^2+\alpha^2)t-x_0)\mid m_\alpha\right) + \mathcal{O}(t^{-1})$$



2. gas-soliton interaction and derivation of the kinetic equations:

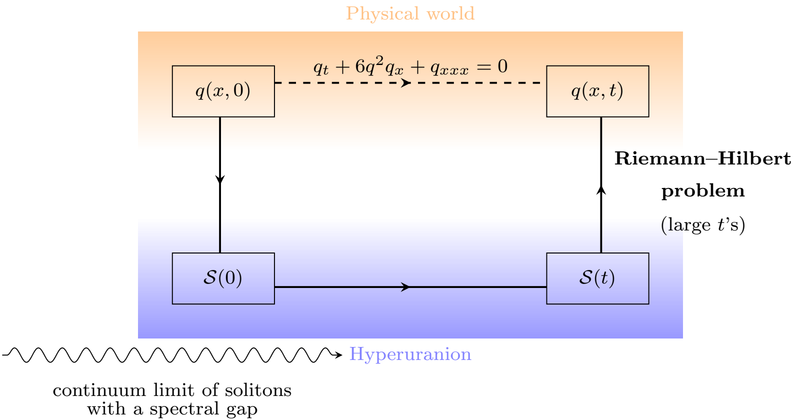
$$\dot{x}_{\text{peak}} = -\left.\frac{2\varphi_t(\mathrm{i}\kappa_0) - \partial_t \ln\Psi(x,t;\kappa_0,\eta_1)}{2\varphi_x(\mathrm{i}\kappa_0) - \partial_x \ln\Psi(x,t;\kappa_0,\eta_1)}\right|_{x=x_{\text{peak}}(t)} + \mathcal{O}\left(t^{-1}\right)$$

and

$$\bar{v}_{\text{sol}}(\kappa_0) = 4\kappa_0^2 + \frac{1}{\kappa_0}\int_{\eta_1}^{\alpha} \ln\left|\frac{\kappa_0 - s}{\kappa_0 + s}\right| (v_{\text{group}}(s) - \bar{v}_{\text{sol}}(\kappa_0))\,\partial_x \varrho(\mathrm{i}s)\,\mathrm{d}s\ .$$

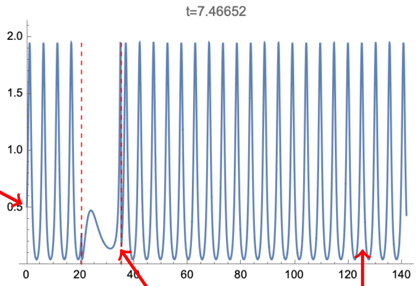
where $\bar{v}_{\text{sol}}$ is the *average velocity* of the soliton peak over one period of the background.

Asymptotic analysis



The full gas

- $q(x,0)$ has elliptic behaviour at both $\pm\infty$:
  $q(x,t) = (\eta_1+\eta_2)\,\mathrm{dn}\left((\eta_1+\eta_2)(x - x_{0,\pm}) \mid m_1\right) + \mathcal{O}\left(x^{-\infty}\right)$ as $x \to \pm\infty$
- as $t \to +\infty$, $q(x,t)$ looks like a genus-1 solution at both $x \to \pm\infty$, with a connecting region that is phase-modulating.

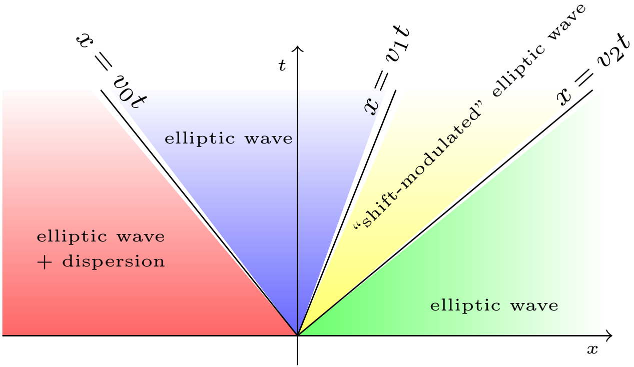


elliptic region $\mathcal{S}_{E_-}$
(+dispersion)

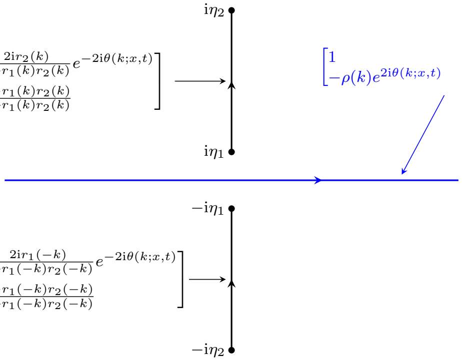
connecting region $\mathcal{S}_C$

elliptic region $\mathcal{S}_{E_+}$

$$q(x,t) = (\eta_1+\eta_2)\,\mathrm{dn}\left((\eta_1+\eta_2)(x - 2(\eta_1^2+\eta_2^2)t - x_0(\tfrac{x}{t})) \mid m_1\right) + \mathcal{O}\left(t^{-\frac{1}{4}+\epsilon}\right)$$


$x = v_0 t$
t
$x = v_1 t$
"shift-modulated" elliptic wave
$x = v_2 t$
elliptic wave
elliptic wave
+ dispersion
elliptic wave
x

The limit RH problem



$$\begin{bmatrix} \frac{1-r_1(k)r_2(k)}{1+r_1(k)r_2(k)} & \frac{2\mathrm{i}r_2(k)}{1+r_1(k)r_2(k)}e^{-2\mathrm{i}\theta(k;x,t)} \\ \frac{2\mathrm{i}r_1(k)}{1+r_1(k)r_2(k)}e^{2\mathrm{i}\theta(k;x,t)} & \frac{1-r_1(k)r_2(k)}{1+r_1(k)r_2(k)} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\overline{\rho(k)}e^{-2\mathrm{i}\theta(k;x,t)} \\ -\rho(k)e^{2\mathrm{i}\theta(k;x,t)} & 1+|\rho(k)|^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1-r_1(-k)r_2(-k)}{1+r_1(-k)r_2(-k)} & \frac{2\mathrm{i}r_1(-k)}{1+r_1(-k)r_2(-k)}e^{-2\mathrm{i}\theta(k;x,t)} \\ \frac{2\mathrm{i}r_2(-k)}{1+r_1(-k)r_2(-k)}e^{2\mathrm{i}\theta(k;x,t)} & \frac{1-r_1(-k)r_2(-k)}{1+r_1(-k)r_2(-k)} \end{bmatrix}$$

$$\boxed{\theta(k;x,t) = 4tk^3 + xk}$$

The core of the RH problem analysis relies on

1. **Small Norm Theorem**: if we have a RHP of the type

$$\boldsymbol{X}_+(k) = \boldsymbol{X}_-(k) \underbrace{\Big(\boldsymbol{I} + \{\text{small terms}\}\Big)}_{\text{(almost) no jumps}} \qquad \text{on the contours}$$

$$\boldsymbol{X}(k) = \boldsymbol{I} + \mathcal{O}\left(k^{-1}\right) \qquad k \to \infty.$$

   then, the solution is $\boldsymbol{X} = \boldsymbol{I} + \{\text{small terms}\}$
   (the approximation can be explicitly estimated!).

2. **Deift–Zhou Steepest Descent method**: perform a sequence of invertible transformations of the original RH problem $\boldsymbol{X}$

$$\boldsymbol{X} \mapsto \boldsymbol{T} \mapsto \boldsymbol{U} \mapsto \ldots \mapsto \boldsymbol{S}$$

   in such away that, in the appropriate regime, the final RH problem $\boldsymbol{S}$ can be solved by an approximating solution $\boldsymbol{W}$ (the "model problem"):

$$\boldsymbol{S} \sim \boldsymbol{W}$$

   (i.e. $\boldsymbol{\mathcal{E}} = \boldsymbol{S}^{-1}\boldsymbol{W}$ fits into the Small Norm theorem setting).

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A closer look at the jumps

The jump on the bands and the phase:

$$\begin{aligned} \boldsymbol{X}_+(k)\boldsymbol{X}_-^{-1}(k) &= \frac{1}{1+r_1(k)r_2(k)}\begin{bmatrix} 1-r_1(k)r_2(k) & 2\mathrm{i}r_2(k)e^{-2\mathrm{i}\theta(k;x,t)} \\ 2\mathrm{i}r_1(k)e^{2\mathrm{i}\theta(k;x,t)} & 1-r_1(k)r_2(k) \end{bmatrix} \\ \theta(k;x,t) &= 4k^3t+kx = 4tk\left(k^2+\frac{x}{4t}\right) \end{aligned}$$

At *each point along* Σ, and for *any value of* $\xi := \frac{x}{t}$,
we always have one of the two exponentials $e^{\pm 2\mathrm{i}\theta(k;x,t)}$ *asymptotically large* ...

The DZ steepest descent business

Massage the RH problem [Deift, Zhou, '92]:

$$\boldsymbol{T}(k) = \boldsymbol{X}(k) e^{-\mathrm{i}g(k)\sigma_3} f(k)^{\sigma_3}$$

The dynamic is driven by the g-function

$$g(k;x,t) = \int_{\Sigma\cup\overline{\Sigma}} \log(k-s)\varrho(s;x,t)\,\mathrm{d}s\ , \quad \Sigma = [\mathrm{i}\eta_1, \mathrm{i}\eta_2]\ .$$

The measure $\rho(s)\,\mathrm{d}s$ is given explicitly

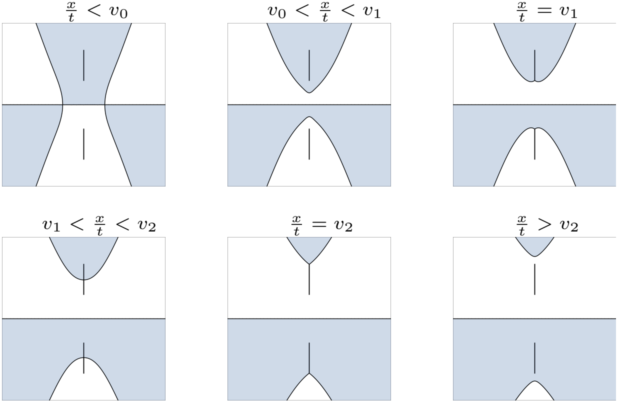
$$\varrho(s;x,t)\mathrm{d}s = -\frac{1}{\pi \mathrm{i}} \frac{12t(s^4 + \frac{1}{2}(\eta_1^2+\eta_2^2)s^2 + c_2) + x(s^2+c_0)}{\sqrt{(s^2+\eta_2^2)(s^2+\eta_1^2)}}\mathrm{d}s$$

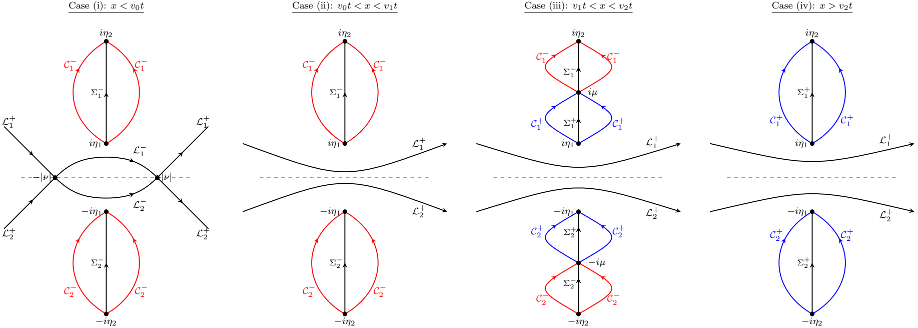
for some constants c_0, c_2 depending on η_1, η_2, uniquely determined by a suitable normalization.

Remark

This is the same g-function that was constructed in [Girotti, Grava, Jenkins, McLaughlin, Minakov, '23].

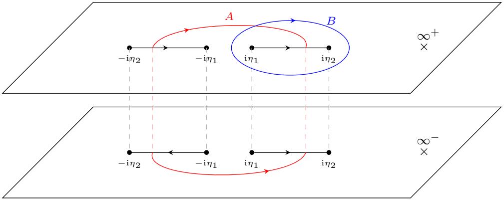
- for $\frac{x}{t} < v_1$ and $\frac{x}{t} > v_2$:
  $q(x,t)$ is a periodic travelling wave with fixed parameters (*elliptic regions* $\mathcal{S}_E$).
- $v_1 < \frac{x}{t} < v_2$:
  $q(x,t)$ is a periodic travelling wave with fixed parameters, but with slowly varying phase (*connecting region* $\mathcal{S}_C$).
- for $\frac{x}{t} < v_0$: contribution of the dispersive tails is present.




Case (i): $x < v_0 t$
Case (ii): $v_0 t < x < v_1 t$
Case (iii): $v_1 t < x < v_2 t$
Case (iv): $x > v_2 t$
$i\eta_2$
$i\eta_1$
$-i\eta_1$
$-i\eta_2$
$i\mu$
$-i\mu$
$-|\nu|$
$|\nu|$
$\mathcal{C}_1^-$
$\mathcal{C}_2^-$
$\mathcal{C}_1^+$
$\mathcal{C}_2^+$
Σ_1^-
Σ_2^-
Σ_1^+
Σ_2^+
$\mathcal{L}_1^+$
$\mathcal{L}_2^+$
$\mathcal{L}_1^-$
$\mathcal{L}_2^-$

The model problem

1. Construct the outer parametric $\boldsymbol{W}$ with ϑ_3-function associated to the genus-1 Riemann surface $\mathfrak{X} = \{(k,\eta) \in \mathbb{C}^2 \mid \eta^2 = (k^2+\eta_1^2)(k^2+\eta_2^2)\}$



2. Construct local parametrices at the endpoints $\boldsymbol{P}_{\pm\eta_j}$ and at the inflection points $\boldsymbol{P}_{\pm\mu}$ (and $\boldsymbol{P}_{\pm\nu}$):

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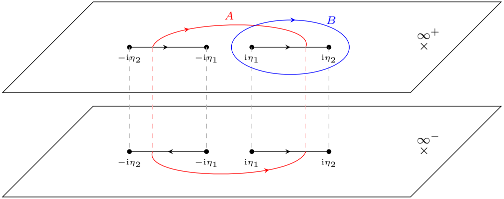

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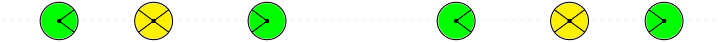

Theorem (large time asymptotics)

The mKdV full gas has the following asymptotic behaviour:

$$q(x,t) = (\eta_1+\eta_2)\,\mathrm{dn}\left((\eta_1+\eta_2)(x-2(\eta_1^2+\eta_2^2)t - x_0(\tfrac{x}{t}))\mid m_1\right) + \mathcal{O}\left(t^{-\frac{1}{4}+\epsilon}\right) \;,$$

$$x_0(\tfrac{x}{t}) = \frac{K(m_1)}{\eta_1+\eta_2}\left(\Delta(\tfrac{x}{t})-1\right) \;,$$

for any $\epsilon>0$*, where* $\mathrm{dn}(x\mid m_1)$ *is the Jacobi elliptic function with modulus* $m_1 = \frac{4\eta_1\eta_2}{(\eta_1+\eta_2)^2}$*,* $K(m) = \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2(\theta)}}$ *is the complete elliptic integral of first kind, and* $\Delta(\xi)$ *is an explicit modulation term:*

$$\Delta(\xi) = \frac{\eta_2}{K(m)}\left[\int_{\Sigma_1^+(\xi)} \frac{\log\left(\frac{2r_2(z)}{1+r_1(z)r_2(z)}\right)}{R_+(z)}\mathrm{d}z - \int_{\Sigma_1^-(\xi)} \frac{\log\left(\frac{2r_1(z)}{1+r_1(z)r_2(z)}\right)}{R_+(z)}\mathrm{d}z + \int_{\mathcal{L}^0(\xi)} \frac{\log(1+|\rho(z)|^2)}{R(z)}\mathrm{d}z\right].$$

Remarks:

1. The solution is equivalent to the elliptic solution

   $$q_{\mathrm{ell}}(x,t) = -\beta_1 - \beta_2 - \beta_3 + \frac{2(\beta_2+\beta_3)(\beta_1+\beta_3)}{\beta_2+\beta_3-(\beta_2-\beta_1)\,\mathrm{cn}^2\left(\sqrt{\beta_3^2-\beta_1^2}(x-2(\beta_1^2+\beta_2^2+\beta_3^2)t)+x_0 \mid m\right)}$$

   here $\beta_1 = 0$, $\beta_2 = \eta_1$ and $\beta_3 = \eta_2$.

2. The error term $\mathcal{O}(t^{-\frac{1}{4}+\epsilon})$ is due to the hyperbolic cylinder parametrix -when present-.
   Otherwise, the error term is $\mathcal{O}(t^{-1/2})/\mathcal{O}(e^{-t})$ (resp. when $\rho \neq 0/\ \rho = 0$).

   Indeed, in our construction we have $r_j(k) \sim |k - \mathrm{i}\eta_j|^{1/2}$ and the local parametrices near the fixed end points are not needed.

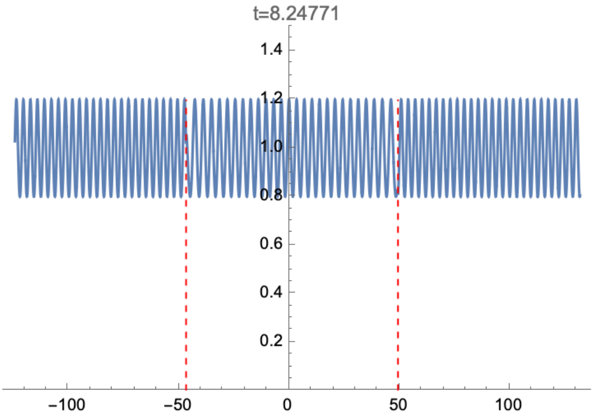

Figure: VIDEO: $\eta_1 = 0.95$ and $\eta_2 = 1$; $\frac{2r_1}{1+r_1 r_2} = e^{iz}$ and $\frac{2r_2}{1+r_1 r_2} = e^{4iz}$.

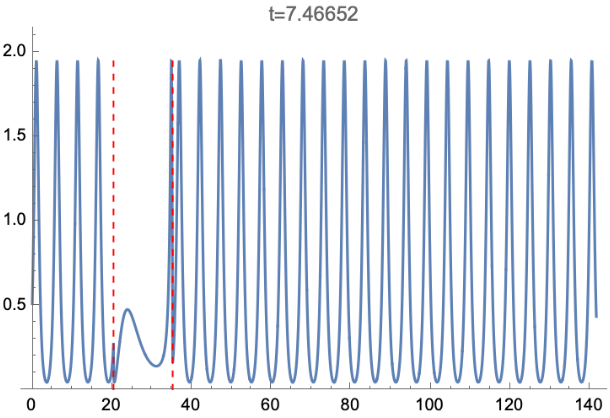

Figure: VIDEO: $\eta_1 = 0.2$ and $\eta_2 = 1$; $\frac{2r_1}{1+r_1 r_2} = e^{iz}$ and $\frac{2r_2}{1+r_1 r_2} = e^{4iz}$.

Conclusive overlook

What we have so far:

- We constructed a new class of solutions to the mKdV equation. This gas is
  - regular
  - dense (condensate)
  - deterministic
- Description of the full soliton gas in the large time regime, over the whole spatial domain.

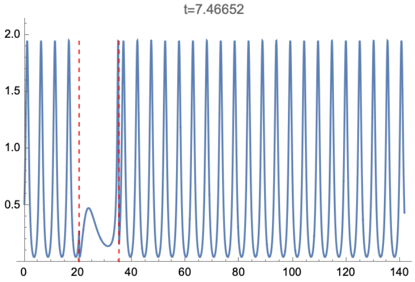

Figure: Snapshot of the Bargmann gas asymptotics

Conclusive overlook

What would be cool to analyze:

- Interaction of the full gas with a big soliton: long time behaviour, kinetic equations, phase shifts, etc.
  - *conjecture*: we'll get kinetic equations for a condensate...
  - *problem*: where do we "initialize" the soliton?
- construct new gasses from soliton+antisoliton limit (i.e. r_1, r_2 may have zeros)
- construct new (full) gasses from different scaling limits as $N \to \infty$ (e.g. reflection coefficients *do not* scale like $\mathcal{O}(N^{-1})$)
  - *conjecture*: we may be able to recover gasses that are *not* condensate

$$\int_\Sigma \ln\left|\frac{s+\kappa}{s-\kappa}\right| u(s)|\mathrm{d}s| + \sigma(k)u(k) = \frac{\pi}{2}$$

$$\int_\Sigma \ln\left|\frac{s+\kappa}{s-\kappa}\right| v(s)|\mathrm{d}s| + \sigma(k)v(k) = -2\pi\kappa^2$$

    where u is the density of states, v is the density of velocities.
  - asymptotic behaviour (a.k.a. g-function) will be different
- adding randomness to reflection coefficients/position of the poles (see Ken's talk!)

Thank you! – Questions?

Conclusive overlook

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