Exactly solvable interacting particle systems

Chiara Franceschini

University of Modena and Reggio Emilia

Joint works with: G. Carinci, R. Frassek, D. Gabrielli, C. Giardinà, F. Redig, D. Tsagkarogiannis

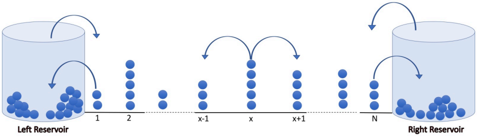
Integrable Systems: Geometrical and Analytical Approaches

Trieste, SISSA, June 4 2024

Finanziato dall'Unione europea
NextGenerationEU

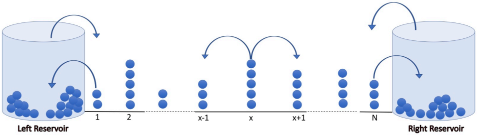
General setting

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.



General setting

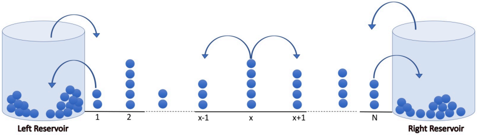
- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.



- **Goal:** Non-equilibrium steady state (NESS).

General setting

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.

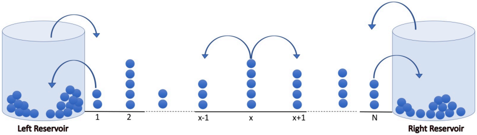


- **Goal:** Non-equilibrium steady state (NESS).

- These models share similar futures:

General setting

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.

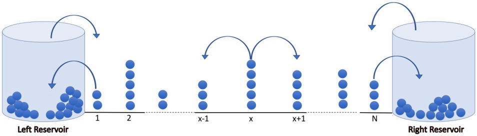


- **Goal:** Non-equilibrium steady state (NESS).

- These models share similar futures:
    - **Markov duality** property.

General setting

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.

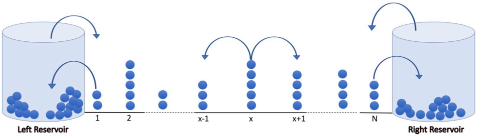


- **Goal:** Non-equilibrium steady state (NESS).

- These models share similar futures:
  - **Markov duality** property.
  - Similar **algebraic description** in terms of the Lie algebra $\mathfrak{su}(1,1)$.

General setting

- Motivation comes from **non-equilibrium statistical mechanics**: open processes with boundary reservoirs, i.e. $L = L_{left} + L_{bulk} + L_{right}$.



- **Goal:** Non-equilibrium steady state (NESS).

- These models share similar futures:
    - **Markov duality** property.
    - Similar **algebraic description** in terms of the Lie algebra $\mathfrak{su}(1,1)$.
    - **Macroscopic** quantities: constant diffusivity and quadratic moblity.

KMP model, [Kipnis, Marchioro, Presutti 1982]

$L^{KMP} = L_{left} + \sum_{x=1}^{N} L_{x,x+1} + L_{right}$ where

$$L_{x,x+1} f(\mathbf{z}) = \int_0^1 dp \left[f(z_1, \ldots, p(z_x + z_{x+1}), (1-p)(z_x + z_{x+1}), \ldots z_N) - f(\mathbf{z}) \right]$$

and for the reservoirs

$$L_{left} f(\mathbf{z}) = \int_0^\infty dz_1' \frac{e^{-z_1'/T_-}}{T_-} \left[f(z_1', \ldots, z_N) - f(\mathbf{z}) \right]$$

$$L_{right} f(\mathbf{z}) = \int_0^\infty dz_N' \frac{e^{-z_1'/T_+}}{T_+} \left[f(z_1, \ldots, z_N') - f(\mathbf{z}) \right]$$

for $z \in (\mathbb{R}^+)^N$, z_x represents the energy at site x

After the redistribution: with $p \sim U([0,1])$
$z_x' = p(z_x + z_{x+1})$ is the new energy at site x and
$z_{x+1}' = (1-p)(z_x + z_{x+1})$ is the new energy at site $x+1$

NESS for the KMP model

Invariant measure in equilibrium ($T_- = T_+ = T$):

Homogeneous product of exponential distribution with mean values T.

$$\nu_{N,T}(z) = \prod_{x=1}^{N} \frac{e^{-z_x/T}}{T}$$

NESS for the KMP model

Invariant measure in equilibrium ($T_- = T_+ = T$):

Homogeneous product of exponential distribution with mean values T.

$$\nu_{N,T}(z) = \prod_{x=1}^{N} \frac{e^{-z_x/T}}{T}$$

Invariant measure in non-equilibrium ($T_- \neq T_+$):

Inhomogeneous product of exponential distribution whose mean values are distributed according to the invariant measure of an auxiliary opinion model. [De Masi, Ferrari, Gabrielli, 2023]

Generalized KMP [Giardinà, Kurchan, Redig, Vafayi 2009]

$L^{KMP} = L_{left} + \sum_{x=1}^{N} L_{x,x+1} + L_{right}$ where

$$L_{x,x+1} f(\mathbf{z}) = \int_0^1 dp \quad \frac{p^{2s-1}(1-p)^{2s-1}\Gamma(4s)}{\Gamma(2s)\Gamma(2s)} \left[f(z_1, \dots, p(z_x + z_{x+1}), (1-p)(z_x + z_{x+1}), \dots z_N) - f(\mathbf{z}) \right]$$

$$L_{left} f(\mathbf{z}) = \int_0^\infty dz_1' \frac{e^{-z_1'/T_-}}{T_-} \frac{(z_1')^{2s-1}}{(T_-)^{2s-1}\Gamma(2s)} \left[f(z_1', \dots, z_N) - f(\mathbf{z}) \right]$$

$$L_{right} f(\mathbf{z}) = \int_0^\infty dz_N' \frac{e^{-z_1'/T_+}}{T_+} \frac{(z_N')^{2s-1}}{(T_+)^{2s-1}\Gamma(2s)} \left[f(z_1, \dots, z_N') - f(\mathbf{z}) \right]$$

for $z \in \left(\mathbb{R}^+\right)^N$, z_x represents the energy at site x

After the redistribution: with $p \sim Beta\,(2s, 2s)$.
Setting $s = 1/2$, we get $p \sim Beta\,(1, 1)$ and we recover the KMP.

NESS for Generalized KMP model

Invariant measure in equilibrium ($T_- = T_+ = T$):

Homogeneous product of Gamma distribution with scale parameter T and shape parameter $2s$

$$\nu_{N,T}(z) = \prod_{x=1}^{N} \frac{z_x^{2s-1}}{\Gamma(2s)} \frac{e^{-z_x/T}}{T^{2s}}$$

NESS for Generalized KMP model

Invariant measure in equilibrium ($T_- = T_+ = T$):

Homogeneous product of Gamma distribution with scale parameter T and shape parameter $2s$

$$\nu_{N,T}(z) = \prod_{x=1}^{N} \frac{z_x^{2s-1}}{\Gamma(2s)} \frac{e^{-z_x/T}}{T^{2s}}$$

Invariant measure in non-equilibrium ($T_- \neq T_+$):

?

A new energy-redistribution rule [Frassek, Giardinà, Kurchan 2020]

The generator is $\mathscr{L} = \mathscr{L}_{left} + \sum_{x=1}^{N} \mathscr{L}_{x,x+1} + \mathscr{L}_{right}$ where

$$\mathscr{L}_{x,x+1} f(\mathbf{y}) = \int_0^{y_x} \frac{d\alpha}{\alpha} \left[f(y_1, \dots, y_x - \alpha, y_{x+1} + \alpha, \dots, y_N) - f(\mathbf{y}) \right] + \int_0^{y_{x+1}} \frac{d\alpha}{\alpha} \left[f(y_1, \dots, y_x + \alpha, y_{x+1} - \alpha, \dots, y_N) - f(\mathbf{y}) \right]$$

$$\mathscr{L}_{left} f(\mathbf{y}) = \int_0^{y_1} \frac{d\alpha}{\alpha} \left[f(y_1 - \alpha, \dots, y_N) - f(\mathbf{y}) \right] + \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\alpha / T_-} \left[f(y_1 + \alpha, \dots, y_N) - f(\mathbf{y}) \right]$$

After the redistribution: with $u \sim Beta(0, 1)$
$y'_x = y_x - u y_x$ is the new energy at site x and
$y'_{x+1} = y_{x+1} + u y_x$ is the new energy at site $x + 1$

Generalized [F., Frassek, Giardinà 2022]

The generator is $\mathscr{L} = \mathscr{L}_{left} + \sum_{x=1}^{N-1} \mathscr{L}_{x,x+1} + \mathscr{L}_{right}$ where

$$\mathscr{L}_{x,x+1} f(\mathbf{y}) = \int_0^{y_x} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_x}\right)^{2s-1} \left[f(y_1, \dots, y_x - \alpha, y_{x+1} + \alpha, \dots, y_N) - f(\mathbf{y})\right]$$
$$+ \int_0^{y_{x+1}} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_{x+1}}\right)^{2s-1} \left[f(y_1, \dots, y_x + \alpha, y_{x+1} - \alpha, \dots, y_N) - f(\mathbf{y})\right]$$

$$\mathscr{L}_{left} f(\mathbf{y}) = \int_0^{y_1} \frac{d\alpha}{\alpha} \left(1 - \frac{\alpha}{y_1}\right)^{2s-1} \left[f(y_1 - \alpha, \dots, y_N) - f(\mathbf{y})\right]$$
$$+ \int_0^{+\infty} \frac{d\alpha}{\alpha} e^{-\alpha/T_-} \left[f(y_1 + \alpha, \dots, y_N) - f(\mathbf{y})\right]$$

After the redistribution: with $u \sim Beta(0, 2s)$
$y'_x = y_x - uy_x$ is the new energy at site x and
$y'_{x+1} = y_{x+1} + uy_x$ is the new energy at site $x+1$

Invariant measures: $s = 1/2$

In equilibrium ($T_- = T_+ = T$):

Homogeneous product of exponential distribution with mean values T.

$$\nu_{N,T}(z) = \prod_{x=1}^{N} \frac{e^{-z_x/T}}{T}$$

Out of equilibrium ($T_- < T_+$):

Theorem (Carinci, F., Gabrielli, Giardinà, Tsagkarogiannis)

$$\nu_{N,T_-,T_+}(z) = \mathbb{E}\left[\prod_{x=1}^{N} \frac{e^{-z_x/T_{x,N}}}{T_{x,N}}\right]$$

above $\mathbb{E}$ is the expectation with respect to the joint distribution of the random variables $(T_{1,N}, \ldots, T_{N,N})$, where $T_{x,N}$ is the x^{th} ordered statistics of the sample of independent uniform distribution in $[T_-, T_+]$.

Markov duality definition

Definition

Let $(z_t)_{t\geq 0}$ and $(\xi_t)_{t\geq 0}$ Markov process on Ω and Ω^{dual} with generators L and L^{dual}, respectively. z_t is **dual** to ξ_t with duality function $D : \Omega \times \Omega^{dual} \to \mathbb{R}$ if $\forall t \geq 0$,

$$\mathbb{E}_z(D(z_t,\xi)) = \mathbb{E}_\xi(D(z,\xi_t)) \qquad \forall (z,\xi) \in \Omega \times \Omega^{dual} .$$

Equivalently,

$$LD(\cdot,\xi)(z) = L^{dual} D(z,\cdot)(\xi)$$

Duality enables to connect, via a duality function, the process of interest to another one which is a **simpler** process.

Markov duality for KMP [Kipnis, Marchioro, Presutti 1982]

The dual process ξ_t describes the motion of particles in a one dimensional $N+1$ sites chain with **absorbing boundary** sites 0 and $N+1$. The generator is $L_{dual}^{KMP} = L_{left} + \sum_{x=1}^{N} L_{x,x+1} + L_{right}$ where

$$L_{x,x+1} f(\boldsymbol{\xi}) = \frac{1}{\xi_x + \xi_{x+1} + 1} \sum_{r=0}^{\xi_x + \xi_{x+1}} \Big[f\left(\xi_0, \dots \xi_{x-1}, r, \xi_x + \xi_{x+1} - r, \dots, \xi_{N+1}\right) - f\left(\boldsymbol{\xi}\right) \Big]$$

$$L_{left} f(\boldsymbol{\xi}) = [f(\xi_0 + \xi_1, 0, \xi_2, \dots, \xi_{N+1}) - f(\boldsymbol{\xi})]$$
$$L_{right} f(\boldsymbol{\xi}) = [f(\xi_0, \xi_1, \dots, \xi_{N-1}, \dots, 0, \xi_N + \xi_{N+1}) - f(\boldsymbol{\xi})]$$

and $\xi_x = \sharp$ particles at x.

The duality function is

$$D(\boldsymbol{z}, \boldsymbol{\xi}) = T_-{}^{\xi_0} \prod_{x=1}^{N} \frac{z_x^{\xi_x}}{\xi_x!} T_+{}^{\xi_{N+1}}$$

Markov duality for the integrable heat conduction [F., Frassek, Giardinà 2023]

The dual generator is

$$\mathscr{L}^{\mathrm{dual}} = \mathscr{L}_1^{\mathrm{dual}} + \sum_{x=1}^{N} \mathscr{L}_{x,x+1}^{\mathrm{dual}} + \mathscr{L}_N^{\mathrm{dual}} \qquad \text{where}$$

$$\mathscr{L}_{x,x+1}^{\mathrm{dual}} f(\boldsymbol{\xi}) = \sum_{k=1}^{\xi_x} \frac{1}{k}\Big[f(\boldsymbol{\xi} - k\boldsymbol{\delta_x} + k\boldsymbol{\delta_{x+1}}) - f(\boldsymbol{\xi})\Big] + \sum_{k=1}^{\xi_{x+1}} \frac{1}{k}\Big[f(\boldsymbol{\xi} + k\boldsymbol{\delta_x} - k\boldsymbol{\delta_{x+1}}) - f(\boldsymbol{\xi})\Big]$$

and

$$\mathscr{L}_1^{\mathrm{dual}} f(\boldsymbol{\xi}) = \sum_{k=1}^{\xi_1} \frac{1}{k}\Big[f(\boldsymbol{\xi} - k\boldsymbol{\delta_1} + k\boldsymbol{\delta_0}) - f(\boldsymbol{\xi})\Big]$$

Duality function

Generalized version has jump rate:

$$\varphi_s(k,n) = \frac{\Gamma(n+1)\Gamma(n-k+2s)}{k\Gamma(n+2s)\Gamma(n-k+1)}$$

Duality function

Generalized version has jump rate:

$$\varphi_s(k,n) = \frac{\Gamma(n+1)\Gamma(n-k+2s)}{k\Gamma(n+2s)\Gamma(n-k+1)}$$

It is not surprising that the duality function in this case is the same as the KMP one:

$$D(\boldsymbol{z},\boldsymbol{\xi}) = T_-{}^{\xi_0} \prod_{x=1}^{N} \frac{z_x^{\xi_x}}{\xi_x!} T_+{}^{\xi_{N+1}}$$

It is not unique!

$$D(\boldsymbol{z},\boldsymbol{\xi}) = (T_- - T)^{\xi_0} \prod_{x=1}^{N} {}_1F_1\left(\begin{matrix}-\xi_x\\ 2s\end{matrix}\,\middle|\,\frac{z_x}{T}\right)(-T)^{\xi_x}(T_+ - T)^{\xi_{N+1}}$$

${}_1F_1\left(\begin{matrix}-n\\ \alpha\end{matrix}\,\middle|\,x\right)$ is Laguerre polynomial.

Relation to previous models of L_{bulk}

For $s = 1/2$, the asymmetric version is the MADM model [Sasamoto, Wadati (1998)]:

$$L^{MADM}_{x,x+1} f(\xi) = \sum_{k=1}^{\xi_x} \frac{1}{[k]_q} \left[f(\xi) - k\boldsymbol{\delta_x} + k\boldsymbol{\delta_{x+1}}) - f(\xi)) \right] + \sum_{k=1}^{\xi_{x+1}} \frac{q^k}{[k]_q} \left[f(\xi) + k\boldsymbol{\delta_x} - k\boldsymbol{\delta_{x+1}}) - f(\xi)) \right]$$

where $[k]_q := \frac{1-q^k}{1-q} \to k$ as $q \to 1$

Further generalized by the $q-$Hahn asymmetric exclusion process [Barraquand, Corwin (2014)]

The absorbing harmonic is also the dual of an open boundary harmonic [Frassek, Giardinà, Kurchan (2020)]

Open Harmonic process ($s = 1/2$)

$$\mathscr{L}^{\text{Harmonic}} = \mathscr{L}_L + \sum_{x=1}^{N} \mathscr{L}_{x,x+1} + \mathscr{L}_R \qquad \text{where}$$

$$\begin{aligned}\mathscr{L}_{x,x+1} f(\boldsymbol{\eta}) &= \sum_{k=1}^{\eta_x} \frac{1}{k} \Big[f(\boldsymbol{\eta} - k\boldsymbol{\delta}_{\boldsymbol{x}} + k\boldsymbol{\delta}_{\boldsymbol{x+1}}) - f(\boldsymbol{\eta}) \Big] \\ &+ \sum_{k=1}^{\eta_{x+1}} \frac{1}{k} \Big[f(\boldsymbol{\eta} + k\boldsymbol{\delta}_{\boldsymbol{x}} - k\boldsymbol{\delta}_{\boldsymbol{x+1}}) - f(\boldsymbol{\eta}) \Big]\end{aligned}$$

and

$$\mathscr{L}_L f(\boldsymbol{\eta}) = \sum_{k=1}^{\eta_1} \frac{1}{k} \Big[f(\boldsymbol{\eta} - k\boldsymbol{\delta}_{\boldsymbol{1}}) - f(\boldsymbol{\eta}) \Big] + \sum_{k=1}^{\infty} \frac{\beta_L^k}{k} \Big[f(\boldsymbol{\eta} + k\boldsymbol{\delta}_{\boldsymbol{1}}) - f(\boldsymbol{\eta}) \Big]$$

$$\mathscr{L}_R f(\boldsymbol{\eta}) = \sum_{k=1}^{\eta_N} \frac{1}{k} \Big[f(\boldsymbol{\eta} - k\boldsymbol{\delta}_{\boldsymbol{N}}) - f(\boldsymbol{\eta}) \Big] + \sum_{k=1}^{\infty} \frac{\beta_R^k}{k} \Big[f(\boldsymbol{\eta} + k\boldsymbol{\delta}_{\boldsymbol{N}}) - f(\boldsymbol{\eta}) \Big]$$

Power of duality

Having that two different models with **open boundary** conditions share the same **purely absorbing dual process** with two different (but meaningful!) duality functions allows to solve one of the two using the solution of the other one.

Power of duality

Having that two different models with **open boundary** conditions share the same **purely absorbing dual process** with two different (but meaningful!) duality functions allows to solve one of the two using the solution of the other one.

Concretely one gets that the NESS of the open Harmonic has the same structure of the NESS of the open integrable KMP where exponential (resp. Gamma) distribution is replaced by geometric (resp. Negative Binomial) distribution.

This is encoded in the structure of the two different duality functions:

$$D(\boldsymbol{z}, \boldsymbol{\xi}) = T_-{}^{\xi_0} \, T_+{}^{\xi_{N+1}} \prod_{x=1}^{N} \frac{z_x^{\xi_x} \Gamma(2s)}{\Gamma(2s+\xi)}$$

for the integrable KMP and

for the Harmonic

$$D(\boldsymbol{\eta}, \boldsymbol{\xi}) = \beta_L{}^{\xi_0} \beta_R{}^{\xi_{N+1}} \prod_{x=1}^{N} \frac{\eta_x!}{(\eta_x - \xi_x)!} \frac{\Gamma(2s)}{\Gamma(2s+\xi)}$$

Summary of the duality results

Open Integrable KMP

Open Harmonic

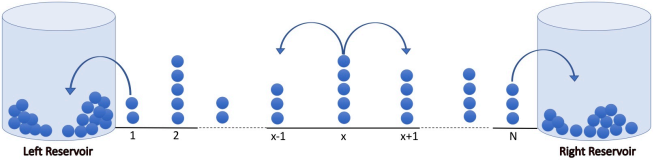
Moments Duality Function

$$D(\boldsymbol{z},\boldsymbol{\xi}) = T_-^{\ \xi_0} T_+^{\ \xi_{N+1}} \prod_{x=1}^{N} \frac{z_x^{\xi_x}\Gamma(2s)}{\Gamma(2s+\xi)}$$

Factorial moments Duality Function

$$D(\boldsymbol{\eta},\boldsymbol{\xi}) = \beta_L^{\ \xi_0} \beta_R^{\ \xi_{N+1}} \prod_{x=1}^{N} \frac{\eta_x!}{(\eta_x-\xi_x)!} \frac{\Gamma(2s)}{\Gamma(2s+\xi)}$$

Absorbing Harmonic



NESS for the Harmonic process $s = 1/2$

In equilibrium ($\beta_L = \beta_R = \beta$):

Homogeneous product of geometric distribution with mean β.

$$\mu_{N,\beta}(\eta) = \prod_{x=1}^{N} \left(\frac{\beta}{1+\beta}\right)^{\eta_x} \left(\frac{1}{1+\beta}\right)$$

NESS for the Harmonic process $s = 1/2$

In equilibrium ($\beta_L = \beta_R = \beta$):

Homogeneous product of geometric distribution with mean β.

$$\mu_{N,\beta}(\eta) = \prod_{x=1}^{N} \left(\frac{\beta}{1+\beta}\right)^{\eta_x} \left(\frac{1}{1+\beta}\right)$$

Out of equilibrium ($\beta_L < \beta_R$):

Theorem (Carinci, F., Gabrielli, Giardinà, Tsagkarogiannis)

$$\mu_{N,\beta_L,\beta_R}(\eta) = \mathbb{E}\left[\prod_{x=1}^{N} \left(\frac{\beta_{x,N}}{1+\beta_{x,N}}\right)^{\eta_x} \left(\frac{1}{1+\beta_{x,N}}\right)\right]$$

above $\mathbb{E}$ is the expectation with respect to the joint distribution of the random variables $(\beta_{1,N}, \ldots, \beta_{N,N})$, where $\beta_{x,N}$ is the x^{th} ordered statistics of the sample of independent uniform distribution in $[\beta_L, \beta_R]$.

NESS for the Harmonic process $2s \in \mathbb{N}$

In equilibrium ($\beta_L = \beta_R = \beta$):

Homogeneous product of negative binomial distribution with shape parameter $2s \in \mathbb{N}$ and mean $2s\beta$.

$$\mu_{N,\beta}(\eta) = \prod_{x=1}^{N} \frac{\Gamma(2s+\eta_x)}{\eta_x!\,\Gamma(2s)} \left(\frac{\beta}{1+\beta}\right)^{\eta_x} \left(\frac{1}{1+\beta}\right)^{2s}$$

NESS for the Harmonic process $2s \in \mathbb{N}$

In equilibrium ($\beta_L = \beta_R = \beta$):

Homogeneous product of negative binomial distribution with shape parameter $2s \in \mathbb{N}$ and mean $2s\beta$.

$$\mu_{N,\beta}(\eta) = \prod_{x=1}^{N} \frac{\Gamma(2s+\eta_x)}{\eta_x!\Gamma(2s)} \left(\frac{\beta}{1+\beta}\right)^{\eta_x} \left(\frac{1}{1+\beta}\right)^{2s}$$

Out of equilibrium ($\beta_L < \beta_R$):

Theorem (Carinci, F., Frassek, Giardinà, Redig)

$$\mu_{N,\beta_L,\beta_R}(\eta) = \mathbb{E}\left[\prod_{x=1}^{N} \frac{\Gamma(2s+\eta_x)}{\eta_x!\Gamma(2s)} \left(\frac{\beta_{2sx,n}}{1+\beta_{2sx,n}}\right)^{\eta_x} \left(\frac{1}{1+\beta_{2sx,n}}\right)^{2s}\right]$$

above $\mathbb{E}$ is the expectation with respect to the joint distribution of the random variables $(\beta_{2s,n}, \ldots, \beta_{2sN,n})$, where $n = 2s(N+1) - 1$. These r.v. are obtained as marginal of $(\beta_{1,n}, \ldots, \beta_{n,n})$ which are the ordered statistics of the sample of n independent uniform distribution in $[\beta_L, \beta_R]$.

Applications: Large Deviations

Macroscopic quantities of interest are

Diffusivity **Mobility**

$$D(\rho) = \frac{1}{2s} \qquad \text{and} \qquad \sigma(\rho) = \frac{\rho}{2s}\left(1 + \frac{\rho}{2s}\right)$$

where the density $\rho : [0,1] \to \mathbb{R}_+$ has to be interpreted as the average number of particles.
The empirical measure of the open symmetric harmonic process is

$$\pi_N(\eta) := \frac{1}{N}\sum_{x=1}^{N} \eta_x \; \delta_{x/N}$$

and the associated pressure, for $h : [0,1] \to \mathbb{R}$ is

$$P(h) := \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{E}_{\mu_N}\left[e^{\sum_{x=1}^{N} \eta_x \, h(x/N)}\right]$$

Pressure

Theorem (Pressure via the explicit characterization of the NESS)

$$P(h) = 2s \log \left(\frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L} \right)$$

Sketch of the proof. $s = 1/2$, constant field h

Recall the MGF of a Geometric $\left(\frac{1}{1+\beta}\right)$: $M_\beta(h) = \dfrac{1}{1 + \beta(1 - e^h)}$

$$\mathbb{E}_{\mu_N}\left[\prod_{x=1}^{N} e^{\eta_x h}\right] = \mathbb{E}\left[\prod_{x=1}^{N} \sum_{\eta} \mu_N(\eta) e^{\eta_x h}\right] = \mathbb{E}\left[\prod_{x=1}^{N} M_{\beta_{x,N}}(h)\right] =$$

$$\mathbb{E}\left[\prod_{x=1}^{N} M_{\beta_x}(h)\right] = \left[\mathbb{E}\left(M_{\beta_1}(h)\right)\right]^N$$

Since $\beta_1 \sim U[\beta_L, \beta_R]$ then

$$\mathbb{E}\left[M_{\beta_1}(h)\right] = \frac{1}{\beta_R - \beta_L} \int_{\beta_L}^{\beta_R} \frac{dx}{1 + x(1 - e^h)} = \frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L}$$

All in all,

$$P(h) := \lim_{N \to +\infty} P_N(h) = \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{E}\left[e^{\sum_{x=1}^{N} \eta_x \, h(x/N)}\right] =$$

$$\lim_{N \to +\infty} \frac{1}{N} \log \left(\frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L} \right)^N =$$

$$\log \left(\frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L} \right)$$

Since $\beta_1 \sim U[\beta_L, \beta_R]$ then

$$\mathbb{E}\left[M_{\beta_1}(h)\right] = \frac{1}{\beta_R - \beta_L} \int_{\beta_L}^{\beta_R} \frac{dx}{1 + x(1 - e^h)} = \frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L}$$

All in all,

$$P(h) := \lim_{N \to +\infty} P_N(h) = \lim_{N \to +\infty} \frac{1}{N} \log \mathbb{E}\left[e^{\sum_{x=1}^{N} \eta_x \, h(x/N)}\right] =$$

$$\lim_{N \to +\infty} \frac{1}{N} \log \left(\frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L} \right)^N =$$

$$\log \left(\frac{1}{(\beta_R - \beta_L)(1 - e^h)} \log \frac{1 + (1 - e^h)\beta_R}{1 + (1 - e^h)\beta_L} \right)$$

For $s = 1/2$ the finite volume pressure P_N does not depend on N and it coincides with the pressure at infinite volume.

Variational problem for the Pressure

The pressure of the open boundary Harmonic process admits the variational expression

$$P(h) = \sup_{\substack{\beta:[0,1]\to\mathbb{R}_+ \\ \text{strictly increasing} \\ \beta(0)=\beta_L \\ \beta(1)=\beta_R}} \left[P(h,\beta) - J(\beta) \right]$$

where

$$P(h,\beta) = \int_0^1 \log\left(\frac{1}{1+(1-e^{h(x)})\beta(x)} \right) dx$$

$$J(\beta) = \int_0^1 \log\left(\frac{\beta'(x)}{\beta_R - \beta_L} \right) dx$$

Variational problem for the Rate Function

$\pi_N(\eta)$ satisfies a functional LDP

$$\mu_N\left(\pi_N(\eta) \sim \rho\right) \approx e^{-N\,\mathrm{I}(\rho)} \quad \text{with} \quad \mathrm{I}(\rho) = \inf_{\substack{\beta:[0,1]\to\mathbb{R}_+ \\ \text{strictly increasing} \\ \beta(0)=\beta_L \\ \beta(1)=\beta_R}} \Big[I(\rho,\beta) - J(\beta) \Big]$$

above

$$I(\rho,\beta) = 2s \int_0^1 \left[\frac{\rho(x)}{2s} \log \frac{\rho(x)}{2s\beta(x)} + \left(1 + \frac{\rho(x)}{2s}\right) \log\left(\frac{1+\beta(x)}{1+\frac{\rho(x)}{2s}}\right) \right] dx$$

is the rate function of LDP for Negative Binomials and

$$J(\beta) = 2s \int_0^1 \log\left(\frac{\beta'(x)}{\beta_R - \beta_L}\right) dx$$

is the rate function of LDP of order statistics of uniform random variables

Last remarks

- When $\beta_L - \beta_R \to 0$ implies $\beta(x)$ constant and the **equilibrium** result is recovered.
- The variational problem for SEP has a maximum instead of a minimum
- The Legendre transform of the rate function is always the pressure
- If $\mathrm{I}(\rho)$ is not convex, the Legendre transform of the pressure is the convex hull of $\mathrm{I}(\rho)$.
- We also proved two additivity principles: one for the pressure and one for the rate function

Last remarks

- When $\beta_L - \beta_R \to 0$ implies $\beta(x)$ constant and the **equilibrium** result is recovered.
- The variational problem for SEP has a maximum instead of a minimum
- The Legendre transform of the rate function is always the pressure
- If $\mathrm{I}(\rho)$ is not convex, the Legendre transform of the pressure is the convex hull of $\mathrm{I}(\rho)$.
- We also proved two additivity principles: one for the pressure and one for the rate function

Thanks for the attention :)