TRIESTE 2024

Bi-flat F-manifolds, Frölicher-Nijenhuis bicomplexes and integrable systems

Paolo Lorenzoni

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Plan of the talk

1. Frölicher-Nijenhuis bicomplex and integrable systems of hydrodynamic type.
   Based on a joint work with **F. Magri** (2005).

2. Frölicher-Nijenhuis bicomplex and Lauricella bi-flat F-maniflolds.
   Based on a joint work with **S. Perletti** (2023).

3. Bi-flat F-manifolds, lifted Frölicher-Nijenhuis bicomplexes and Gauss-Manin connections.
   Based on a joint work with **A. Arsie** (2024).

PART I

Frölicher-Nijenhuis bicomplex and integrable systems of hydrodynamic type.

Based on a joint work with **F. Magri** (2005).

Nijenhuis operators and Frölicher-Nijenhuis bicomplex

Let L be a $(1,1)$ tensor field L with vanishing Nijenhuis torsion:

$$[LX, LY] - L\,[X, LY] - L\,[LX, Y] + L^2\,[X, Y] = 0.$$

Define d_L as

$$(d_L\omega)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i (LX_i)(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_k)) +$$

$$+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \omega([X_i, X_j]_L, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_k),$$

where

$$[X_i, X_j]_L = [LX_i, X_j] + [X_i, LX_j] - L[X_i, X_j].$$

Frölicher-Nijenhuis bicomplex

The Frolicher-Nijeenhuis bicomplex is $(d, d_L, \Omega(M))$ where

- $\Omega(M)$is the Grasmann algebra of differential forms on M.
- d is the usual de Rham differential,
- $d_L^2 = 0$ due to the vanishing of the Nijenhuis torsion.
- Since $L + I$ has vanishing Nijenhuis torsion we have

$$d \cdot d_L + d_L \cdot d = 0.$$

Classical Lenard-Magri recursion

Starting from a solution of

$$d \cdot d_L a_0 = 0$$

the recursion

$$da_{k+1} = d_L a_k$$

is well defined. Indeed, due to anticommutativity of d and d_L at each step we have

$$d \cdot d_L a_k = -d_L \cdot da_k = -d_L^2 a_{k-1} = 0.$$

Generalized Lenard-Magri recursion

Starting from a solution of

$$d \cdot d_L\, a_0 = 0$$

it is not difficult to prove that also the recursion

$$da_{k+1} = d_L a_k - a_k da_0$$

is well defined. The sequence of functions $a_0 . a_1, a_2, ...,$ obtained in this way satisfy the equation

$$d \cdot d_L a_k = da_k \wedge da_0 .$$

Similar recursion appears in the geometric theory of separation of variables (Falqui, Magri and Pedroni).

Generalized Lenard-Magri chains and integrable systems of hydrodynamic type

Theorem (P.L and F. Magri 2005)

Let $a_0, a_1, a_2, \ldots$ be a sequence of functions obtained applying generalized Lenard-Magri recursion and let V_k be the tensor fields of type $(1,1)$ defined recursively by

$$V_{k+1} = V_k L - a_k I,$$

starting from $V_0 = I$, then the flows

$$\boldsymbol{u}_{t_k} = V_k \boldsymbol{u}_x \qquad k = 0, 1, 2, \ldots$$

commute. Moreover the densities of conservation laws h satisfy the equation

$$d \cdot d_L h = dh \wedge da_0.$$

Generalized ε-system

The system of hydrodynamic type

$$\begin{bmatrix} u^1_{t_1} \\ \vdots \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 - \sum_{k=1}^n \varepsilon_k u^k & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & u^n - \sum_{k=1}^n \varepsilon_k u^k \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^n_x \end{bmatrix}$$

has been obtained by Pavlov as finite component reduction of an infinite hydrodynamic chain. It can be written as

$$\mathbf{u}_{t_1} = (L - a_0 I)\mathbf{u}_x$$

with $a_0 = \sum_{k=1}^n \varepsilon_k u^k$ and $L = \mathrm{diag}(u^1, ..., u^n)$. The system for densities of conservation laws

$$d \cdot d_L h = dh \wedge da_0$$

reduces to the Euler-Darboux-Poisson system that appears in the study of Whitham equations (F.R.Tian 1994, T. Grava 2001).

Kodama-Konopelchenko system, 2015

The system of hydrodynamic type

$$\begin{bmatrix} u^1_{t_1} \\ u^2_{t_1} \\ \vdots \\ u^{n-1}_{t_1} \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 & 1 & 0 & \dots & 0 \\ 0 & u^1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^1 & 1 \\ 0 & \dots & 0 & 0 & u^1 \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^{n-1}_x \\ u^n_x \end{bmatrix} \qquad (1)$$

can be written as $\mathbf{u}_{t_1} = (L - a_0 I)\mathbf{u}_x$ with $a_0 = -u^1$ and

$$L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

(see also Konopelchenko-Ortenzi 2017)

Clearly L has vanishing Nijenhuis torsion and $dd_L a_0 = 0$. Applying the first step of the recursive procedure we have

$$\begin{aligned}
\partial_1 a_1 &= -a_0 \partial_1 a_0 = -u^1 \\
\partial_2 a_1 &= \partial_1 a_0 - a_0 \partial_2 a_0 = -1 \\
\partial_3 a_1 &= 0 \\
&\vdots \\
\partial_n a_1 &= 0.
\end{aligned}$$

This implies (up to an inessential constant) $a_1 = -u^2 - \frac{(u^1)^2}{2}$. Therefore the first commuting flow

$$\mathbf{u}_t = (L^2 - a_0 L - a_1 I)\mathbf{u}_x.$$

PART II

Frölicher-Nijenhuis bicomplex and Lauricella bi-flat F-maniflolds.

Based on a joint work with **S. Perletti** (2023).

Flat F-manifolds (Manin)

Definition

A **flat F-manifold** $(M, \nabla, \circ, e)$ a manifold equipped with a product

$$\circ : TM \times TM \to TM$$

with structure functions c^i_{jk}, a connection ∇ with Christoffel symbols a^i_{jk} and a distinguished vector field e s.t.

- the one parameter family of connections ∇_λ

  $$a^i_{jk} - \lambda c^i_{jk}$$

  is flat and torsionless for any λ.
- e is the unit of the product.
- e is flat: $\nabla e = 0$.

For a given λ the torsion and the curvature are

$$\begin{aligned} T_{ij}^{(\lambda)k} &= a_{ij}^k - a_{ji}^k + \lambda(c_{ij}^k - c_{ji}^k) \\ R_{ijl}^{(\lambda)k} &= R_{ijl}^k + \lambda(\nabla_i c_{jl}^k - \nabla_j c_{il}^k) + \lambda^2(c_{im}^k c_{jl}^m - c_{jm}^k c_{il}^m), \end{aligned}$$

We obtain

1. the connection ∇ is torsionless,
2. the product $\circ$ is commutative,
3. the connection ∇ is flat,
4. the tensor field $\nabla_l c_{ij}^k$ is symmetric in the low indices,
5. the product $\circ$ is associative.

The above conditions imply

$$c_{jk}^i = \partial_j \partial_k F^i.$$

Bi-flat F manifolds (A. Arsie, P.L 2012)

Definition

A *bi-flat* F-manifold is a manifold M equipped with a pair of flat structures $(\nabla, \circ, e)$ and $(\nabla^*, *, E)$ related by the following conditions:

- E is an Euler vector field for the first structure

$$[e, E] = e, \qquad \mathcal{L}_E \circ = \circ$$

  and at a generic point the operator $E\circ$ is assumed to be invertible.

- $*$ is the product defined by E, i.e. $X * Y = (E\circ)^{-1} X \circ Y$.

- $(d_\nabla - d_{\nabla^*})(X \circ) = 0$, where d_∇ is the exterior covariant derivative of vector-valued differential forms.

Regular bi-flat F-manifolds

Definition

A bi-flat F-manifold is called regular if at a generic point the Jordan canonical form of the operator $L = E\circ$ has n Jordan blocks of sizes $m_1, ..., m_n$ with distinct eigenvalues.

Semisimple bi-flat F-manifolds are regular bi-flat F-manifolds where $m_1 = \cdots = m_n = 1$.

In semisimple case there exist local coordinates (called canonical coordinates) such that:

$$e = \sum_{s=1}^{n} \frac{\partial}{\partial u^s}, \qquad E = \sum_{s=1}^{n} u^s \frac{\partial}{\partial u^s}, \qquad c^i_{jk} = \delta^i_j \delta^i_k$$

David-Hertling canonical coordinates

Theorem
Let $(M, \nabla, \circ, e, \nabla^*, *, E)$ be a regular bi-flat F-manifold. Then there exists local coordinates such that

$$
\begin{aligned}
e &= \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^{m_1+1}} + \frac{\partial}{\partial u^{m_1+m_2+1}} + \cdots + \frac{\partial}{\partial u^{m_1+\cdots+m_{n-1}+1}}, \\
E &= \sum_{s=1}^{m_1+\cdots+m_n} u^s \frac{\partial}{\partial u^s}, \\
c^l_{ij} &= \delta^l_{i+j-1}, \quad i, j, l = m_1 + \cdots + m_{k-1} + 1, ..., m_1 + \cdots + m_{k-1} + m_k \\
c^l_{ij} &= 0, \quad \text{otherwise}.
\end{aligned}
$$

Remark: the result holds true under weaker assumptions.

Operator of multiplication by E

The tensor field $L = E\circ$ contains n blocks $L_1, ..., L_n$ of dimension $m_1, ..., m_n$ respectively. Each block has the form

$$L_k = \begin{bmatrix} u^{k,1} & 0 & \dots & 0 \\ u^{k,2} & u^{k,1} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ u^{k,m_k} & \dots & u^{k,2} & u^{k,1} \end{bmatrix}$$

where $u^{k,l} = u^{m_1+\cdots+m_{k-1}+l}$.

Dubrovin-Frobenius manifolds

To define a Dubrovin-Frobenius manifold we need

- a metric η satisfying the conditions

$$\eta_{il}c^l_{jk} = \eta_{jl}c^l_{ik}, \qquad \nabla\eta = 0.$$

  It turns out that $c^i_{jk} = \eta^{il}\partial_l\partial_j\partial_k F$ where F is a solution of WDVV equations.

- Duality for Dubrovin-Frobenius manifold (almost-dual structure, Dubrovin 2004): it is defined by the data $(g, *, E)$ where E is the Euler vector field,

$$g = (E\circ)\eta^{-1}, \qquad X * Y = (E\circ)^{-1}X \circ Y$$

  Notice that $\nabla^{(g)}E \neq 0$ but there exists μ such that $\nabla^{(g)} - \nabla^* = \mu\, *$.

Integrable hierarchy associated with flat F-manifolds

Given a flat F-manifold $(M, \nabla, \circ, e)$ the flows of the associated integrable hierarchy have the form

$$\boldsymbol{u}_t = X \circ \boldsymbol{u}_x$$

where X is a solution of the equation

$$d_\nabla(X\circ) = 0.$$

A sequence of solutions of this equation can be obtained starting from flat vector fieds X_0 by means of the following recursive relation

$$d_\nabla X_{(\alpha+1)} = X_{(\alpha)} \circ .$$

This sequence is called **principal hierarchy**.

Some results on bi-flat F-manifolds

Bi-flat F-manifolds structures naturally appear:

- on the orbit space of reflection groups:
  M. Kato, T. Mano, J. Sekiguchi (2015),
  A. Arsie and P. L (2016),
  Y.Konishi, S. Minabe and Y. Shiraishi (2016).
- in relation with Painlevé transcendents:
  A.Arsie, P.L (2012 and 2015)
  P.L (2013),
  H. Kawakami and T. Mano (2017).
- as genus 0 part of F-cohomological field theories:
  A.Arsie, A. Buryak, P.L and P. Rossi (2020).

Problem

Given a bi-flat F-structure $(\nabla, \circ, e, \nabla^*, *, E)$ is well known that the operator

$$L = E\circ$$

has vanishing Nijenhuis torsion. In the regular case in David-Hertling canonical coordinates L has block diagonal form and each block has lower triangular Toeplitz form. Consider

$$V = L - a_0 I, \qquad d \cdot d_L a_0 = 0$$

Question: there exists a bi-flat F-manifolds such that

$$d_\nabla V = 0, \qquad ?$$

Regular Lauricella bi-flat F-manifolds

Theorem (P. L, S. Perletti 2023)

For any choice of $\varepsilon_1, \dots, \varepsilon_r$ *there exists a unique regular bi-flat structure* $(\nabla, \nabla^*, \circ, *, e, E)$ *with canonical coordinates* $\{u^1, \dots, u^n\}$ *such that* $d_\nabla(E \circ -a_0 I) = 0$*, where* r *is the number of the Jordan blocks (of sizes* $m_1, \dots, m_r$*) of* $E\circ$ *and, set* $m_0 = 0$*,*

$$a_0 = \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha u^{1(\alpha)} = \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha u^{m_0+m_1+\dots+m_{\alpha-1}+1}.$$

The case of a 2×2 Jordan block

In this case

$$L = \begin{bmatrix} u^1 & 0 \\ u^2 & u^1 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} \quad a_0 = 2\varepsilon_1 u^1.$$

The non vanishing Christoffel symbol of $\nabla^{(1)}$ is

$$\Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2}.$$

3×3 Jordan block

$$L = \begin{bmatrix} u^1 & 0 & 0 \\ u^2 & u^1 & 0 \\ u^3 & u^2 & u^1 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1}, \quad a_0 = 3\varepsilon_1 u^1$$

The non vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22} = \Gamma^3_{23} = -\frac{3\varepsilon_1}{u^2}, \ \Gamma^3_{22} = \frac{3\varepsilon_1 u^3}{(u^2)^2}$$

$2 \times 2 + 1 \times 1$ Jordan blocks

$$L = \begin{bmatrix} u^1 & 0 & 0 \\ u^2 & u^1 & 0 \\ 0 & 0 & u^3 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^3}, \quad a_0 = 2\varepsilon_1 u^1 + \varepsilon_3 u^3$$

The non vanishing Christoffel symbols Γ^i_{jk} (up to exchange of j with k) are

$$\Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2}, \, \Gamma^1_{13} = \Gamma^2_{23} = -\Gamma^1_{11} = -\Gamma^1_{33} = -\Gamma^2_{12} = \frac{\varepsilon_3}{u^1 - u^3}$$

$$\Gamma^3_{11} = \Gamma^3_{33} = -\Gamma^3_{13} = \frac{2\varepsilon_1}{u^1 - u^3}, \, \Gamma^2_{11} = \Gamma^2_{33} = -\Gamma^2_{13} = \frac{\varepsilon_3 u^2}{(u^1 - u^3)^2}$$

Work in progress

What happens for arbitrary solutions of the equation

$$d \cdot d_L\, a_0 = 0 \qquad ?$$

Conjectural answer: $a_0 \to (\nabla, \circ, e)$ where ∇ is still compatible with the product but the flatness condition is replaced by

$$Z \circ R(W, Y)(X) + W \circ R(Y, Z)(X) + Y \circ R(Z, W)(X) = 0.$$

Joint work with with **S. Perletti** and **K. van Gemst.**

PART III

Bi-flat F-manifolds, lifted Frölicher-Nijenhuis bicomplexes and Gauss-Manin connections.

Based on a joint work with **A. Arsie** (2024).

From bi-flat structures to a differential bicomplex

$$\begin{array}{ccc} & (M, \nabla, \circ, e, \nabla^*, *, E) & \\ \swarrow\!\!\nearrow & & \nwarrow\!\!\searrow \\ (d_\nabla, d_{E\circ\nabla^*}, \mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^\bullet_M) & \longleftrightarrow & \nabla_{GM} \end{array}$$

The bicomplex $(d_\nabla, d_{E\circ\nabla^*}, \mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^\bullet_M)$ can be thought as a "lift" of the Frölicher-Nijenhuis bicomplex on the space of vector valued forms.

Exterior covariant derivative

d_∇ acts on $\mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^k_M$ as follows

$$(d_\nabla \omega)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \nabla_{X_i}(\omega(X_0, \dots, \hat{X}_i, \dots, X_k)) +$$

$$+ \sum_{0 \leqslant i < j \leqslant k} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_k).$$

On an open ball the cohomology (for $k \geqslant 1$) of d_∇ is trivial this follows from the triviality of the usual de Rham cohomology).

It turns out $d^2_\nabla = 0$ due to the flatness of ∇.

L-exterior covariant derivative (A. Arsie, P.L 2012)

Given a flat connection ∇ and a $(1,1)$-tensor field with vanishing Nijenhuis torsion it is possible to define another differential on $\mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^k_M$. $d_{L\nabla}$ acts as follows

$$(d_{L\nabla}\omega)(X_0,\dots,X_k) = \sum_{i=0}^{k}(-1)^i \nabla_{LX_i}(\omega(X_0,\dots,\hat{X}_i,\dots,X_k))+$$

$$+\sum_{0\leqslant i<j\leqslant k}(-1)^{i+j}\omega([X_i,X_j]_L,X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k).$$

It turns out $d^2_{L\nabla}=0$ due to the flatness of ∇ and the vanishing of the Nijenhuis torsion of L.

A differential bicomplex (A. Arsie, P.L 2024)

Theorem

On any bi-flat F-manifold M, d_∇ and $d_{L\nabla^*}$ determine a differential bicomplex structure on $\mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega^\bullet_M$.

Due to the previous results it remains to prove that d_∇ and $d_{L\nabla^*}$ anticommute. We skip the proof since it is equivalent to the vanishing of the curvature of a family of connections we are going to introduce.

Lenard-Magri chains

Let $X_{(0)}$ be a local section of $\mathcal{T}_M$ for ∇ satisfying the equation

$$d_\nabla \cdot d_{L\nabla*} X_{(0)} = 0. \qquad (2)$$

Then we can define "higher" local sections of $\mathcal{T}_M$ by using the following recurrence relations

$$d_\nabla X_{(\alpha+1)} = d_{L\nabla*} X_{(\alpha)}, \quad \alpha \in \mathbb{N} \qquad (3)$$

Indeed, equation (2) tell us that $d_{L\nabla*} X_{(0)}$ is a d_∇-cocycle in degree 1. Therefore, there exists locally a vector field $X_{(1)}$ such that

$$d_\nabla X_{(1)} = d_{L\nabla*} X_{(0)}.$$

Since d_∇ and $d_{L\nabla*}$ anticommute we have

$$d_{L\nabla*} \cdot d_\nabla X_{(1)} = -d_\nabla \cdot d_{L\nabla*} X_{(0)} = 0$$

and we can repeat the previous argument.

Gauss-Manin connections

The Gauss-Manin connections associated with a bi-flat structure are

$$\nabla^{GM}_X Y = \nabla^*_X Y + \lambda(\nabla^*_{(E\circ-\lambda e\circ)^{-1}X} Y - \nabla_{(E\circ-\lambda e\circ)^{-1}X} Y).$$

Denoting by a^i_{jk} and b^i_{jk} the Christoffel symbols of the connections ∇ and ∇^* respectively, the Christoffel symbols are

$$\Gamma^j_{hk} := b^j_{hk} + \lambda((L_\lambda)^{-1})^s_h(b^j_{sk} - a^j_{sk}), \tag{4}$$

or

$$\Gamma^j_{hk} = b^j_{hk} + \lambda((L_\lambda)^{-1})^s_h c^l_{sk}\nabla^*_l e^j, \tag{5}$$

or

$$\Gamma^j_{hk} = b^j_{hk} - \lambda((L_\lambda)^{-1})^s_h c^{*l}_{sk}\nabla_l E^j, \tag{6}$$

Theorem

(A. Arsie, P.L 2024) The family of Gauss-Manin connections is flat and torsionless for any fixed λ on the open set where L_λ is invertible.

$$
\begin{aligned}
&(R^{GM})^j_{lik}(L_\lambda)^l_h(L_\lambda)^i_t = \\
&(R_{\nabla *})^j_{ilk}(L_\lambda)^l_h(L_\lambda)^i_t + \lambda^2((R_{\nabla *})^j_{htk} - (R_\nabla)^j_{htk}) \\
&+\lambda\left(L^l_h((R_{\nabla *})^j_{tlk} - (R_\nabla)^j_{tlk}) + L^l_t((R_{\nabla *})^j_{hlk} - (R_\nabla)^j_{hlk})\right) \\
&-\lambda(\nabla_t c^s_{hk} - \nabla_h c^s_{tk})\nabla_s E^j - \lambda(c^s_{hk}\nabla_t\nabla_s E^j - c^s_{tk}\nabla_h\nabla_s E^j) \\
&+\lambda(N_L)^m_{th}\left((L_\lambda)^{-1}\right)^s_m (L^{-1})^r_s c^t_{rk}\nabla_t E^j = 0.
\end{aligned}
$$

From this formula it follows that ∇^* can be replaced with $\nabla^* + \mu *$. In the case of Dubrovin-Frobenius manifolds with a suitable choice of μ one gets the Levi-Civita connection of g_λ.

Theorem

Let X be a local section $X \in \mathcal{T}_M$ that is flat with respect to ∇_{GM}. Then, viewing X is as an element of $\mathcal{T}_M[[\lambda^{-1}]]$, then the formal power series coefficients satisfy the Lenard-Magri recurrence relations (3)*.*

From $\nabla^{GM} X = 0$, using arbitrary sections $Y \in \mathcal{T}_M$, we have

$$\begin{aligned}\nabla^{GM}_{L_\lambda Y} X &= \nabla^*_{L_\lambda Y} X + \lambda(\nabla^*_Y X - \nabla_Y X) \\ &= \nabla^*_{LY} X - \lambda \nabla^*_Y X + \lambda(\nabla^*_Y X - \nabla_Y X) \\ &= \nabla^*_{LY} X - \lambda \nabla_Y X = 0.\end{aligned}$$

Substituting

$$X = X_{(0)} + \frac{X_{(1)}}{\lambda} + \frac{X_{(2)}}{\lambda^2} + \cdots$$

we get Lenard-Magri recurrence relations (3).

Extended Gauss-Manin system: dependence on λ

Proposition

Let $(t^1, ..., t^n)$ be flat coordinates of ∇ such that $e = \frac{\partial}{\partial t^1}$. In such coordinates, the solutions of the system $\nabla^{GM}\theta = 0$ have the form

$$\theta = \theta(t^1 - \lambda, t^2, ..., t^n).$$

Moreover

$$(L - \lambda I)^j_i \partial_\lambda \theta_j = -L^l_i b^j_{l1} \theta_j.$$

Bibliography

1. P. Lorenzoni and F. Magri, *A cohomological construction of integrable hierarchies of hydrodynamic type*, IMRN 34 (2005).

2. P. Lorenzoni and S. Perletti, *Integrable systems, Frölicher-Nijenhuis bicomplexes and Lauricella bi-flat structures*, Nonlinearity 36 (2023).

3. A, Arsie and P. Lorenzoni, *Bi-flat F-structures as differential bicomplexes and Gauss-Manin connections*, arXiv:2405.12649 (2024).