Biorthogonal measures, polymer partition functions, and random matrices

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Integrable Systems: Geometrical and Analytical Approaches
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Plan of the seminar

- A motivating example : KPZ & the Airy_2 process
- A relevant class of integral kernels
- Applications to random matrix theory
- Multiplicative statistics
- Applications to polymers and their small temperature limit

"Biorthogonal measures, polymer partition functions, and random matrices."
with T. Claeys arXiv : 2401.10130.

KPZ and the Airy$_2$ point process

$$\mathbb{E}_{\mathrm{KPZ}}\left[\mathrm{e}^{-\mathrm{e}^{\mathcal{H}(2T,X)-\frac{X^2}{4T}+\frac{T}{12}+sT^{1/3}}}\right]=\mathbb{E}_{\mathrm{Ai}_2}\left[\prod_{j\geq 1}\left(1-\sigma\left(T^{1/3}(\zeta_j+s)\right)\right)\right]$$

Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn, (2010).

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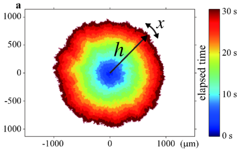
$\mathcal{H}(T,X)$ narrow wedge solution of the KPZ equation

$$\frac{\partial}{\partial T}\mathcal{H}(T,X) = \frac{1}{2}\frac{\partial^2}{\partial X^2}\mathcal{H}(T,X)+\frac{1}{2}\left(\frac{\partial}{\partial X}\mathcal{H}(T,X)\right)^2+\xi(T,X)$$

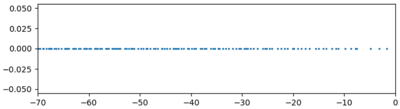


Takeuchi, Sano, Sasamoto, Spohn

$\{\zeta_j\}_{j\geq 1}$ realization of the Airy$_2$ determinantal point process with kernel

$$K(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(y+z)\mathrm{d}z,$$

$$\sigma(z) = \frac{1}{1+\mathrm{e}^{-z}}$$



Rescaled GUE process, $N = 10^4$

Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn, (2010).

Determinants and integrability

$$Q(s,T) := \mathbb{E}_{\mathrm{Ai}_2}\left[\prod_{j\geq 1}\left(1-\sigma\left(T^{1/3}(\zeta_j+s)\right)\right)\right]$$

is a Fredholm determinant, it can be studied through the lens of *integrability*.

$$U(x,t) := \partial_x^2 \log Q(x,t) + \frac{x}{2t}, \quad x := sT^{-1/6},\ t := T^{-1/2}$$

is a solution of the *KdV equation*

$$U_t + 2UU_x + \frac{1}{6}U_{xxx} = 0,$$

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$$U_t + 2UU_x + \frac{1}{6}U_{xxx} = 0,$$

As such, it is amenable to asymptotics analysis, yielding information about *large deviations* of $Q(s,T)$.

$$\log Q(s,T) = -T^2\phi(sT^{-2/3}) - \frac{1}{6}\sqrt{1+\frac{\pi^2 s}{T^{2/3}}} + \mathcal{O}(\log^2 s) + \mathcal{O}(T^{1/3}),\ s\to+\infty$$

$$\phi(y) := \frac{4}{15\pi^6}(1+\pi^2 y)^{5/2} - \frac{4}{15\pi^6} - \frac{2}{3\pi^4}y - \frac{1}{2\pi^2}y^2.$$

$$M^{-1} \leq T \leq Ms^{3/2},$$

C.-Claeys, C.-Claeys-Ruzza, (2021).

Other instances (incomplete list)

Similar formulas relating stochastic models to determinantal point process are available for :

- O'Connel-Yor directected random polymer (relation to a signed bi-orthogonal measure) (Imamura-Sasamoto, 2016)
- Asymmetric Exclusion Process (relation to the discrete Laguerre Ensemble) (Borodin-Olshanski, 2017)
- Stochastic higher spin six vertex model (relation to MacDonald measure) (Borodin, 2018)
- Deformed PNG model and Poissonized Plancherel measure (Aggarwal-Borodin-Wheeler, 2018, Imamura-Mucciconi-Sasamoto, 2022)
- . . .

Our aim

Single out and study a large class of (signed) biorthogonal measures, and apply them to the study of random matrices, polymer models and their relations.

$$\frac{1}{Z_N} \det \left(f_m(x_k)\right)_{k,m=1}^N \det \left(g_m(x_k)\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k.$$

For a first application to the Log-Gamma polymer and its large deviations, see Tom's talk !

A class of integral kernels

$$L_N(x,x') := \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_N(v)}{W_N(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u}$$

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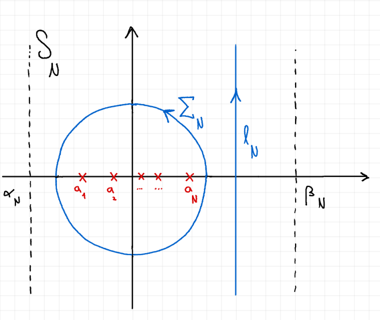

- W_N analytical for $\alpha_N < \Re z < \beta_N$
- $W_N(z) = \mathcal{O}(|z|^{-\epsilon}), \ |z| \to \infty$
- $a_1, \ldots, a_N$ zeros of W.

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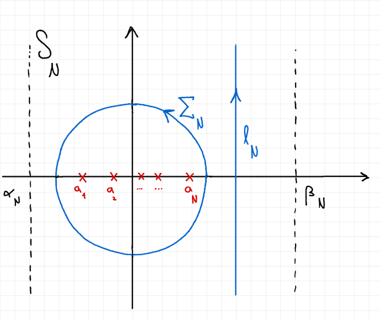

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$$\mathrm{d}\mu_N(\vec{x}) \equiv \mathrm{d}\mu_N(x_1,\dots,x_N) := \frac{1}{N!} \det\left(L_N(x_m,x_k)\right)_{m,k=1}^{N} \prod_{k=1}^{N} \mathrm{d}x_k,$$

Bi-orthogonal structure of $\mathrm{d}\mu_N$

Theorem (M.C., T. Claeys)

1. When $a_1, \cdots, a_N$ are all distinct,

$$L_N(x,x') = \sum_{m=1}^{N} \mathrm{e}^{a_m x'} \psi_m(\mathrm{e}^x), \quad \textit{where} \quad \psi_m(y) := \frac{1}{2\pi i W_N'(a_m)} \int_{\ell_N} \frac{W_N(y) y^{-v}}{v - a_m} \mathrm{d}v,$$

$$\int_{\mathbb{R}} \mathrm{e}^{a_m x} \psi_k(\mathrm{e}^x) \mathrm{d}x = \delta_{k,m}, \quad k,m = 1,\dots,N.$$

$\mathrm{d}\mu_N$ *is then given explicitily by*

$$\mathrm{d}\mu_N(\vec{x}) = \frac{1}{N!} \det\left(\mathrm{e}^{a_m x_k}\right)_{k,m=1}^{N} \det\left(\psi_m(\mathrm{e}^{x_k})\right)_{m,k=1}^{N}$$

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2. *In the completely confluent case* $a_1 = a_2 = \ldots = a_N \equiv a$*,*

$$\mathrm{d}\mu_N(\vec{x}) = \frac{1}{Z_N} \Delta(\vec{x}) \det\left(\phi_m(\mathrm{e}^{x_k})\right)_{m,k=1}^{N} \prod_{k=1}^{N} \mathrm{d}x_k$$

$$\textit{where} \quad \phi_m(y) := \frac{y^a}{2\pi \mathrm{i}} \int_{\ell_N} \frac{W_N(v) y^{-v}}{(v-a)^{N-m+1}}$$

Comments on the proof

Step 1 :

$$L(x,x') := \frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v\, \frac{W_N(v)}{W_N(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u} = \sum_{m=1}^{N} \mathrm{e}^{a_m x'} \psi_m(\mathrm{e}^x)$$

simply using the residue theorem.

Recall that $a_1, \cdots, a_m$ are the zeros of W inside Σ_N, and that

$$\psi_m(y) = \frac{1}{2\pi i W_N'(a_m)} \int_{\ell_N} \frac{W_N(y) y^{-v}}{v - a_m} \mathrm{d}v$$

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simply using the residue theorem.

Step 2 :

Using inverse and direct Mellin transform

$$\mathcal{M}[g](v) := \int_0^\infty y^{v-1} g(y)\mathrm{d}y, \qquad \mathcal{M}^{-1}[G](y) := \frac{1}{2\pi \mathrm{i}}\int_\ell G(v) y^{-v} \mathrm{d}v$$

you find out that ψ_m is the inverse Mellin transform of $\dfrac{W_N(v)}{W_N'(a_m)(v-a_m)}$.

Consequently,

$$\int_{-\infty}^{\infty} \mathrm{e}^{a_k x}\psi_m(\mathrm{e}^x)\mathrm{d}x = \int_0^\infty s^{a_k-1}\psi_m(s)\mathrm{d}s = \frac{1}{W'(a_k)}(\mathcal{M}\circ\mathcal{M}^{-1})\left[\frac{W_N(\cdot)}{\cdot - a_m}\right](a_k) = \delta_{k,m}.$$

Comments on the proof

Step 1 :

$$L(x,x') := \frac{1}{(2\pi i)^2}\int_{\Sigma_N} du \int_{\ell_N} dv \, \frac{W_N(v)}{W_N(u)} \frac{e^{-vx+ux'}}{v-u} = \sum_{m=1}^{N} e^{a_m x'} \psi_m(e^x)$$

simply using the residue theorem.

Step 2 :

$$\int_{-\infty}^{\infty} e^{a_k x}\psi_m(e^x) dx = \int_0^{\infty} s^{a_k-1}\psi_m(s) ds = \frac{1}{W'(a_k)} (\mathcal{M}\circ\mathcal{M}^{-1}) \left[\frac{W_N(\cdot)}{\cdot - a_m}\right](a_k) = \delta_{k,m}.$$

Step 3 :

The rest of the proof exploits the bi-orthogonality condition above. In particular, the kernel L_N is self-reproducing and

$$\int_{\mathbb{R}} L_N(x,x) dx = N.$$

LUE with external potential

$N \times N$ positive-definite Hermitian matrix with probability distribution

$$\frac{1}{Z_N}(\det M)^{\nu} \mathrm{e}^{-\mathrm{Tr}((I-B)M)} \mathrm{d}M, \quad \nu \geq 0, \quad B = \mathrm{diag}(b_1, \ldots, b_N).$$

$M = \frac{1}{N} HH^*$, with H a $N \times (N + \nu)$ matrix whose columns are Gaussian distributed with covariance matrix $(I - B)^{-1}$.

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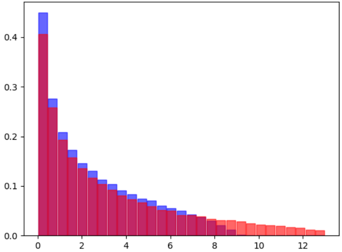
The eigenvalues of M realize a determinantal point process with kernel

$$L_N(x,x') = \frac{1}{(2\pi\mathrm{i})^2}\int_{\Sigma_N}\mathrm{d}u\int_{\ell_N}\mathrm{d}v\,\frac{W_N^{\mathrm{LUE}+}(v)}{W_N^{\mathrm{LUE}+}(u)}\frac{\mathrm{e}^{-vx+ux'}}{v-u}, \quad W_N^{\mathrm{LUE}+}(v) := \frac{\prod_{j=1}^N(z-b_j)}{(z-1)^{N+\nu}}$$

(Baik-Ben Arous-Péché, '04)



Distribution of the eigenvalues of a LUE matrix of size 2×10^3 without (blue) and with (orange) external potential.

GUE with external potential

$N \times N$ Hermitian matrix with probability distribution

$$\frac{1}{Z_N} \mathrm{e}^{-\mathrm{Tr}\left(\frac{M^2}{2\tau} - AM\right)} \mathrm{d}M, \quad A = \mathrm{diag}(a_1, \ldots, a_N).$$

$M = \frac{1}{2}(H + H^*) + A$, with H a $N \times N$ matrix whose columns are Gaussian distributed.

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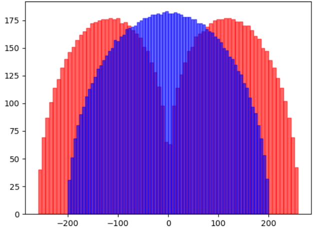
$$L_N(x, x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_N^{\mathrm{GUE}+}(v)}{W_N^{\mathrm{GUE}+}(u)} \frac{e^{-vx+ux'}}{v-u}, \quad W_N^{\mathrm{GUE}+}(v) := \prod_{j=1}^{N} (z - a_j) e^{\tau z^2/2}$$

(Brezin-Hikami, '97)



Distribution of the eigenvalues of a LUE matrix of size 10^4 without (blue) and with (orange) external potential.

LUE with external potential plus GUE

We consider M_1 a LUE matrix with external potential $B = \mathrm{diag}(b_1, \dots, b_N)$ and $\nu = 0$, M_2 a GUE matrix ; and

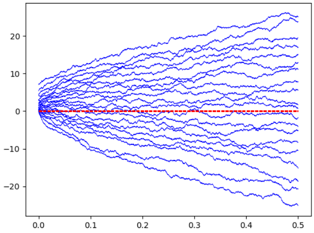
$$Q := M_1 + \sqrt{\tau N} M_2.$$

Proposition (M.C., T. Claeys)

The eigenvalues of Q realize a determinantal point process with kernel

$$L_N(x, x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_N^{\mathrm{GLUE}+}(v)}{W_N^{\mathrm{GLUE}+}(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u}, \qquad W_N^{\mathrm{GLUE}+}(v) := \frac{z^N \mathrm{e}^{\tau z^2/2}}{\prod_{k=1}^N (z-1+b_k)}.$$

This is Dyson diffusion applied to the eigenvalues of the Laguerre Unitary Ensemble with external potential !



Multiplicative statistics associated to $\mathrm{d}\mu$

Given a function $\sigma : \mathbb{R} \longrightarrow \mathbb{R}$, we now define

$$\mu_N[\sigma] := \int_{\mathbb{R}^N} \prod_{k=1}^{N} (1 - \sigma(x_k))\, \mathrm{d}\mu(x_1, \ldots, x_N).$$

All the examples will be related to the function

$$\sigma(x) = \sigma_t(x) = \frac{1}{1 + \mathrm{e}^{-x-t}}$$

If $\mathrm{d}\mu_N$ is positive-valued, i.e. associated to a point process with realization $\{\zeta_1, \ldots, \zeta_N\}$, then

$$\mu_N[\sigma] = \mathbb{E}\left[\prod_{k=1}^{N} (1 - \sigma(\zeta_k))\right].$$

Three different Fredholm identities

$$L_N(x,x') = \frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma_N}\mathrm{d}u\int_{\ell_N}\mathrm{d}v\,\frac{W_N(v)}{W_N(u)}\frac{\mathrm{e}^{-vx+ux'}}{v-u},\quad \mathrm{d}\mu_N(\vec{x}) = \frac{1}{N!}\det\left(L_N(x_m,x_k)\right)_{m,k=1}^N\prod_{k=1}^N \mathrm{d}x_k,$$

$$\mu_N[\sigma] = \int_{\mathbb{R}^N}\prod_{k=1}^N(1-\sigma(x_k))\mathrm{d}\mu(x_1,\dots,x_N).$$

We suppose that

- $\sigma = \sigma_t$
- $W_N(v) = \mathcal{O}(|v|^{-1-\epsilon}),\ v\to\infty, v\in\mathcal{S}_N$
- $a_{\max} - a_{\min} < 1$

Three different Fredholm identities

Theorem (M.C., Tom Claeys)

Under the hypothesis above, we can write $\mu_N[\sigma]$ as Fredholm determinant in three different ways :

1) $\mu_N[\sigma] = \det(1 - \sigma L_N)_{L^2(\mathbb{R})}$
2) $\mu_N[\sigma] = \det(1 - H_N^\sigma)_{L^2(0,\infty)}$ (*"finite temperature kernel"*), *where*

$$H_N^\sigma(y, y') := \int_{\mathbb{R}} \sigma(x) \Psi_1(\mathrm{e}^{y+x}) \Psi_2(\mathrm{e}^{y'+x}) \mathrm{d}x, \ \ y, y' > 0,$$

$$\textit{with} \quad \Psi_1(s) := \int_{\Sigma_N} \frac{s^u \mathrm{d}u}{W_N(u)} \quad \textit{and} \quad \Psi_2(s) = \mathcal{M}^{-1}[W_N](s)$$

3) $\mu_N[\sigma] = \det(1 - K_N)_{L^2(\Sigma_N)}$

$$\textit{with} \quad K_N(u, u') = \frac{1}{2\pi \mathrm{i}} \int_{\ell_N} \frac{\pi \mathrm{e}^{t(v-u)}}{\sin \pi(u-v)} \frac{W_N(v)}{W_N(u)} \frac{\mathrm{d}v}{v - u'}$$

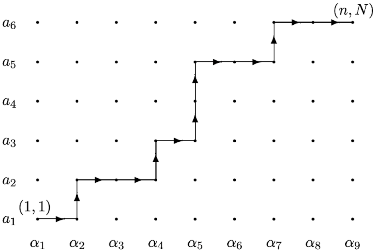
Connecting to polymers

General strategy :

We use the Fredholm determinant representation of polymer partition functions, existing in the literature, to connect to one of the three representations shown above.

The Log Gamma polymer (a random walk in a random environment)

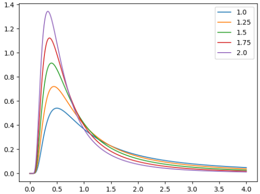
Seppäläinen, (2012)



$$n, N > 0,$$

$$(\alpha_j)_{j=1}^{n},\ (a_k)_{k=1}^{N} \quad \text{s.t.} \quad \alpha_j - a_k > 0$$

$$\mathbb{P}(d_{j,k} \le y) = \frac{1}{\Gamma(\alpha_j - a_k)} \int_0^y x^{-\alpha_j + a_k - 1} \mathrm{e}^{-1/x} \mathrm{d}x$$



$$d_{j,k} \sim \Gamma^{-1}(\alpha_j - a_k)$$

$$Z_{n,N}^{\mathsf{Log}\Gamma}(\vec{\alpha}, \vec{a}) := \sum_{\pi:(1,1)\nearrow(n,N)} \prod_{(j,k)\in\pi} d_{j,k}$$

The biorthogonal structure of the Log Gamma polymer

Corollary (M.C., T. Claeys, using Borodin-Corwin-Remenik, (2013))

Take

$$L_N(x,x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_{n,N}^{\mathrm{Log}\Gamma}(v)}{W_{n,N}^{\mathrm{Log}\Gamma}(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u}, \quad W_{n,N}^{\mathrm{Log}\Gamma}(z) := \frac{\prod_{j=1}^{n} \Gamma(\alpha_j - z)}{\prod_{k=1}^{N} \Gamma(z - a_k)}.$$

Then

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^t Z_{n,N}^{\mathrm{Log}\Gamma}(\vec{\alpha},\vec{a})}\right] = \mu_N[\sigma_t], \quad \sigma_t(x) = \frac{1}{1+\mathrm{e}^{-x-t}}.$$

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Then

$$\mathbb{E}\left[e^{-e^t Z_{n,N}^{\mathrm{Log}\Gamma}(\vec{\alpha},\vec{a})}\right] = \mu_N[\sigma_t], \quad \sigma_t(x) = \frac{1}{1+e^{-x-t}}.$$

The biorthogonal structure of the measure is given explicitly by

$$d\mu_{n,N}^{\mathrm{Log}\Gamma}(\vec{x};\vec{\alpha},\vec{a}) = \frac{1}{Z_N}\det\left(e^{a_m x_k}\right)_{k,m=1}^N \det\left(G_{0,n+N}^{n,0}\left(\begin{matrix} - \\ \vec{\alpha};[1]_m+\vec{a}-\vec{e}_m\end{matrix}\,\middle|\, e^{-x_k}\right)\right)_{m,k=1}^N \prod_{k=1}^N dx_k.$$

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$$\mathrm{d}\mu_{n,N}^{\mathrm{Log}\Gamma}(\vec{x};\vec{\alpha},\vec{a}) = \frac{1}{Z_N}\det\left(\mathrm{e}^{a_m x_k}\right)_{k,m=1}^N \det\left(G_{0,n+N}^{n,0}\left(\begin{matrix} - \\ \vec{\alpha};[1]_m+\vec{a}-\vec{e}_m\end{matrix}\,\middle|\, \mathrm{e}^{-x_k}\right)\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k.$$

In the confluent case :

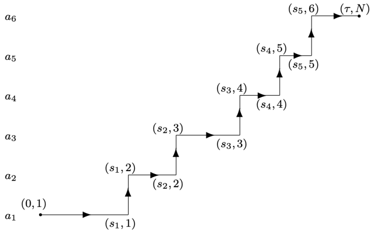
$$\mathrm{d}\mu_{n,N}^{\mathrm{Log}\Gamma}(\vec{x};\vec{\alpha},\vec{0}) = \frac{1}{Z_N}\Delta(\vec{x})\det\left(G_{0,n+N}^{n,0}\left(\begin{matrix} - \\ \vec{\alpha};[0]_{N-m+1};[1]_{m-1}\end{matrix}\,\middle|\, \mathrm{e}^{-x_k}\right)\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k.$$

Reminder, Meijer G-Functions

$$G^{m,n}_{p,q}\left(\begin{smallmatrix} a_1,\dots,a_p \\ b_1,\dots,b_q \end{smallmatrix}\,\middle|\, z\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{\ell=1}^{m}\Gamma(b_\ell - s)\prod_{\ell=1}^{n}\Gamma(1-a_\ell+s)}{\prod_{\ell=m}^{q-1}\Gamma(1-b_{\ell+1}+s)\prod_{\ell=n}^{p-1}\Gamma(a_{\ell+1}-s)} z^s \mathrm{d}s$$

The O'Connel-Yor polymer

O'Connel-Yor, (2001)



$\tau \in \mathbb{R}_{>0}, N > 0,$

$(a_k)_{k=1}^N$ real parameters

$\{B_i\}$ one-dimensional Brownian motion with drift $\{a_i\}$

$$E(\phi) = \sum_{k=1}^{N} B_i(s_i) - B_i(s_{i-1})$$

$$Z_N^{\mathrm{OY}}(\vec{a}, \tau) := \int_{0<s_1<\ldots<s_{N-1}<\tau} \mathrm{e}^{E(\phi)} \mathrm{d}s_1 \cdots \mathrm{d}s_{N-1}$$

The biorthogonal structure of the O'Connel-Yor polymer

Corollary (Imamura - Sasamoto, M.C. - T. Claeys, using Borodin-Corwin (2014))

Take

$$L_N(x,x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_N^{\mathrm{OY}}(v)}{W_N^{\mathrm{OY}}(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u}, \qquad W_N^{\mathrm{OY}}(z) := \frac{\mathrm{e}^{\frac{\tau z^2}{2}}}{\prod_{k=1}^{N} \Gamma(z-a_k)}.$$

Then

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^t Z_N^{\mathrm{OY}}(\vec{a},\tau)}\right] = \mu_N[\sigma_t], \quad \sigma_t(x) = \frac{1}{1+\mathrm{e}^{-x-t}}.$$

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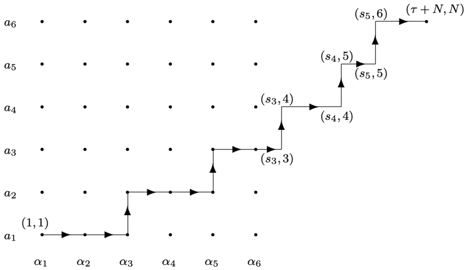
The biorthogonal structure is given explicitily in terms of contour integrals of exponential and Gamma functions. The case $a=0$ was already found by Imamura and Sasamoto.

The mixed polymer

Borodin-Corwin-Ferrari-Vető, (2015)



$$n, N \in \mathbb{Z}_{>0}, \quad \tau \in \mathbb{R}_{>0}, \quad \{\alpha_1, \dots, \alpha_n\}, \quad \{a_1, \dots, a_N\} \quad \text{s.t.} \quad \alpha_j - a_k > 0.$$

$$Z_{n,N}^{\mathrm{Mixed}}(\vec{\alpha}, \vec{a}, \tau) = \sum_{k=1}^{N} Z_{k,N}^{\mathrm{Log}\Gamma}(\vec{\alpha}, a_1, \dots, a_k) Z_{N-k}^{\mathrm{OY}}(a_{k+1}, \dots, a_N, \tau).$$

The biorthogonal structure of the mixed polymer

Corollary (M.C. - T. Claeys, using Imamura-Sasamoto (2017))

Take

$$L_N(x,x') = \frac{1}{(2\pi \mathrm{i})^2} \int_{\Sigma_N} \mathrm{d}u \int_{\ell_N} \mathrm{d}v \, \frac{W_N^{\mathrm{Mixed}}(v)}{W_N^{\mathrm{Mixed}}(u)} \frac{\mathrm{e}^{-vx+ux'}}{v-u},$$

$$W_N^{\mathrm{Mixed}}(z) := \frac{\mathrm{e}^{\frac{\tau z^2}{2}} \prod_{k=1}^{N} \Gamma(\alpha_j - z)}{\prod_{k=1}^{N} \Gamma(z - a_k)}.$$

Then

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^t Z_N^{\mathrm{Mixed}}(\vec{\alpha},\vec{a},\tau)}\right] = \mu_N[\sigma_t], \quad \sigma_t(x) = \frac{1}{1+\mathrm{e}^{-x-t}}.$$

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Small temperature polymers and RMT

Tuning appropriately their parameters with a rescaling by $T \to 0$, the partition functions of polymers are described by last particle distribution of RMT :

Corollary (M.C., T. Claeys)

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^{t/T} Z_{n,N}^{\mathrm{Log}\Gamma}([T]_n, T\vec{b})}\right] \longrightarrow \mathbb{P}_{\mathrm{LUE}+}\left(\max\{x_1,\dots,x_N\} \leq -t; \vec{b}, \nu = n-N\right)$$

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^{t/T} Z_{n,N}^{\mathrm{OY}}(T\vec{a}, \tau/T^2)}\right] \longrightarrow \mathbb{P}_{\mathrm{GUE}+}\left(\max\{x_1,\dots,x_N\} \leq -t; \vec{a}, \tau\right)$$

$$\mathbb{E}\left[\mathrm{e}^{-\mathrm{e}^{t/T} Z_{N,N}^{\mathrm{Mixed}}([T]_N - T\vec{b}, [0]_N, \tau/T^2)}\right] \longrightarrow \mathbb{P}_{\mathrm{GLUE}+}\left(\max\{x_1,\dots,x_N\} \leq -t; \vec{b}, \tau\right).$$

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Remark

The first two limits are known in the literature (last passage percolation models). For the mixed polymer, the appearance of LUE with external source plus GUE seems new.

Sketch of the proof (Log Gamma polymer)

$$W^{\mathrm{Log}\Gamma}_{n,N}(T\zeta;[T]_n,T\vec{b}) = \frac{\Gamma(T(1-\zeta))^n}{\prod_{m=1}^N \Gamma(T(\zeta-b_m))} = T^{N-n}\frac{\prod_{m=1}^N(\zeta-b_m)}{(1-\zeta)^n}\,\frac{\Gamma(1+T(1-\zeta))^n}{\prod_{m=1}^N\Gamma(1+T(\zeta-b_m))}$$

$$\Downarrow$$

$$\lim_{T\to 0}\frac{1}{T}L^{\mathrm{Log}\Gamma}_{n,N}\left(\frac{x}{T},\frac{x'}{T};[T]_n,T\vec{b}\right) = L^{\mathrm{LUE}+}_N(x,x';\vec{b},\nu=n-N)$$

$$\Downarrow$$

$$\det\left(\frac{\sigma_{t/T}(x_j/T)}{T}L^{\mathrm{Log}\Gamma}_{n,N}\left(\frac{x_j}{T},\frac{x_k}{T};[T]_n,T\vec{b}\right)\right)_{j,k=1}^N \to \det\left(1_{(-t,+\infty)}L^{\mathrm{LUE}+}_N\left(x_j,x_k;\vec{b},n-N\right)\right)_{j,k=1}^N.$$

and then you integrate over $\mathbb{R}^N$, using dominated convergence.

Conclusions

- Polymers and random matrices (with external sources) share a common *biorthogonal structure*, which can be used to compare them more easily.
- It is also useful, combined with marking and conditioning of point processes [Thm 1.10 and (Claeys-Glesner, 2023)], to study asymptotics (Tom's talk).

What's next

- Use the three different Fredholm determinant representation to give a RH approach to the study of the polymers.
- Connect them to inverse scattering theory in integrable PDEs.

Thanks !