# Affine Gaudin Bethe Ansatz and the ODE/IM correspondence

Davide Masoero
Grupo de Física Matemática da Universidade de Lisboa
With the support of the FCT Project UIDB/00208/2020

Integrable Systems: Geometrical and Analytical Approaches

SISSA, 3-7 July 2024

Fundação para a Ciência e a Tecnologia

Grupo de Física Matemática da Universidade de Lisboa

# References

Recent papers and collaborators:


- D.M., E. Mukhin and A. Raimondo *Q-functions for lambda opers*. arXiv 2023
- G. Cotti, D. Guzzetti and D. M. *Asymptotic solutions for linear ODEs with not-necessarily meromorphic coefficients: a Levinson type theorem on complex domains, and applications*. arXiv 2023
- R. Conti and D.M., *On solutions of the Bethe Ansatz for the Quantum KdV model*. Commm. Math. Phys, 2023
- R. Conti and D.M., *Counting Monster Potentials*. JHEP 2021
- D.M. and Andrea Raimondo, *Opers for higher states of quantum KdV models*, Commm. Math. Phys, 2020.


+ ongoing work with G. Degano, G. Ruzza and previous work with D. Valeri.

# Plan of the talk

- Introduction on the ODE/IM correspondence
- $Q$-functions from Affine Gaudin Bethe Equations
- Conjectures and perspectives.

It is a general fact that many interesting objects in mathematical physics can be expressed via the (generalised) monodromy data of a of linear ODE in the complex plane.

Quantum field theories & related geometrical objects often correspond to equations with irregular singularities (e.g. Frobenius manifolds and TQFT, ODE/IM correspondence).
This correspondence is somehow mediated by the Nonlinear Stokes Phenomenon (Wall-Crossing).

# Monster potentials

We consider the linear ODE

$$-\Psi''(x) + \left(x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - 2\frac{d^2}{dx^2}\log P(x^{2\alpha+2}) - E\right)\Psi(x) = 0.$$

- $\alpha > 0, x \in \widetilde{\mathbb{C}^*}, E, \ell \in \mathbb{C}$.
- $P$ is a polynomial such that i) $P(0) \neq 0$, and ii) the monodromy is trivial about the singularities $x \neq 0, \infty$.

Assuming roots $z_1, \dots, z_N$ of $P$ are distinct: the trivial monodromy is equivalent to the BLZ system

$$\sum_{j\neq k} \frac{z_k\left(z_k^2+(3+\alpha)(1+2\alpha)z_k z_j+\alpha(1+2\alpha)z_j^2\right)}{(z_k-z_j)^3} - \frac{\alpha z_k}{4(1+\alpha)} + \Delta(\ell,\alpha) = 0,\ k=1,\dots,N.$$

Dorey-Tateo (1998) $\ell = 0, N = 0$,
Bazhanov-Lukyanov-Zamolodchikov (1998 and 2003).

As a (entire) function of $E$, the central connection coefficients $Q_{\pm}(E)$ and the Stokes multiplier $T(E)$ satisfy the Quantum Wronskian, TQ relations and Bethe Equation of Quantum KdV:

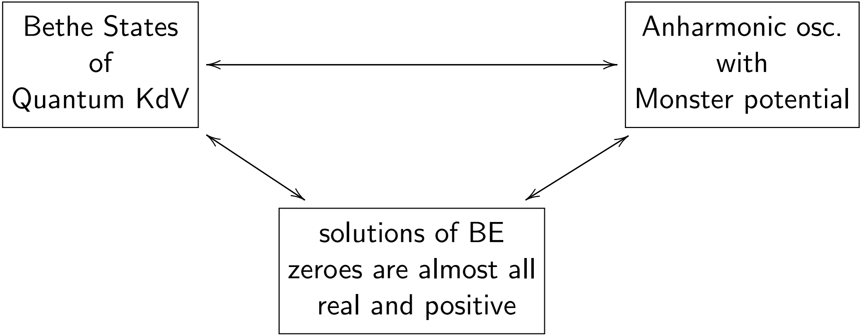
$$\gamma\, Q_{+}(qE)\, Q_{-}(q^{-1}E) - \gamma^{-1}\, Q_{+}(q^{-1}E)\, Q_{-}(qE) = 1,$$

$$T(E)Q_{\pm}(E) = \gamma^{\pm 1} Q_{\pm}(q^{2}E) + \gamma^{\mp 1} Q_{\pm}(q^{-2}E)$$

$$\gamma^{-2}\frac{Q_{+}\left(E^{*}\, q^{-2}\right)}{Q_{+}\left(E^{*}\, q^{2}\right)} = -1,\ \forall E^{*} \text{ s.t. } Q_{+}(E^{*}) = 0.$$

Here $\gamma, q$ depends on the paramters $\ell, \alpha$.

# The ODE/IM Conjecture for Quantum KdV



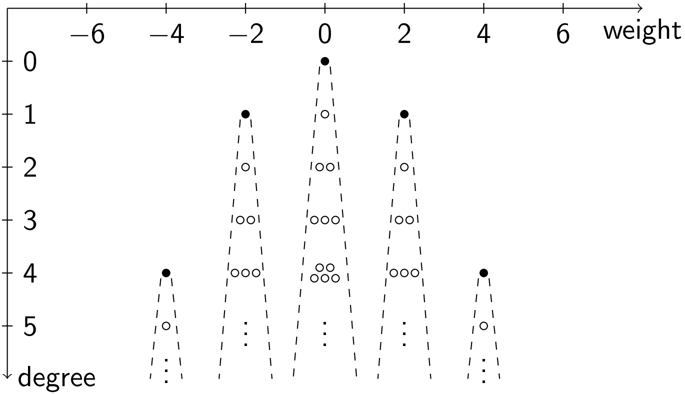

Remark: 1. Quantum KdV (Bazhanov-Lukyanov-Zamolodchikov '94-'96) is a CFT with $c = 1 - \frac{6\alpha^2}{\alpha+1}, \Delta = \frac{(2\ell+1)^2 - 4\alpha^2(\alpha+1)}{4\alpha+4}$.
2. Substantial progress on the bijection monster potentials/solution of Bethe Equations in Conti, M. (2021,2023).

# Affine Gaudin Model Bethe Ansatz Equation

- Quantum KdV should be related to the (hypothetical) Affine Gaudin Model (a Quantum Integrbale Model).
- The algebra is $\widehat{\mathfrak{sl}}_2$ and the module $L_{2m,2l} \otimes L_{1,0}$.
- The Hamiltonians should act on the space of singular vectors of the module.

Feigin-Frenkel 2011, Feigin-Jimbo-Miwa-Mukhin 2017, Lacroix-Vicedo-Young 2020, Kotousov-Lukyanov 2021.

# The $\widehat{\mathfrak{sl}}_2$ module $L_{2m,2l} \otimes L_{1,0}$: Sugawara Construction




After Sugawara, $\left(L_{2m,2l} \otimes L_{1,0}\right)_{sing} \cong \oplus_{r\in\mathbb{Z}} M_{c,\Delta_r}$ as a Virasoro module, with $c = 1 - \frac{6}{(k+2)(k+3)}$, $\Delta_r = \frac{(l-2r-kr)(l+1-2r-kr)}{(k+2)(k+3)}$.

# Affine Gaudin Model Bethe Ansatz Equation

The BAE for the $\widehat{\mathfrak{sl}}_2$-module $L_{2m,2l} \otimes L_{1,0}$ is an algebraic system on variables $\{s_i\}_{i=1}^{d_0}$, $\{t_j\}_{j=1}^{d_1}$:

$$\frac{1/2}{s_i - 1} + \frac{k/2 - l}{s_i} + \sum_{j=1}^{d_1} \frac{1}{s_i - t_j} - \sum_{j=1, j \neq i}^{d_0} \frac{1}{s_i - s_j} = 0,$$

$$\frac{l}{t_i} + \sum_{j=1}^{d_0} \frac{1}{t_i - s_j} - \sum_{j=1, j \neq i}^{d_1} \frac{1}{t_i - t_j} = 0.$$

Here $l \in \mathbb{C}$ and $k := 2m + 2l > -2$.
The integer $r$ (sector) in Sugawara should be $d_1 - d_0$.

Given a solution to the BAE one can cosntruct two affine opers $L_0$ and $L_1$ with apparent singularities at $\{t_j\}_{j=1}^{d_1}$ and $\{s_i\}_{i=1}^{d_0}$.
In particular,

$$L_1 = -\partial_x^2 + \left( x^k (x-1)\lambda^2 + \frac{l(l+1)}{x^2} + \sum_{i=1}^{d_0} \frac{2}{(x-s_i)^2} + \sum_{i=1}^{d_0} \frac{k + s_i/(s_i-1)}{x(x-s_i)} \right).$$

Here $(x,\lambda) \in \widetilde{\mathbb{C}^*} \times \widetilde{\mathbb{C}^*}, l \in \mathbb{C}, k > -2$.

Above cited works + Gaiotto-Lee-Vicedo-Wu 2022.

We are interested in solution analytic in $x, \lambda$:
$\mathcal{A} = \{\psi : (x,\lambda) \in \widetilde{\mathbb{C}^*} \times \widetilde{\mathbb{C}^*} \to \mathbb{C}, L_1\psi(x,\lambda) = 0\}$.

1. Frobenius solutions: for generic $l$, $\exists! \chi_\pm \in \mathcal{A}$,

$$\chi_\pm(x,\lambda) = x^{\frac{1\pm(2l+1)}{2}} \left( \sum_{j=0}^{\infty} x^j g_j(x^{k+2}\lambda) \right), \; g_j : \mathbb{C} \to \mathbb{C}, g_0(0) = 1.$$

They are eigensolution of the (Quantum) Monodromy:

$$M\chi_\pm(e^{2\pi i}x, e^{-\pi(k+2)i}\lambda) = -e^{\pm\frac{2l+1}{2}\pi i}\chi_\pm(x,\lambda).$$

M-Mukhin-Raimondo (2023).

Stokes sectors

$$\Sigma_j[\lambda] = \left\{ x \in \widetilde{\mathbb{C}^*} : \left| \arg \lambda + \frac{k+3}{2} \arg x - \pi j \right| \leq \frac{\pi}{2} \right\}, \, j \in \mathbb{Z}.$$

For every $j \in \mathbb{Z}$, $\exists!$ Sibuya solution $\psi^{(j)} \in \mathcal{A}$ such that $\forall \lambda \in \widetilde{\mathbb{C}^*}$

$$\lim_{x \to \infty} \psi^{(j)}(x, \lambda) \, / \, \left( x^{-\frac{k+1}{4}} e^{(-1)^{j+1} \frac{2\lambda}{k+3} x^{\frac{k+3}{2}} + \dots} \right) = 1 \, ,$$

in any proper subsector of the sector $\Sigma_{j-1}[\lambda] \cup \Sigma_j[\lambda] \cup \Sigma_{j+1}[\lambda]$.

M-M-R (2023), using a theorem by Cotti-Guzzetti-M (2023).

The central and lateral (=Stokes multiplier) connection coefficients

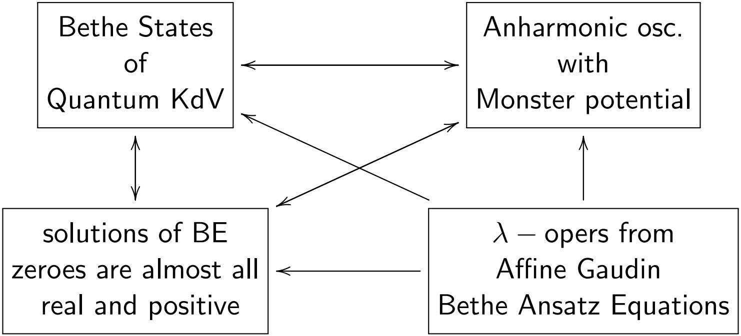
$$Q_{\pm}(\lambda) = \lambda^{-\frac{k+3}{4} \pm (l + r + \frac{1}{2})} Wr_x \left( \chi^{\pm}(x, \lambda^{\frac{k+3}{2}}), \psi^{(0)}(x, \lambda^{\frac{k+3}{2}}) \right).$$

$$T(x, \lambda) = Wr_x \left( \psi^{(-1)}(x, \lambda^{\frac{k+3}{2}}), \psi^{(1)}(x, \lambda^{\frac{k+3}{2}}) \right)$$

are analytic functions $Q_{\pm}, T : \mathbb{C}^* \to \mathbb{C}$.

They satisfy the $TQ$ and $QQ$ system of quantum KdV for the Virasoro parameters $c = 1 - \frac{6}{(k+2)(k+3)}$, $\Delta_r = \frac{(l-2r-kr)(l+1-2r-kr)}{(k+2)(k+3)}$

# The ODE/IM Conjecture for Quantum KdV, revised



# $\lambda \to 0$ limit

The $\lambda \to 0$ limit is very singular: the irregular singularity disappear. The limit depends on $r$, namely on the variables $t_j$, which do not appear in the equation.

**Theorem (M-M-R 2023)**

$Q_+$ and $T$ extends to entire functions $\mathbb{C} \to \mathbb{C}$.

**Conjecture (M-M-R 2023)**

If $I$ is generic: $Q_-$ extends to an entire function and $Q_\pm(0) \neq 0$.

# $\lambda \to +\infty$ limit. Ground state $d_0 = 0$, and $l$ real.

1. The zeroes $\lambda_j, j \in \mathbb{N}$ of $Q_+(\lambda; l)$ are simple, real and positive. Moreover,

$$(\lambda_j)^{\frac{k+3}{2}} = \frac{k+2}{2} \frac{\sqrt{\pi}\Gamma\left(\frac{k+5}{2}\right)}{\Gamma\left(\frac{k+4}{2}\right)} \left(2j + 1 + \frac{l + \frac{1}{2}}{k+2}\right) + O(j^{-1}),\ j \to +\infty,$$

2. Fixed $j$, as $l \to +\infty$

$$(\lambda_j)^{\frac{k+3}{2}} = \frac{\left(k+3\right)^{\frac{k+3}{2}}}{\left(k+2\right)^{\frac{k+2}{2}}}(l + \frac{1}{2}) + \left(\frac{8(2+k)^{1+k}}{(3+k)^{4+k}}\right)^{-\frac{1}{2}}(2j+1) + O\left(\hat{\ell}^{-1}\right).$$

From the physics literature, full proof in Degano-M. to appear.

What about large $\lambda, l$ asymptotics for other opers?

Given a partition $(\mu_1,\dots,\mu_j)\vdash d_0$, one defines the

$$P^{\mu}(x)=Wr[H_{\mu_1+j-1}(x),\dots,H_{\mu_j}(x)]=C^{\mu}\prod_{i=1}^{d_0}(x-v_i^{\mu}).$$

**Conjecture (MMH, following Conti-M.). Partially proven**

1. For generic $l$, the number of solutions of the Gaudin BAE coincides with the number of partitions of $d_0$.
2. For every partition $\mu$ of $d_0$, there exists a solution $\{s_i^{\mu},t_i^{\mu}\}_{i=1}^{d_0}$ with the asymptotics

$$s_i^{\mu}=\frac{k+2}{k+3}\left(1+\frac{2^{1/4}}{(k+2)^{1/4}(k+3)^{1/4}}\,v_i^{\mu}\,l^{-\frac{1}{2}}+O(l^{-1})\right).$$

# Parenthesis: A proven identity

**Conjecture Conti-M., proven in Felder-Veselov 2024**

Let $\mu \vdash N$, assume $P^\mu$ has $N$ distinct zeroes Consider the Jacobian

$$J^\mu_{ij} = \delta_{ij}\left(1+\sum_{l\neq j}\frac{6}{(v^\mu_i - v^\mu_j)^4}\right) - (1-\delta_{ij})\frac{6}{(v^\mu_i - v^\lambda_j)^4}, \, i,j=1,\dots,N.$$

The eigenvalues of $J^\mu$ are are square of the hook-lengths of the partition $\mu$.

# $\lambda, l \to +\infty$ limit. $d_0 > 0$

**Conjecture (MMH, following Conti-M.)**

Fix $d_0 = d_1$ and a solution $s_i^\mu$ of the Gaudin BAE.
1. If $l$ is large enough, the zeroes $\lambda_j^\mu, j \in \mathbb{N}$ of $Q_+(\lambda; l)$ are simple, real and positive. Moreover,

$$(\lambda_j)^{\frac{k+3}{2}} = \frac{k+2}{2} \frac{\sqrt{\pi}\Gamma\left(\frac{k+5}{2}\right)}{\Gamma\left(\frac{k+4}{2}\right)} \left(2j+1+\frac{l+\frac{1}{2}}{k+2}\right) + O(j^{-1}),\ j \to +\infty.$$

2. Fixed $j$, as $l \to +\infty$,

$$\left(\lambda_j^\mu\right)^{\frac{k+2}{3}} = \frac{(k+3)^{\frac{k+3}{2}}}{(k+2)^{\frac{k+2}{2}}} (l+\frac{1}{2}) + \left(\frac{8(2+k)^{1+k}}{(3+k)^{4+k}}\right)^{-\frac{1}{2}} (2n_j^\mu+1) + O(l^{-1}),$$

where $n_j^\mu = j - \mu_{j+1}^t$ (Maya diagram) .

RMK: The zeroes of the $Q_+$ function for monster potentials satisfy the same asymptotic expansions (Conti-M. 2021).

# Classification of solution to Bethe Equation for Quantum KdV

**Theorem. Conti-M. (2023).**

Let $-2 < k < -1$ and $d_0 > 0$ be fixed.
1. If $l > 0$ is large enough, solutions to the Bethe Equations for Quantum KdV, with purely real roots satisfying the large $\lambda$ asymptotics of the zeroes of $Q_+$, are classified by partitions of $d_0$.
2. The solution associated with the partition $\mu$ satisfies the large momentum asymptotics of the zeroes of the $Q_+$ functions.

**Corollary. MMH**

Assuming the conjectures on the asymptotic of the zeroes of $Q_+$ hold.
If $l$ is large, for every sector $r$, there is a bijection among $\lambda$-opers and monster potentials such that the spectral determinants coincide.

## The Big ODE/IM Conjecture

If a QFT is Bethe Integrable then the corresponding solutions of the Bethe Equations are **spectral determinants of linear differential operators**.

$\rightarrow$ Bethe Roots are eigenvalues of a (possibly self-adjoint) differential operator (cf. Hilbert-Pólya Conjecture).

## M - Raimondo- Valeri) (2016,20717), M.-Raimonfo (2020,2021,2023), after Feigin-Frenkel (2011)

$\widehat{\mathfrak{g}}$ an affine Kac-Moody Lie-algebra and ${}^L\widehat{\mathfrak{g}}$ the Langlands dual,

$$\left\{ \text{Bethe states of } \widehat{\mathfrak{g}} - \text{quantum KdV} \right\} \dashleftarrow\dashrightarrow \left\{ {}^L\widehat{\mathfrak{g}} - \text{opers on } \mathbb{C}^* \right\}.$$

# Open questions? A lot

The affine Gaudin model gives a an important hint on the nature of the ODE/IM correspondence.
Even so, the ODE/IM correspondence for Quantum KdV is just a tiny piece of an enormous field of research of which we know a lot but still very little.

- How do we guess which ODE (if any) corresponds to a given Quantum Field Theory?
- Once, they are found, how do we prove them?
- Why the ODE/IM correspondence? Can we find a theory? Why is the nonlinear Stokes phenomenon that important?