Quantum intersection numbers on $\overline{\mathcal{M}}_{g,n}$ and the Gromov–Witten invariants of $\mathbb{CP}^1$

Alexandr Buryak

National Research University Higher School of Economics and
Skolkovo Institute of Science and Technology
Russian Federation

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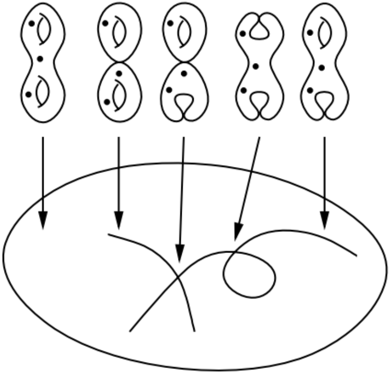
Outline

- The Kontsevich–Witten theorem for the intersection numbers on $\overline{\mathcal{M}}_{g,n}$.
- The Hamiltonians of the KdV hierarchy and the double ramification cycles on $\overline{\mathcal{M}}_{g,n}$.
- A quantization of the KdV hierarchy through the double ramification cycle. Quantum tau-function of the KdV hierarchy. A theorem of X. Blot.
- Quantum intersection numbers and the stationary Gromov–Witten invariants of $\mathbb{CP}^1$.

Intersection numbers



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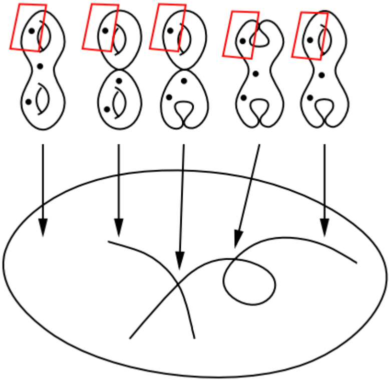

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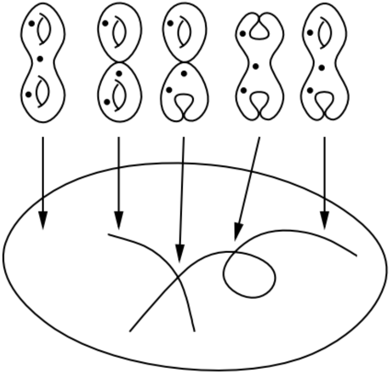

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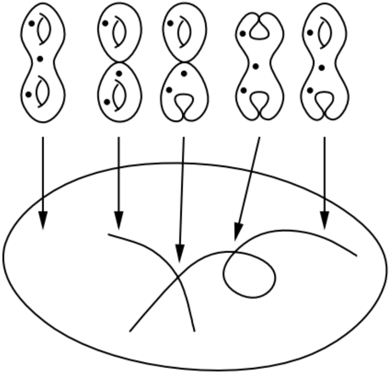

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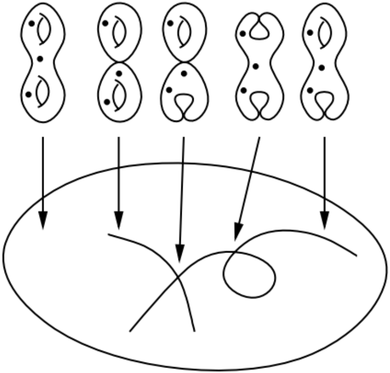

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The generating series of intersection numbers:

$$\mathcal{F}(t_0, t_1, t_2, \dots, \varepsilon) := \sum_{g,n \geq 0} \varepsilon^{2g} \sum_{d_1,\dots,d_n \geq 0} \langle \tau_{d_1} \tau_{d_2} \dots \tau_{d_n} \rangle_g \frac{t_{d_1} t_{d_2} \dots t_{d_n}}{n!}$$

Witten's conjecture

Obviously, $\langle \tau_{d_1} \tau_{d_2} \ldots \tau_{d_n} \rangle_g = \mathrm{Coef}_{\varepsilon^{2g}} \frac{\partial^n \mathcal{F}}{\partial t_{d_1} \ldots \partial t_{d_n}} \Big|_{t_* = 0}$

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Witten's conjecture (1991, proved by Kontsevich in 1992)

- $u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the KdV equation (we identify $x = t_0$)

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- Moreover, u is a solution of the whole hierarchy of infinitesimal symmetries of the KdV equation

$$\frac{\partial u}{\partial t_2} = \frac{u^2 u_x}{2} + \varepsilon^2\left(\frac{u u_{xxx}}{12} + \frac{u_x u_{xx}}{6}\right) + \varepsilon^4 \frac{u_{xxxxx}}{240},$$
$$\frac{\partial u}{\partial t_n} = \frac{u^n u_x}{n!} + \ldots, \quad n \geq 3.$$

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Together with the string equation $\frac{\partial \mathcal{F}}{\partial t_0} = \sum_{k\geq 0} t_{k+1}\frac{\partial \mathcal{F}}{\partial t_k} + \frac{t_0^2}{2}$, this determines all the intersection numbers.

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The Hamiltonian structure of the KdV hierarchy

The KdV hierarchy is Hamiltonian:

$$\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\} = \partial_x \frac{\delta \overline{h}_n}{\delta u}, \qquad \{\overline{h}_m, \overline{h}_n\} = 0.$$

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The first few Hamiltonians are

$$\begin{aligned} \overline{h}_0 &= \int \frac{u^2}{2} dx, \\ \overline{h}_1 &= \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} \right) dx, \\ \overline{h}_2 &= \int \left(\frac{u^4}{24} + \frac{\varepsilon^2}{48} u^2 u_{xx} + \frac{\varepsilon^4}{480} u_{xx}^2 \right). \end{aligned}$$

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The Hamiltonian $\overline{h}_1$ gives the KdV equation
$\{u, \overline{h}_1\} = \partial_x \frac{\delta \overline{h}_1}{\delta u} = u u_x + \frac{\varepsilon^2}{12} u_{xxx}$.

More on the intersection numbers

$u = \frac{\partial^2 \mathcal{F}}{\partial t_0^2}$ is a solution of the KdV hierarchy $\frac{\partial u}{\partial t_n} = \{u, \overline{h}_n\}$, with the initial condition $u|_{t_{\geq 1}=0} = x$ (recall that we identify $t_0 = x$).

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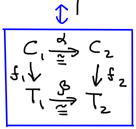
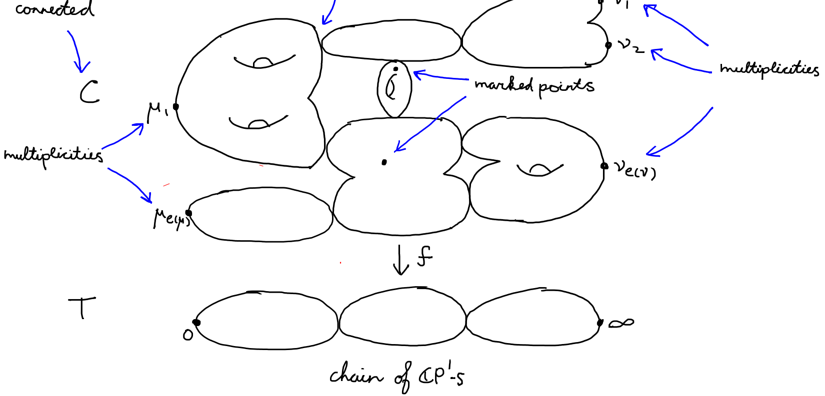
> We will give an explicit formula for the KdV Hamiltonians $\overline{h}_n$, as elements of $\widehat{\mathcal{B}}$, using the moduli space $\overline{\mathcal{M}}_{g,n}$.

Moduli space of stable relative maps $\overline{\mathcal{M}}_{g,n_0}(\mathbb{CP}^1,\mu,\nu)^{\sim}$

μ and ν are partitions of the same integer

The moduli space $\overline{\mathcal{M}}_{g,n_0}(\mathbb{CP}^1,\mu,\nu)^{\sim}$ parameterizes the isomorphism classes of stable relative maps $C \xrightarrow{f} T$:





Double ramification cycle

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The partition μ is formed by the positive numbers among the a_i-s. The partition ν is formed by the negatives of the negative numbers among the a_i-s. n_0 is the number of zeroes among the a_i-s.

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The partition μ is formed by the positive numbers among the a_i-s. The partition ν is formed by the negatives of the negative numbers among the a_i-s. n_0 is the number of zeroes among the a_i-s.

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The space of holomorphic differentials on any stable curve is g-dimensional. $\rightsquigarrow$ Rank g vector bundle $\mathbb{E}$ over $\overline{\mathcal{M}}_{g,n}$ called the Hodge bundle.

The Hodge classes $\lambda_i := c_i(\mathbb{E}) \in H^{2i}(\overline{\mathcal{M}}_{g,n})$.

The KdV Hamiltonians and the double ramification cycles

Here is an explicit geometric formula for the KdV Hamiltonians, as elements of $\widehat{\mathcal{B}}$.

Theorem (B., 2015)

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \frac{(-\varepsilon^2)^g}{n!} \sum_{\substack{a_1,\ldots,a_n\in\mathbb{Z}\\ \sum a_i=0}} \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d \lambda_g \mathrm{DR}_g(0,a_1,\ldots,a_n) \right) \prod_{i=1}^n p_{a_i}.$$

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Example: consider

$$\overline{h}_1 = \int \left(\frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx}\right) dx = \sum_{a_1+a_2+a_3=0} \frac{p_{a_1}p_{a_2}p_{a_3}}{6} - \varepsilon^2 \sum_{a\in\mathbb{Z}} a^2 \frac{p_a p_{-a}}{24}$$

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Towards quantum KdV

Recall $\widehat{\mathcal{B}} = \mathbb{C}[p_1, p_2, \ldots][[p_0, p_{-1}, p_{-2}, \ldots, \varepsilon]]$ with bracket

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We want to quantize the KdV hierarchy:

construct $\overline{H}_n = \overline{h}_n + O(\hbar) \in \widehat{\mathcal{B}}^\hbar$ such that $[\overline{H}_m, \overline{H}_n] = 0$.

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The moduli space $\overline{\mathcal{M}}_{g,n}$ gives a beautiful solution!

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Recall

$$\overline{h}_d = \sum_{g\geq 0,\, n\geq 2} \sum_{\substack{\overline{a}=(a_1,\dots,a_n)\in\mathbb{Z}^n \\ \sum a_i=0}} \frac{\prod_{i=1}^n p_{a_i}}{n!} \left(\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^d (-\varepsilon^2)^g \lambda_g \mathrm{DR}_g(0,\overline{a}) \right)$$

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Obviously, $\overline{H}_n = \overline{h}_n + O(\hbar)$.

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Theorem (B.–Rossi, 2016)

$$[\overline{H}_m, \overline{H}_n] = 0.$$

Quantum KdV hierarchy II

The integral

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For example,

$$\overline{H}_1 = \int\left(\frac{u^3}{6} + \frac{\varepsilon^2}{24}uu_{xx}\right)dx,$$

$$\overline{H}_2 = \int\left(\frac{u^4}{24} + \varepsilon^2\frac{u^2u_{xx}}{48} + \varepsilon^4\frac{uu_{xxxx}}{480} - i\hbar\frac{2uu_{xx}+u^2}{48}\right)dx.$$

Quantum intersection numbers

Recall

$$\left.\frac{\partial^{n+1}\mathcal{F}}{\partial t_0 \partial t_{d_1} \ldots \partial t_{d_n}}\right|_{t_*=0} = \left.\left\{\left\{\ldots\left\{\left\{\frac{\delta \overline{h}_{d_1}}{\delta u}, \overline{h}_{d_2}\right\}, \overline{h}_{d_3}\right\}, \ldots\right\}, \overline{h}_{d_n}\right\}\right|_{u_k=\delta_{k,1}}.$$

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Introduce quantum intersection numbers:

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Relation with the classical intersection numbers:

$\langle\tau_{d_1}\dots\tau_{d_n}\rangle_{g,0}=\langle\tau_{d_1}\dots\tau_{d_n}\rangle_g.$

Theorem of X. Blot

Denote $S(z) := \frac{e^{z/2}-e^{-z/2}}{z}$.

Theorem (X. Blot, 2022)

For $g, n \geq 0$ *satisfying* $2g-2+n > 0$, *we have*

$$\sum_{d_1,\ldots,d_n \geq 0} \langle \tau_{d_1} \ldots \tau_{d_n} \rangle_{0,g} \, \mu_1^{d_1} \cdots \mu_n^{d_n} =$$

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The proof is remarkably complicated!

Relation with one-part double Hurwitz numbers

X. Blot interpreted the theorem in the following way.

For two tuples $\mu = (\mu_1, \ldots, \mu_k) \in \mathbb{Z}^k_{\geq 1}$ and $\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{Z}^m_{\geq 1}$, $k, m \geq 1$, with $\sum \mu_i = \sum \nu_j$, denote by $H^g_{\mu,\nu}$ the double Hurwitz number.

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Goulden–Jackson–Vakil (2005):

$$H^g_{\sum\mu_j,(\mu_1,\ldots,\mu_n)}=r!\left(\sum\mu_j\right)^{r-1}\mathrm{Coef}_{z^{2g}}\left(\frac{\prod_{j=1}^n S(\mu_j z)}{S(z)}\right),$$

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Equivalent formulation of the theorem:

$$\langle\tau_{d_1}\ldots\tau_{d_n}\rangle_{0,g}=\mathrm{Coef}_{\mu_1^{d_1}\ldots\mu_n^{d_n}}\left(\frac{H^g_{\sum\mu_j,(\mu_1,\ldots,\mu_n)}}{r!\sum\mu_j}\right).$$

More properties of the quantum intersection numbers

X. Blot proved and conjectured more properties of the quantum intersection numbers.

In particular, he conjectured that $\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l}$ vanishes if $\sum d_i < 2g-3+n-l$ and that

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{l,g-l} = (-1)^g \int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \lambda_l \psi_1^{d_1} \cdots \psi_n^{d_n} \quad \text{if } \sum d_i = 2g-3+n-l.$$

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So mysteriously various invariants from curve-counting theories are collected in the formal power series $\mathcal{F}^{(q)}$!

Relation with stationary Gromov–Witten invariants of $\mathbb{CP}^1$

For $\mu=(\mu_1,\ldots,\mu_k)\in\mathbb{Z}^k_{\geq 1}$ and $\nu=(\nu_1,\ldots,\nu_m)\in\mathbb{Z}^m_{\geq 1}$, $k,m\geq 1$, with $\sum\mu_i=\sum\nu_j$, denote by $\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,\mu,\nu)$ the moduli space of stable relative maps to $(\mathbb{CP}^1,0,\infty)$.

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Theorem (Blot–B., arXiv:2403.05190)

Let $g, l \geq 0$, $n \geq 1$, and $\overline{d} = (d_1, \dots, d_n) \in \mathbb{Z}^n_{\geq 0}$.
1. For any $k \geq 1$, the integral

$$P_{g,\overline{d}}(a_1, \dots, a_k) := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1,\dots,a_k))]^{\mathrm{vir}}} \lambda_l \prod_{j=1}^n \psi_j^{d_j} \mathrm{ev}_j^*(\omega), \quad a_1, \dots, a_k \in \mathbb{Z}_{\geq 1},$$

is a polynomial in $a_1, \dots, a_k$. Here $A := \sum a_i$.

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2. Let $k := \sum d_j - 2g + l + 1$. *Then*

$$\langle \tau_0 \tau_{d_1} \ldots \tau_{d_n} \rangle_{l,g-l} = \begin{cases} \boxed{\frac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} P_{g,\overline{d}}}, & \text{if } k \geq 1, \\ (-1)^g \int_{\overline{\mathcal{M}}_{g,2}} \lambda_g \lambda_l \psi_1^{d_1}, & \text{if } k = 0 \text{ and } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

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Corollary

We have

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{d_1} \cdots \psi_n^{d_n} =$$

$$= \frac{1}{k!} \mathrm{Coef}_{a_1 \cdots a_k} \int_{\left[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \dots, a_k))\right]^{\mathrm{vir}}} \lambda_g \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega)$$

where $k = 2g - 2 + n$.

More on the theorem of X. Blot

In the opposite case $l = 0$, the previous theorem says that
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where $k = \sum d_j - 2g + 1 \geq 1$.

The integral $\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1, A, (a_1, \dots, a_k))]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega)$ is a stationary relative Gromov–Witten invariant of $\mathbb{CP}^1$, for which Okounkov and Pandharipande (in 2006) presented an explicit formula using the infinite wedge technique.

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For an operator $A \colon \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ denote by $\langle A \rangle$ the coefficient of $v_\emptyset$ in $A v_\emptyset$.

Okounkov–Pandharipande formula

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{CP}^1,\mu,\nu)^{\bullet}]^{\mathrm{vir}}} \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{ev}_j^*(\omega) =$$
$$\frac{1}{\prod_{i=1}^{l(\mu)} \mu_i \prod_{j=1}^{l(\nu)} \nu_j} \mathrm{Coef}_{z_1^{d_1+1} \dots z_n^{d_n+1}} \left\langle \prod_{i=1}^{l(\mu)} \alpha_{\mu_i} \prod_{j=1}^{n} \mathcal{E}_0(z_j) \prod_{i=1}^{l(\nu)} \alpha_{-\nu_i} \right\rangle .$$

In a joint work with X. Blot (arXiv:2403.05190), using the Okounkov–Pandharipande formula, we give a short proof of the theorem of X. Blot.