Discrete Painlevé equations in random partitions and planar maps

Sofia Tarricone

Institut de Physique Théorique, CEA Paris-Saclay

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Outline

1. Introduction on discrete Painlevé equations
2. dPII and discrete probabilities in random partitions
3. dPI and combinatorics of planar quadrangulations with fixed distance

The first two discrete Painlevé equations

The first and second discrete Painlevé equations are the second order nonlinear discrete equations

$$\begin{aligned} \text{dPI:} \quad & y_n(y_{n+1} + y_n + y_{n-1}) = c_1 + c_2 n + c_3 y_n \\ \text{dPII:} \quad & (1 - y_n^2)(y_{n+1} + y_{n-1}) = (c_1 + c_2 n) y_n + c_3, \quad c_i \in \mathbb{C} \end{aligned}$$

admitting as continuous limit (for different scaling limits) the classical first and second Painlevé equations

$$\begin{aligned} \text{PI:} \quad & u''(t) = 6u^2(t) + t \\ \text{PII:} \quad & u''(t) = 2u^3(t) + tu(t) + \alpha, \quad \alpha \in \mathbb{C}. \end{aligned}$$

Their main properties (for this talk)

[Cresswell - Joshi, 1998] Both the first and second discrete Painlevé equations can be extended to a hierarchy i.e. a sequence of higher order equations, as their continuous versions.

The k-th equation of each hierarchy is encoded by discrete Lax pairs i.e. linear systems of type

$$\Phi_{n+1}^{(k)}(\lambda) = L_n(\lambda; y_n)\,\Phi_n^{(k)}(\lambda)$$
$$\frac{\partial}{\partial\lambda}\Phi_n^{(k)}(\lambda) = M_n^{(k)}\left(\lambda; \{y_\ell\}_{\ell=n-k}^{n+k}\right)\Phi_n^{(k)}(\lambda)$$

where L_n, M_n are rational in λ (eventually matrix-valued) with coefficients depending on y_ℓ. The compatibility condition of this system

$$\frac{\partial}{\partial\lambda}L_n + L_n M_n^{(k)} - M_{n+1}^{(k)} L_n = 0$$

corresponds to the k-th equation of the hierarchy.

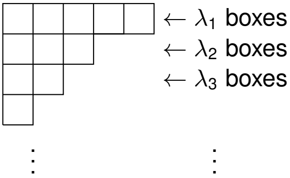
Outline

Random partitions models

For a given $M \in \mathbb{N}$ a partition of M is a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$ with $\lambda_i \in \mathbb{N}$

$$\sum_{i \geq 1} \lambda_i = M \ (= |\lambda|).$$

We can represent a given partition λ via the Young diagram of shape λ



And a standard Young tableau (SYT) of shape λ is obtained by filling in the boxes of the Young diagram of shap λ with numbers $1, \dots, |\lambda|$ with increasing sequences in both directions $\rightarrow$ and $\downarrow$.

A random partition model is then given by the definition of a probability measure on the set of partitions.

Why? The Ulam problem

Consider the symmetric group S_M taken with uniform distribution so that for any $\pi_M \in S_M$ we have

$$\mathbb{P}(\pi_M) = \frac{1}{M!}$$

and denote $\ell(\pi_M)$ the length of the longest increasing sub-sequence of π_M.

Example $\pi_5 = 4 \quad 3 \quad 1 \quad 2 \quad 5$ and $\ell(\pi_5) = 3$.

Ulam problem (1961)

Describe the behavior of $\ell(\pi_M)$ for $M \to \infty$.

The Poissonized Plancherel measure

Uniform random permutations of M elements are equivalent to a model of random partitions of M thanks to the Robinson–Schensted correspondence, i.e. the bijection

$$RS : \pi_M \ni S_M \to RS(\pi_M) \in \{(P, Q) \in \mathrm{SYT}_M \times \mathrm{SYT}_M,\ \mathrm{sh}(P) = \mathrm{sh}(Q)\}.$$

$$\downarrow$$

The uniform measure on S_M corresponds on the set of partitions of M to the Plancherel measure

$$\mathbb{P}_{\mathrm{Pl.}}(\lambda) = \frac{F_\lambda^2}{M!},\ \text{with } F_\lambda = \#\{P \in \mathrm{SYT}_M,\ \mathrm{sh}(P) = \lambda\}.$$

$$\downarrow$$

Its Poissonization consists of taking on the set of all partitions the measure

$$\mathbb{P}_{\mathrm{P.Pl.}}(\lambda) = \mathrm{e}^{-\theta^2} \left(\frac{\theta^{|\lambda|} F_\lambda}{|\lambda|!} \right)^2,\ \text{where } |\lambda| = \mathrm{weight}(\lambda).$$

Remark [Schensted, 1961] Moreover, in the RS correspondence $\ell(\pi_M) = \lambda_1(\pi_M)$.

Distribution of first parts and Toeplitz determinants

In the Poissonized Plancherel model, the distributions of first parts are given by the Gessel's formula

$$\mathbb{P}_{\text{P.Pl.}}(\lambda_1 \leq k) = \mathrm{e}^{-\theta^2} D_{k-1}(\varphi)$$

where $D_k(\varphi)$ are Toeplitz determinants associated to the symbol $\varphi = \varphi[\theta](z) = \mathrm{e}^{w(z)}$ for $w(z) = v(z) + v(z^{-1})$ and $v(z) = \theta z$. In particular

$$\boxed{D_k := \det(T_k(\varphi))}$$

with $T_k(\varphi)$ being the k-th Toeplitz matrix associated to the symbol $\varphi(z)$

$$T_k(\varphi)_{i,j} := \varphi_{i-j}, \quad i,j = 0, \dots, k$$

where for every $\ell \in \mathbb{Z}$, φ_ℓ is the ℓ-th Fourier coefficient of $\varphi(z)$, namely

$$\varphi_\ell = \int_{-\pi}^{\pi} e^{-i\ell\theta} \varphi(e^{i\theta}) \frac{d\theta}{2\pi}, \text{ so that } \sum_{\ell \in \mathbb{Z}} \varphi_\ell z^\ell = \varphi(z).$$

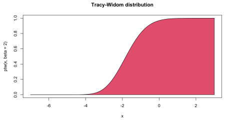
The Baik-Deift-Johansson result

[Baik - Deift - Johansson, 1999] The limiting behavior of the lenght of the longest increasing subsequence of a random permutation is

$$\lim_{M\to\infty} \mathbb{P}\left(\frac{\ell(\pi_M) - 2\sqrt{M}}{M^{1/6}} \leq s\right) = F(s),$$

where $F(s)$ is the (GUE) Tracy-Widom distribution

$$\begin{cases} F(s) = \exp\left(-\int_s^{+\infty}(r-s)u^2(r)dr\right), \text{with} \\ u''(s) = su(s) + 2u^3(s), \quad u(s) \sim_{s\to\infty} \mathrm{Ai}(s). \end{cases}$$



Remark The result of B–D–J was obtained by studying large k scaling limit behaviour of the Toeplitz determinants and thereafter using a *de-Poissonization* procedure.

More integrability result

[Borodin, Adler - Van Moerbeke, Baik, 2000] For every $k \geq 1$ we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where x_k solves the so called discrete Painlevé II equation, which corresponds to the second order nonlinear difference equation

$$\theta(x_{k+1} + x_{k-1})(1 - x_k^2) + k x_k = 0$$

with initial conditions $x_0 = -1, x_1 = \varphi_1/\varphi_0$.

In the limit for $\theta \to \infty$ and for $k = s\theta^{1/3} + 2\theta$ (or $s = (k - 2\theta)\theta^{-1/3}$), then

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 \quad = \quad -x_k^2, \qquad\qquad x_{k+1} + x_{k-1} = -\frac{k x_k}{\theta(1 - x_k^2)}$$

$$\text{B–D–J}\downarrow \boxed{x_k = (-1)^k \theta^{-1/3} u(s)} \qquad\qquad \downarrow \boxed{x_k = (-1)^k \theta^{-1/3} u(s)}$$

$$\partial_s^2 \log F(s) = -u^2(s), \qquad\qquad \underbrace{u''(s) = 2u^3(s) + s u(s)}_{\text{Painlevé II equation}}$$

Multicritical random partitions

[Okunkov, 2001] On the set of partitions consider the *Schur* measures (n parameters)

$$\mathbb{P}_{\mathrm{Sc.}}(\lambda) = Z^{-1} s_\lambda \left[\theta_1, \ldots, \theta_n\right]^2 ,$$

where s_λ can be computed as

$$s_\lambda \left[\theta_1, \ldots, \theta_n\right] = \det_{i,j} h_{\lambda_i - i + j} \left[\theta_1, \ldots, \theta_n\right] ,$$

with $\sum_{k\geq 0} h_k z^k = \mathrm{e}^{v(z)}$, $v(z) = \sum_{i=1}^n \frac{\theta_i}{i} z^i$ and $Z = \mathrm{e}^{\sum_{i=1}^n \frac{\theta_i^2}{i}}$.

Remark For $n = 1$ with $\mathbb{P}_{\mathrm{P.Pl.}}(\lambda) = \mathbb{P}_{\mathrm{Sc.}}(\lambda)$ with $\theta_1 = \theta$.

The probability distribution of the first part of such a random partition is given again by Toeplitz determinants

$$\mathbb{P}_{\mathrm{Sc.}}\left(\lambda_1 \leq k\right) = \mathrm{e}^{-\sum_i^n \hat{\theta}_i^2/i} D_{k-1}\left(\varphi^{(n)}\left[\hat{\theta}_1, \ldots, \hat{\theta}_n\right]\right) ,$$

where the symbol is

$$\varphi^{(n)}\left[\hat{\theta}_1, \ldots, \hat{\theta}_n\right](z) = \mathrm{e}^{w(z)},\ w(z) = v(z) + v(z^{-1}),\ \theta_i \to \hat{\theta}_i = (-1)^{i+1}\theta_i.$$

In the multicritical setting: $\hat{\theta}_1 = \theta, \hat{\theta}_i = \frac{(n-1)!(n+1)!}{(n-i)!(n+i)!}\theta, i = 2, \ldots, n.$

Our main result

Theorem (Chouteau - T., 2023)

For any fixed $n \geq 1$, for the Toeplitz determinants $D_k, k \geq 1$, we have

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$

where now x_k solves the $2n$ order nonlinear difference equation

$$kx_k + \left(v_k + v_k Perm_k - 2x_k \Delta^{-1}\left(x_k - (\Delta + I)x_k Perm_k\right)\right) L^n(0) = 0$$

where L is a discrete recursion operator that acts as follows

$$L(u_k) := \left(x_{k+1}\left(2\Delta^{-1} + I\right)\left((\Delta + I)\, x_k Perm_k - x_k\right) + v_{k+1}\,(\Delta + I) - x_k x_{k+1}\right) u_k,$$

and $L(0) = \theta_n x_{k+1}$. Here $v_k := 1 - x_k^2$, Δ denotes the difference operator $\Delta : u_k \to u_{k+1} - u_k$ and $Perm_k$ is the transformation

$$\begin{array}{rccc} Perm_k : & \mathbb{C}\left[(x_j)_{j\in[[0,2k]]}\right] & \longrightarrow & \mathbb{C}\left[(x_j)_{j\in[[0,2k]]}\right] \\ & P\left((x_{k+j})_{-k\leqslant j\leqslant k}\right) & \longmapsto & P\left((x_{k-j})_{-k\leqslant j\leqslant k}\right). \end{array}$$

The first equations of the hierarchy

$$n = 1: \quad kx_k + \theta_1(x_{k+1} + x_{k-1})(1 - x_k^2) = 0, \leftarrow$$ discrete Painlevé II equation

$$n = 2: \quad kx_k + \theta_1(1 - x_k^2)\,(x_{k+1} + x_{k-1})$$
$$+ \theta_2(1 - x_k^2)\left(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2) - x_k(x_{k+1} + x_{k-1})^2\right) = 0,$$

$$n = 3: \quad kx_k + \theta_1(1 - x_k^2)\,(x_{k+1} + x_{k-1})$$
$$+ \theta_2(1 - x_k^2)\left(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2) - x_k(x_{k+1} + x_{k-1})^2\right)$$
$$+ \theta_3(1 - x_k^2)\left(x_k^2(x_{k+1} + x_{k-1})^3 + x_{k+3}(1 - x_{k+2}^2)(1 - x_{k+1}^2) + x_{k-3}(1 - x_{k-2}^2)(1 -\right.$$
$$+ \theta_3(1 - x_k^2)\left(-2x_k(x_{k+1} + x_{k-1})(x_{k+2}(1 - x_{k+1}^2) + x_{k-2}(1 - x_{k-1}^2))\right)$$
$$+ \theta_3(1 - x_k^2)\left(-x_{k-1}x_{k-2}^2(1 - x_{k-1}^2) - x_{k+1}x_{k+2}^2(1 - x_{k+1}^2)\right)$$
$$+ \theta_3(1 - x_k^2)\,(-x_{k+1}x_{k-1}(x_{k+1} + x_{k-1})) = 0.$$

Remark Similar discrete equations appeared previously in [Periwal-Schewitz, 1990] in the study of some unitary matrix integrals.

The limiting behavior

[Betea–Bouttier–Walsh , 2021] For the $\theta_i, i = 1, \ldots, n$ in the multicritical regime, then the limiting behavior of the distribution of the first part is described, for certain $b = b(n), d = d(n)$, by

$$\lim_{\theta \to \infty} \mathbb{P}_{\mathrm{Sc.}} \left(\frac{\lambda_1 - b\theta}{(\theta d)^{\frac{1}{2n+1}}} < s \right) = F_n(s).$$

where now $F_n(s)$ is the n-th order (GUE) Tracy-Widom distribution characterized by [Cafasso - Claeys - Girotti, 2019] by the formula

$$\frac{d^2}{ds^2} \ln F_n(s) = -u^2((-1)^{n+1} s)$$

here u solves the n-th member of the homogeneous Painlevé II hierarchy with boundary condition $u(s) \sim Ai_n(s)$ for $s \to +\infty$.

Continuous limit

Recall the B–B–W result for $n = 2$: $\lim_{\theta\to\infty} \mathbb{P}_{\text{Sc.}}\left(\frac{\lambda_1 - \frac{3}{2}\theta}{(4^{-1}\theta)^{1/5}} \le s\right) = F_2(s)$.

In the limit for $\theta \to \infty$, taking $k = s\left(\frac{\theta}{4}\right)^{1/5} + \frac{3}{2}\theta$ (or $s = \left(k - \frac{3}{2}\theta\right)\theta^{-\frac{1}{5}}4^{\frac{1}{5}}$)

$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2,$$

$$\xrightarrow{\text{B–B–W}} \boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s)}$$

$$\partial_s^2 \log F_2(s) = -u^2(s),$$

$$k x_k + \theta_1 v_k \left(x_{k+1} + x_{k-1}\right) + \theta_2 v_k \left(x_{k+2} v_{k+1} + x_{k-2} v_{k-1} - x_k (x_{k+1} + x_{k-1})^2\right) = 0$$

$$\downarrow \boxed{x_k = (-1)^k \left(\frac{\theta}{4}\right)^{-1/5} u(s)\ \theta_1 = \theta,\ \theta_2 = \frac{\theta}{4}}$$

$$\underbrace{u'''' - 10u(u')^2 - 10u^2u'' + 6u^5 = -su}_{\text{2nd eq. of the Painlevé II hierarchy}}$$

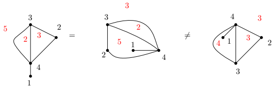
which recovers the generalized Tracy-Widom formula for $n = 2$ [Cafasso–Claeys–Girotti, 2019]. And so on ...

Idea of the proof

1. We consider the family $\{p_k(z) = \kappa_k z^k + \dots\}_{k\in\mathbb{N}}$ of orthogonal polynomials on the unit circle (OPUC) w.r.t. the measure
$$\mathrm{d}\mu(\alpha) = \varphi(\mathrm{e}^{i\alpha})\frac{\mathrm{d}\alpha}{2\pi} = \mathrm{e}^{w(\mathrm{e}^{i\alpha})}\frac{\mathrm{d}\alpha}{2\pi}.$$
2. This family of orthogonal polynomials $\{p_k(z)\}$ can be characterized by a 2×2 matrix-valued Riemann–Hilbert problem (part of Baik-Deift-Johansson's results for the generalized weight).
3. From the explicit form of the solution of the Riemann–Hilbert problem, one can easly deduce the formula
$$\frac{D_k D_{k-2}}{D_{k-1}^2} - 1 = -x_k^2$$
where $x_k = \frac{1}{\kappa_k} p_k(0)$.
4. The solution to the Riemann–Hilbert problem allows to construct a *Lax pair* (sort of linear representation) for the discrete Painlevé II hierarchy for x_k.
5. This Lax pair is mapped into the original Lax pair obtained by Cresswell and Joshi in 1998 which first introduced the discrete Painlevé II hierarchy.

Outline

Two-point function for planar quadrangulations

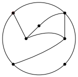


A planar map is a connected graph embedded in the sphere, so

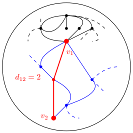
$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = 2,$$

considered up to continuous deformation.

A quadrangulation is a map in which all faces have degree 4.



The 2-point function is the generating function for maps with two marked points at given distance $\ell \rightsquigarrow$ the integrated two-point function is the generating function R_ℓ for maps with two marked points at distance at most ℓ.



Remark The analogue question is: what is the average distance between two points chosen uniformly on a random (uniform) planar map?

dPI and planar quadrangulations

[Bouttier - Di Francesco - Guitter (2003)] By using bijective combinatorics methods, it is shown that R_ℓ, in the case of quadrangulations, satisfies the recursion

$$R_\ell = 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1}),\ \ell \geq 1,\quad R_0 = 0.$$

with $\lim_{\ell\to\infty} R_\ell = R$ and $3gR^2 + 1 = R$.

- The scaling limit of R_ℓ when $\ell \to +\infty$, and $g = \frac{1}{12}\mathrm{e}^{-\epsilon\lambda}$, $\epsilon \to 0$ at fixed rate $r = \ell\epsilon^{1/4}$ was computed by continuous limit of the discrete equation for R_ℓ, with an appropriate ansatz.
- In the case of general bipartite planar maps other higher order equations are satisfied by R_ℓ.

Remark[Bessis - Itzykson - Zuber (1980)] . . . [Bleher - Gharakhloo - McLaughlin (2022)] while studying the expansion of matrix integrals (of size N) using orthogonal polynomials on the real line with mesure $\mathrm{e}^{-N(x^2/2+gx^4/4)}dx$, a similar equation was found

$$x_\ell + gx_\ell(x_{\ell+1} + x_\ell + x_{\ell-1}) = \frac{\ell}{N},$$

in relation with the generating function of 4-valent planar maps .

Two-point function characterization in terms of Hankel determinants

[Bouttier - Guitter (2010)] R_ℓ is written as

$$R_\ell = \frac{D_\ell D_{\ell-2}}{D_{\ell-1}^2}$$

where D_ℓ are the Hankel determinants $D_\ell = \det_{i,j=0}^{\ell} F_{i+j}$. The entries of the Hankel matrices F_ℓ, in the case of quadrangulations, are computed as

$$F_\ell = \begin{cases} R(1-2Rg)\mathrm{cat}_k R^k - Rg\mathrm{cat}_{k+1} R^{k+1}, \ \ell = 2k, \\ 0, \ \text{otherwise.} \end{cases}$$

Remark The result is true also in the general case of bipartite planar maps, by replacing the formula for F_ℓ with a more convoluted expression. Moreover [Bergère - Eynard - Guitter - Oukassi (2023)] recently generalized this type of result in the case of hypermaps.

Catalan numbers as moments of the Wigner semi-circle law

The Wigner semicircle law defined over the interval $[-s, s]$, $s > 0$ is denoted by

$$f_s(x) = \frac{2}{\pi s^2}\sqrt{s^2 - x^2}.$$

Property For every $k \geq 0$ the k-th moment of the measure $f_s(x)\mathrm{d}x$ is

$$m_k^{(f_s)} = \begin{cases} \left(\frac{s}{2}\right)^{2n} \mathrm{cat}_n \text{ for } k = 2n \\ 0 \text{ otherwise.} \end{cases}$$

Consequence Using this property of the Catalan's numbers, we deduce that

$$F_\ell = \begin{cases} m_{2k}^{(R,4)} & \ell = 2k \\ 0, & \text{otherwise} \end{cases}$$

where $m_{2k}^{(R,4)}$ denotes the $2k$-th moment of the measure $\mu^{(R,4)}(x)\mathrm{d}x$ with

$$\mu^{(R,4)}(x) = \frac{1}{2\pi}\sqrt{4R - x^2}\left(1 - 2gR - gx^2\right) \quad -2\sqrt{R} \leq x \leq 2\sqrt{R}.$$

Connection with orthogonal polynomials

- The (integrated) two-point function for planar quadrangulations was written in terms of Hankel determinants

$$R_\ell = \frac{D_\ell D_{\ell-2}}{D_{\ell-1}^2}, \; D_\ell = \det_{i,j=0}^{\ell} F_{i+j}$$

and F_{i+j} is the $i+j$-th moment of the measure $\mu^{(R,4)}(x)\mathrm{d}x$.

- Scaling by $2\sqrt{R}$ we obtain the normalized measure on $[-1,1]$ given by

$$\rho^{(R,4)}(x) = \frac{2}{\pi}\sqrt{1-x^2}R\left(1-2gR-4gRx^2\right).$$

The family of orthogonal polynomials associated to it, satisfies a three terms recurrence relation of type $x\pi_\ell(x) = \pi_{\ell+1}(x) + r_\ell \pi_{\ell-1}(x)$ where

$$r_\ell = \frac{H_{\ell-2}H_\ell}{H_{\ell-1}^2}, \;\; H_\ell = \det_{i,j=0}^{\ell} m_{i+j}^{(\rho^{(R,4)})} \quad \Longrightarrow \quad 4R\, r_\ell = R_\ell.$$

- We can see this family of orthogonal polynomials as a deformation of Jacobi polynomials since

$$\rho^{(R,4)}(x) = (1-x^2)^{1/2}(t^2-x^2)k(t), \; t^2 = \frac{R+2}{4(R-1)}.$$

Ongoing work (with J. Bouttier)

- Analytical proof of the discrete equation satisfied by R_ℓ

$$R_\ell = 1 + gR_\ell(R_{\ell+1} + R_\ell + R_{\ell-1})$$

using the relation $4Rr_\ell = R_\ell$ and the techniques from orthogonal polynomials. In particular these are semi-classical orthogonal polynomials and derivation of *Laguerre-Freud* equation for r_ℓ was developed in many works.

- Computation of the scaling limit for $\ell \to +\infty$ and $t = 1 + k/\ell^2$ of the coefficients of the three terms recurrence relation $r_\ell(t)$ of the family of orthogonal polynomials w.r.t. $\rho^{(R,4)}(x)\mathrm{d}x = (1-x^2)^{1/2}(t^2 - x^2)k(t)$ to confirm the one known for R_ℓ by continuous limit of the discrete equation.

- Extension of these results to the case of general bipartite maps with bounded degree face. Indeed the connection with orthogonal polynomials still holds, by replacing $\mu^{(R,4)}(x)$ by

$$\mu^{(R,2N)}(x) = \frac{1}{2\pi}\sqrt{4R - x^2}P_{2N}(x; R, g = g_4, \ldots, g_{2N}).$$

Conclusion

1. We can compute the distributions of first parts of multicritical random partitions via solutions of the discrete Painlevé II hierarchy.
2. We can count bipartite planar maps with fixed graph distance between two vertices with a deformed version of the discrete Painlevé I hierarchy.
3. Orthogonal polynomials are always behind the scene.

Thank you!