Quasimodular functions from Donaldson–Thomas Invariants

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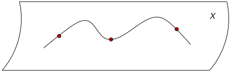
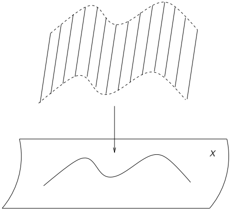
Overview

- Classical questions in algebraic geometry: counting problems.

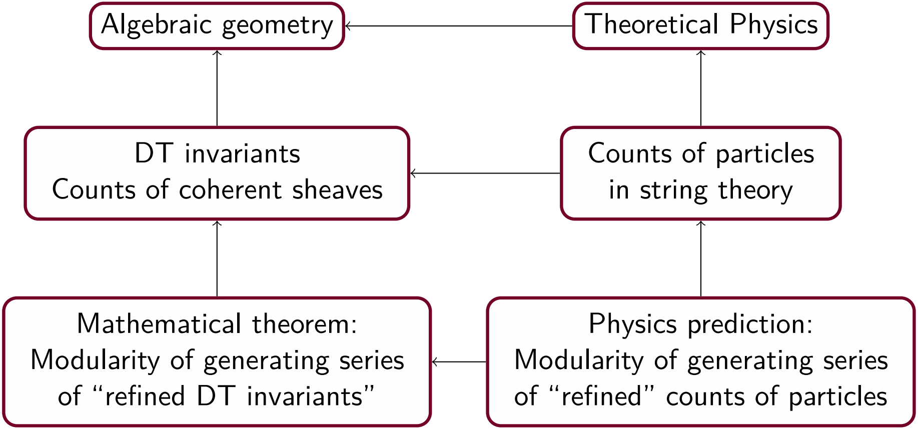
- What are counts of curves in a space X?
  - Gromov–Witten invariants.



- What are counts of vector bundles, or more generally coherent sheaves, in a space X?
  - Donaldson–Thomas (DT) invariants.



Physics: relations with counts of particles in string theory.


Algebraic geometry
Theoretical Physics
DT invariants
Counts of coherent sheaves
Counts of particles
in string theory
Mathematical theorem:
Modularity of generating series
of “refined DT invariants”
Physics prediction:
Modularity of generating series
of “refined” counts of particles

Plan of the talk

- The notion of "modularity" of a generating series
- Refined Donaldson–Thomas (DT) Invariants
  - Betti numbers of moduli spaces of coherent sheaves

Main result:
Modularity of generating series of refined DT invariants

- Proof: Scattering diagrams and Gromov–Witten invariants

Modularity

- (a_n): a sequence of numbers
- Form a generating series (formal power series in a formal variable q)

$$\sum_n a_n q^n \xrightarrow{q=e^{2i\pi\tau}} \sum_n a_n e^{2i\pi n\tau}$$

Definition

$f(\tau) := \sum_n a_n e^{2i\pi n\tau}$ is called a ***modular function*** of weight k for a subroup $G \subset SL(2,\mathbb{Z})$ if it satisfies

- Convergence property: $f(\tau)$ is a holomorphic function on the upper half-plane $\mathbb{H} := \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\tau > 0\}$
- Symmetry property:

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \text{ for every } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G\,.$$

- If f is modular, it is enough to know $f(\tau)$ for τ in the fundamental domain, to recover $f(\tau)$ for any value of τ by symmetry.

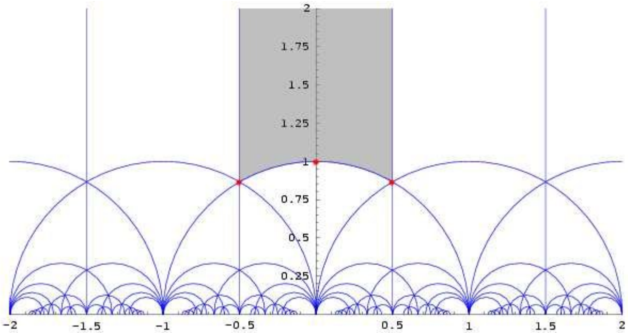


Figure: Fundamental domains for the action of $SL(2,\mathbb{Z})$ on the upper half-plane $\mathbb{H}$

Example

The **Eisenstein series**

$$E_{2k}(\tau) := 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \frac{n^{2k-1} q^n}{1-q^n} = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(m+n\tau)^{2k}}$$

where $q = e^{2\pi i \tau}$, is modular of weight $2k$ for $2k \geq 4$.

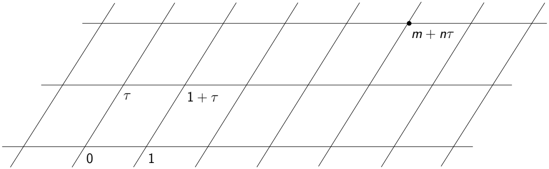


Figure: The lattice $\mathbb{Z} + \tau\mathbb{Z}$

Constructing modular functions

- Consider the elliptic curve $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ given by a cubic equation,

$$y^2 = 4x^3 - \frac{4}{3}\pi^4 E_4(\tau)x - \frac{8}{27}\pi^6 E_6(\tau) \quad \text{where} \quad E_4, E_6 : \mathbb{H} \to \mathbb{C}$$

  - The discriminant satisfies

$$\Delta(\tau) = E_4(\tau)^3 - 27E_6(\tau)^2 = (2\pi)^{12}\eta^{24}(\tau)$$

    where $\eta(\tau)$ is the Dedekind eta function:

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(1 - q - q^2 + q^5 + q^7 - q^{12} + \dots)$$

  for $q = e^{2\pi i\tau}$.

  - $\eta(\tau)$ is a modular function of weight $1/2$ for $\mathrm{SL}(2,\mathbb{Z})$.
  - We'll construct further modular functions in terms of $\eta(\tau)$.

Example: modular functions for $\Gamma_1(3)$

- Congruence subgroup $G = \Gamma_1(3) \subset SL(2,\mathbb{Z})$:

$$\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \right\}$$

- Modular of weight 1 for $\Gamma_1(3)$:

$$A(\tau) := \left(\frac{\eta(\tau)^9}{\eta(3\tau)^3} + 27 \frac{\eta(3\tau)^9}{\eta(\tau)^3} \right)^{\frac{1}{3}} = 1+6q+6q^3+6q^4+12q^7+6q^9+\dots$$

- Modular of weight 3 for $\Gamma_1(3)$:

$$C(\tau) := \frac{\eta(\tau)^9}{\eta(3\tau)^3} = 1-9q+27q^2-9q^3-117q^4+216q^5+27q^6-450q^7+\dots$$

for $q = e^{2\pi i \tau}$.

Quasimodularity

- $f(\tau) = \sum_n a_n e^{2i\pi n\tau}$ is called ***quasimodular of weight*** k for G if the symmetry property is satisfied up to a correction term of a specific form.

Example

- Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n \geq 1} \frac{nq^n}{1-q^n} = 1 - 24q - 72q^2 - 96q^3 - 168q^4 - 144q^5 + \dots$$

is quasimodular of weight 2 for $SL(2,\mathbb{Z})$, but not modular:

$$(c\tau + d)^{-2} E_2\left(\frac{a\tau+b}{c\tau+d}\right) = E_2(\tau) + \frac{6}{i\pi}\left(\frac{c}{c\tau+d}\right)$$

- Quasimodular of weight 2 for $\Gamma_1(3)$:

$$B(\tau) := \frac{1}{4}\left(E_2(\tau) + 3E_2(3\tau)\right) = 1 - 6q - 18q^2 - 42q^3 - 42q^4 + \dots$$

Modular/Quasimodular Functions in Mathematics

- Modular and quasimodular functions appear in many places in mathematics (number theory, combinatorics, geometry, representation theory, algebra,...).
- To be modular/quasimodular is an extremely strong constraint: the space of quasimodular functions of given weight (and with growth condition at infinity) is finite dimensional. In particular, the entire sequence (a_n) can be recovered from finitely many elements.

Main result: Express generating series of (refined) DT invariants, defined geometrically, in terms of the modular functions $A(\tau)$, $C(\tau)$ and the quasimodular function $B(\tau)$. Hence, obtain quasi-modular functions for $\Gamma_1(3)$ from (refined) DT invariants.

Geometry: Donaldson-Thomas (DT) invariants

- X: Calabi–Yau 3-fold
  - Complex manifold of dimension 3 which admits a nowhere vanishing holomorphic 3-form.
- Coherent sheaves on X:
  - Holomorphic vector bundles on X
  - Holomorphic vector bundles supported on complex submanifolds of X.
- DT invariants of X: counts of coherent sheaves on X
  - Euler characteristics of the moduli spaces of coherent sheaves on X

Finitely many coherent sheaves on X with fixed topology How many? DT invariants.	$\longleftrightarrow$	String theory on $\mathbb{R}^4 \times X$ Counts of particles determined by the geometry of X.

DT invariants of $K_{\mathbb{P}^2}$

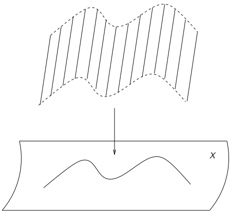
- $\mathbb{P}^2$: complex projective plane, complex dimension 2. Coordinates:

$$[x_0 : x_1 : x_2] \sim [\lambda x_0, \lambda x_1, \lambda x_2].$$

- $X = K_{\mathbb{P}^2}$: total space of the canonical line bundle on $\mathbb{P}^2$
  - Non-compact Calabi-Yau 3-fold.
  - Zero-section $\mathbb{P}^2 \subset K_{\mathbb{P}^2}$.
- DT invariants of $K_{\mathbb{P}^2}$
  - Counts of coherent sheaves on $K_{\mathbb{P}^2}$ supported on curves inside $\mathbb{P}^2$.
  - Euler characteristic of the moduli spaces of coherent sheaves supported on curves inside $\mathbb{P}^2$.
- Refined DT invariants of $K_{\mathbb{P}^2}$
  - Betti numbers of the moduli spaces of coherent sheaves supported on curves inside $\mathbb{P}^2$.

Moduli spaces of coherent sheaves on $\mathbb{P}^2$

- M_d: moduli space of (stable) coherent sheaves on $\mathbb{P}^2$ supported on degree d curves.
  - Variety of dimension d^2+1, classically studied (Simpson, Le Potier, $\sim$ 1990).



- There is a fibration of M_d over the space B_d of degree d curves in $\mathbb{P}^2$.

$$\pi\colon M_d \to B_d = \mathbb{P}^{\frac{d(d+3)}{2}}$$
$$F \mapsto \mathrm{supp}(F)$$

Example

- For $d=1$: $a_0x_0 + a_1x_1 + a_2x_2 = 0 \implies B_1 = \mathbb{P}^2 = \mathbb{P}^{\frac{1(1+3)}{2}}$
- For $d=2$: $B_2 = \mathbb{P}^5 = \mathbb{P}^{\frac{2(2+3)}{2}}$
- For $d=3$: $B_3 = \mathbb{P}^9 = \mathbb{P}^{\frac{3(3+3)}{2}}$

Moduli spaces of coherent sheaves on $\mathbb{P}^2$

- General fiber of $\pi : M_d \to B_d$ over a point corresponding to a degree d smooth curve C: all possible line bundles on C, i.e. $\mathrm{Jac}(C)$
  - $\mathrm{Jac}(C)$ is a compact torus of dimension $g = \frac{(d-1)(d-2)}{2}$.
- $\exists$ singular fibers $\implies$ challenging to compute Betti numbers of M_d.

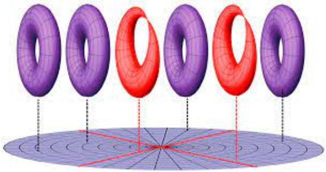


Figure: The fibration $\pi : M_d \to B_d$

Refined DT invariants of $K_{\mathbb{P}^2}$: Betti numbers $b_j(M_d) := \dim H^j(M_d, \mathbb{Q})$.

Examples of refined DT invariants of $K_{\mathbb{P}^2}$

- $M_1 = \mathbb{P}^2$: moduli space of coherent sheaves over lines in $\mathbb{P}^2$.

$$\sum_{j=0}^{4} b_j(M_1) y^{\frac{j}{2}} = 1 + y + y^2$$

- $M_2 = \mathbb{P}^5$: moduli space of coherent sheaves over conics in $\mathbb{P}^2$.

$$\sum_{j=0}^{10} b_j(M_2) y^{\frac{j}{2}} = 1 + y + y^2 + y^3 + y^4 + y^5$$

- M_3: moduli space of coherent sheaves over cubics in $\mathbb{P}^2$.
  - $M_3 \to B_3 = \mathbb{P}^9$ with generic fibers complex tori of dimension 1.

$$\sum_{j=0}^{20} b_j(M_3) y^{\frac{j}{2}} = 1+2y+3y^2+3y^3+3y^4+3y^5+3y^6+3y^7+3y^8+2y^9+y^{10}$$

- Generating series of DT invariants of $K_{\mathbb{P}^2}$

$$\sum_{d\geq 1} \sum_{j=0}^{\dim_{\mathbb{R}} M_d} b_j(M_d) y^{\frac{j}{2}} q^d \,.$$

- We need a change of variables $y = e^{\hbar}$ and expand in powers of $\hbar$:

$$y = e^{\hbar} = \sum_{n\geq 0} \frac{\hbar^n}{n!}$$

Definition

Define generating series of refined DT invariants, $F_g(q) \in \mathbb{Q}[\![q]\!]$ by the change of variables $y = e^{\hbar}$:

$$\sum_{d\geq 1} \sum_{j=0}^{\dim_{\mathbb{R}} M_d} b_j(M_d) y^{\frac{j}{2}} q^d = \sum_{g\geq 0} F_g(q) \hbar^{2g-1} \,.$$

Main result

Theorem [B., B.-Fan-Guo-Wu]

Let $q = e^{2i\pi\tau}$.

- For every $g \geq 0$, $F_g(\tau)$ is a quasimodular function of weight 0 for $\Gamma_1(3)$. Moreover, for every $g \geq 2$, we have

$$F_g \in C^{-(2g-2)} \cdot \mathbb{Q}[A, B, C]_{6g-6} \,.$$

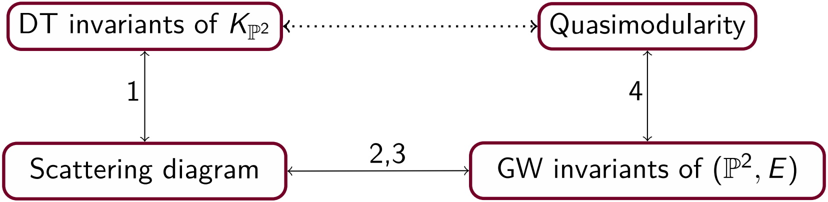
- This proves a conjecture from the physics of string theory on $\mathbb{R}^4 \times K_{\mathbb{P}^2}$ (Huang-Klemm, 2010)

Example

$$F_2 = \frac{1}{11520 C^2}(-37A^6 + 5A^4 B + 48 A^3 C - 16 C^2) \,.$$

Outline of the proof

- Relate DT invariants to **Gromov–Witten (GW) invariants**: counts of degree d, genus g curves satisfying a given list of constraints.



[1] *Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$*, Bousseau, accepted for publication in the **Journal of Algebraic Geometry**.

[2] *Refined tropical curve counting from higher genera and lambda classes*, Bousseau, **Inventiones Mathematicae**, 215(1), 1-79, 2019

[3] *A proof of N.Takahashi's conjecture for $(\mathbb{P}^2, E)$ and a refined sheaves/Gromov-Witten correspondence*, Bousseau, **Duke Mathematical Journal**, 172.15 (2023): 2895-2955

[4] *Holomorphic anomaly equation for $(\mathbb{P}^2, E)$ and the Nekrasov-Shatashvili limit of local $\mathbb{P}^2$*, Bousseau, Fan, Guo, Wu, **Forum of Mathematics, Pi**, 9, E3, 2021

Scattering diagrams in $\mathbb{R}^2$

- G : a group.

Definition

A **scattering diagram** in $\mathbb{R}^2$ is a collection of pairs $(\mathfrak{d}, f_{\mathfrak{d}})$ given by

- $\mathfrak{d}$: rays/lines in $\mathbb{R}^2$, called **walls**
- $f_{\mathfrak{d}}$: an element of G, called **wall-crossing transformations**.

- In the framework of mirror symmetry (Kontsevich-Soibelman, Gross-Siebert): scattering diagrams with G group of formal automorphisms of the torus $(\mathbb{C}^*)^2$
  - Encode data of genus 0 Gromov–Witten invariants.
- In our set-up I work with "quantum scattering diagrams": G group associated to a quantum torus Lie algebra.
  - Encode data of DT invariants and higher genus Gromov–Witten invariants.

Example: the initial scattering diagram

- Initial walls $\mathfrak{d}$: lines of slope n corresponding to line bundles $\mathcal{O}(n)$ on $\mathbb{P}^2$.
- Initial wall crossing automorphisms $f_{\mathfrak{d}}$: power series with coefficients DT invariants.

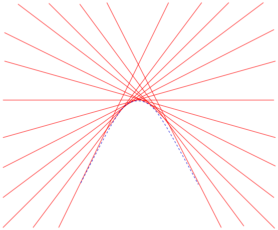


Figure: The initial scattering diagram for $\mathbb{P}^2$

- To obtain the data of all DT invariants: need to construct a **consistent scattering diagram** out of this initial scattering diagram.

Consistent scattering diagram

Definition

A scattering diagram is called **consistent** if for any point where walls intersect, the composition of wall-crossing transformations on all walls passing through this point is the identity.

- **Algorithm**: Systematically insert walls to complete the initial scattering diagram to a consistent scattering diagram.
  - This is a purely algebraic algorithm.

The consistent scattering diagram

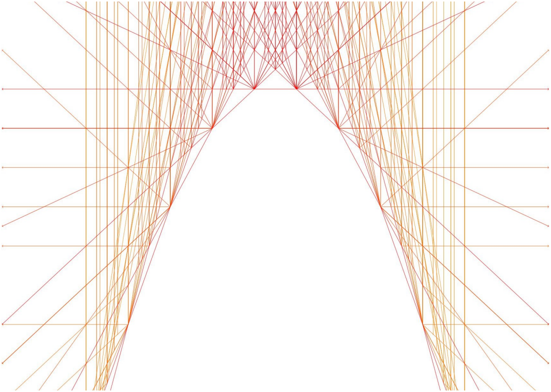


Figure: The consistent scattering diagram for $\mathbb{P}^2$

- New walls : coherent sheaves on $\mathbb{P}^2$, attached wall-crossing transformations encode refined DT invariants.

Theorem [B.][1]

Consider any wall inserted to the initial scattering diagram for $\mathbb{P}^2$ to complete it to a consistent one. Then, the attached wall crossing transformation is a power series with coefficients refined DT invariants.

- All Betti numbers $b_j(M_d)$ are computed algorithmically from the consistent scattering diagram for $\mathbb{P}^2$.

- Proof: Reconstruction of the consistent scattering diagram from the initial walls is analogous to the reconstruction of general coherent sheaves using resolution by a complex of direct sum of line bundles.

Example

L: line in $\mathbb{P}^2$, resolution:

$$0 \to \mathcal{O}(-1) \to \mathcal{O} \to \mathcal{O}_L \to 0$$

[1] *Scattering diagrams, stability conditions, and coherent sheaves on* $\mathbb{P}^2$, Bousseau, accepted for publication in the **Journal of Algebraic Geometry**.

Step 2: The scattering diagram for $\mathbb{P}^2$ and GW invariants

- **New geometry**: fix a smooth cubic curve E in $\mathbb{P}^2$.
- $GW_{g,d}$: count of genus g degree d curves in $\mathbb{P}^2$ intersecting E in a single point.
- **Idea**: Compute $GW_{g,d}$ using tropical geometry (combinatorial tools for enumerating curves using their piecewise-linear analogues)

Figure: A holomorphic cubic curve and its tropical analogue

- Notions of genus and degree for tropical analogues of holomorphic curves.

Step 2: The scattering diagram for $\mathbb{P}^2$ and GW invariants

Theorem [B.][1,2]

The Gromov–Witten invariants $GW_{g,d}$ of degree d genus g curves in $\mathbb{P}^2$ touching $E \subset \mathbb{P}^2$ at a single point equal to the count of tropical curves of genus g, degree d.

- Generalization of the tropical correspondence theorem between counts of genus zero curves in toric situations (Mikhalkin, Nishinou-Siebert).

Theorem [B.][1,2]

The Gromov–Witten invariant $GW_{g,d}$ can be computed algorithmically from the consistent scattering diagram for $\mathbb{P}^2$.

- Proof: the consistent scattering diagram for $\mathbb{P}^2$ contains all tropical curves of degree d and genus g.

[1] *Refined tropical curve counting from higher genera and lambda classes*, **Inventiones Mathematicae**, 215(1), 1-79, 2019

[2] *A proof of N.Takahashi's conjecture for* $(\mathbb{P}^2, E)$ *and a refined sheaves/Gromov-Witten correspondence*, Bousseau, **Duke Mathematical Journal**, 172.15 (2023): 2895-2955

Combining the first two steps we obtain:

Theorem [B.][1]

For every $g \geq 0$, we obtain the generating series F_g defined from refined DT invariants for $K_{\mathbb{P}^2}$

$$F_g = \sum_{d \geq 1} GW_{g,d} q^d \,.$$

[1] *A proof of N.Takahashi's conjecture for* $(\mathbb{P}^2, E)$ *and a refined sheaves/Gromov-Witten correspondence*, Bousseau, **Duke Mathematical Journal**, 172.15 (2023): 2895-2955

Step 3: Modularity from the Gromov-Witten side

Theorem [B.-Fan-Guo-Wu][2]

For every $g \geq 0$, $\sum_{d\geq 1} GW_{g,d} q^d$ is quasimodular of weight 0 for $\Gamma_1(3)$.

Proof uses

- Expressing $GW_{g,d}$ in terms of GW invariants of the elliptic curve and $K_{\mathbb{P}^2}$.
- Modularity results for Gromov-Witten invariants of the elliptic curve (Okounkov-Pandharipande, 2003).
- Modularity results for Gromov-Witten invariants of $K_{\mathbb{P}^2}$ (Lho-Pandharipande, Coates-Iritani, 2018).

Conclusion

Theorem [B., B.-Fan-Guo-Wu]

For every $g \geq 0$, the generating series F_g defined from the refined DT invariants of $K_{\mathbb{P}^2}$ using the change of variables $y = e^{\hbar}$ is quasimodular of weight 0 for $\Gamma_1(3)$.

- New appearance of modular functions in DT theory.
- Future questions:
  - More general Calabi-Yau 3-folds X? Expectation: replace $\Gamma_1(3)$ with $\mathrm{Aut}(D^b\mathrm{Coh}(X))$.
  - Further refinement F_{g_1,g_2} using the perverse filtration on $H^\star(M_d,\mathbb{Q})$ induced by the fibration $\pi\colon M_d \to B_d$?

Thank you for your attention !