The extended Ablowitz-Ladik hierarchy and a generalized Frobenius manifold

Youjin Zhang

Tsinghua University

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1. Introduction

The Ablowitz-Ladik equation

$$ir_{n,t} = r_{n+1} + r_{n-1} - 2r_n \pm r_n^* r_n (r_{n+1} + r_{n-1})$$

was proposed by Ablowitz and Ladik as a discretization of the nonlinear Schrödinger equation and was solved by the inverse scattering method in 1975. It has close relations with some other important soliton equations such as the 2D Toda lattice equation, the relativistic Toda lattice equation and the Toeplitz lattice equation. It also characterizes some elementary geometric properties of the motion of discrete curves on the N-dimensional spheres. Our study of this equation is motivated by its relation with the Gromov-Witten invariants of local $\mathbb{CP}^1$ that is discovered by Andrea Brini in 2010.

The Ablowitz-Ladik equation can be represented by the compatibility condition of the following linear system:

$$\Psi_{n+1} = L_n \Psi_n, \quad \Psi_{n,t} = M_n \Psi_n,$$

where

$$\begin{aligned} L_n &= \begin{pmatrix} \lambda & q_n \\ r_n & \lambda^{-1} \end{pmatrix}, \quad q_n = \mp r_n^*, \\ M_n &= M_{n,-1} + M_{n,0} + M_{n,1} \\ &= i \begin{pmatrix} \lambda^2 - q_n r_{n-1} & q_n \lambda \\ r_{n-1}\lambda & 0 \end{pmatrix} + i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + i \begin{pmatrix} 0 & -q_{n-1}\lambda^{-1} \\ -r_n \lambda^{-1} & -\lambda^{-2} + q_{n-1} r_n \end{pmatrix} \end{aligned}$$

The flow $\frac{\partial}{\partial t}$ can be represented as sum of the flows $\frac{\partial}{\partial t_{-1}}$, $\frac{\partial}{\partial t_0}$, $\frac{\partial}{\partial t_1}$ given by the following evolution of the wave function:

$$\Psi_{n,t_{-1}} = M_{n,-1}\Psi_n, \quad \Psi_{n,t_0} = M_{n,0}\Psi_n, \quad \Psi_{n,t_1} = M_{n,1}\Psi_n.$$

We can introduce other positive flows $\frac{\partial}{\partial t_k}$ $(k \geq 1)$ and negative flows $\frac{\partial}{\partial \hat{t}_{-k}}$ $(k \geq 1)$ by imposing appropriate evolutionary conditions on the wave function, these flows form the positive and negative flows of the Ablowitz-Ladik hierarchy (Suris 2003).

We define the variables $q(x)$, $r(x)$ by applying an ϵ-interpolation to the discrete variables of q_n, r_n such that

$$q_n = q(x)|_{x=n\epsilon}, \quad r_n = r(x)|_{x=n\epsilon},$$

then the AL hierarchy can be rewritten as evolutionary equations of $q(x)$, $r(x)$.

Now we introduce a pair of new unknown functions

$$P = \frac{q}{q^-}, \quad Q = \frac{q}{q^-}(1 - q^- r^-),$$

here and in what follows we denote $f^+(x) = \Lambda f(x)$, $f^-(x) = \Lambda^{-1} f(x)$ with $\Lambda = e^{\epsilon \partial_x}$.

In terms of the functions (P, Q) the positive and negative flows of the AL hierarchy can be represented by the following Lax equations:

$$\frac{\partial L}{\partial t_k} = \frac{1}{\varepsilon(k)!}[(L^k)_+, L], \quad k \geq 1,$$
$$\frac{\partial L}{\partial t_{-k}} = \frac{(-1)^{k-1}(k-1)!}{\varepsilon}[(M^k)_-, L], \quad k \geq 1,$$

where the Lax operators L and M have the forms

$$L = (1 - Q\Lambda^{-1})^{-1}(\Lambda - P),$$
$$M = (\Lambda - P)^{-1}(1 - Q\Lambda^{-1}).$$

This integrable hierarchy is called the relativistic Toda hierarchy in the literature. We still call it the Ablowitz-Ladik hierarchy

The Ablowitz-Ladik hierarchy was shown by Adrea Brini to be related with the equivariant Gromov-Witten invariants of the resolved conifold

$$X = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

with anti-diagonal C^*-action on the fibers. He conjectured that the generating function of these Gromov-Witten invariants yields a tau function of the Ablowitz-Ladik hierarchy, and proved his conjecture at the genus one approximation. Andrea Brini, Guido Carlet, Paolo Rossi further showed that the dispersionless limit of the positive flows of the Ablowitz-Ladik hierarchy belong to the Principal Hierarchy of a generalized Frobenius manifold M_{AL} with non-flat unity, which is almost dual to the Frobenius manifold associated with the Gromov-Witten invariants of the resolved conifold with anti-diagonal action.

We are to construct a certain extension of the Ablowitz-Ladik hierarchy by including to it some additional flows, and to show that the extended Ablowitz-Ladik hierarchy possesses a set of Virasoro symmetries that act linearly on its tau function, and from this fact it follows that it is the topological deformation of the Principal Hierarchy of the generalized Frobenius manifold M_{AL}.

2. The Principal Hierarchy of a generalized Frobenius manilold M_{AL}

The generalized Frobenius manifold that is related with the Ablowitz-Ladik hierarchy has the potential

$$F = \frac{1}{2}(v^1)^2 v^2 + v^1 e^{v^2} + \frac{1}{2}(v^1)^2 \log v^1$$

and flat metric

$$\eta = (\eta_{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The non-flat unity e and the Euler vector field E of M_{AL} are given by

$$e = \frac{v^1 \partial_{v^1} - \partial_{v^2}}{v^1 - e^{v^2}}, \quad E = v^1 \partial_{v^1} + \partial_{v^2},$$

On M_{AL} we have the deformed flat connection $\widetilde{\nabla}$ defined by

$$\widetilde{\nabla}_a b = \nabla_a b + z a \cdot b, \quad \forall a, b \in \mathrm{Vect}(M),$$

where ∇ is the Levi-Civita connection of the flat metric η, and the multiplication $a \cdot b$ of the vector fields a, b is defined by $\partial_\alpha \cdot \partial_\beta = c^\gamma_{\alpha\beta} \partial_\gamma$ with $\partial_\alpha = \frac{\partial}{\partial v^\alpha}$ and

$$c^\gamma_{\alpha\beta} = \eta^{\gamma\xi} \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\xi}, \quad (\eta^{\alpha\beta}) = (\eta_{\alpha\beta})^{-1}.$$

This deformed connection can be extended to a flat connection on $M_{\mathsf{AL}} \times \mathbb{C}^*$ by the formulae

$$\widetilde{\nabla}_{\frac{\mathrm{d}}{\mathrm{dz}}} b = \partial_z b + E \cdot b - \frac{1}{z} \mu v,$$

where

$$\mu = \frac{2-d}{2} - \nabla E = \mathrm{diag}\left(-\frac{1}{2}, \frac{1}{2}\right).$$

A basis of the horizontal sections of the connection $\widetilde{\nabla}$ yields a system of deformed flat coordinates $(\tilde{v}^1, \tilde{v}^2)$ of M_{AL}, which can be chosen to take the form

$$(\tilde{v}_1, \tilde{v}_2) = (\theta_1(v, z), \theta_2(v, z))z^{\mu} z^R,$$

where the constant matrix

$$R = R_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$$

is part of the monodromy data of M_{AL} at $z = 0$, and the functions $\theta_\alpha(v, z)$ are analytic at $z = 0$ and have the expansions

$$\theta_\alpha(v, z) = \sum_{k \geq 0} \theta_{\alpha,k}(v) z^k, \quad \theta_{\alpha,0}(v) = v_\alpha := \eta_{\alpha\gamma} v^\gamma, \quad \alpha = 1, 2.$$

The coefficients $\theta_{\alpha,k}(v)$ satisfy the equations

$$\partial_\gamma\partial_\beta\theta_{\alpha,k+1} = c^{\varepsilon}_{\gamma\beta}\partial_\varepsilon\theta_{\alpha,k}, \quad \alpha,\beta,\gamma = 1,2,\ k\geq 0;$$
$$\partial_E\nabla\theta_{\alpha,k} = (k+\mu_\alpha-\mu)\nabla\theta_{\alpha,k} + (R_1)^{\varepsilon}_{\alpha}\nabla\theta_{\varepsilon,k-1}, \quad \alpha,\gamma = 1,2,\ k\geq 1.$$

To define the Principal Hierarchy of M_{AL} we need to introduce another set of functions $\theta_{0,\ell}$ for $\ell\in\mathbb{Z}$. They are determined by

$$\theta_{0,0} = \varphi = v^2 - \log\left(e^{v^2} - v^1\right),$$

and the relations

$$\partial_\alpha\partial_\beta\theta_{0,\ell} = c^{\gamma}_{\alpha\beta}\partial_\gamma\theta_{0,\ell-1}, \quad \alpha,\beta = 1,2,\ \ell\in\mathbb{Z};$$
$$\partial_E\theta_{0,k} = k\theta_{0,k} + \theta_{2,k-1}, \quad \partial_E\theta_{0,-k} = -k\theta_{0,-k}, \quad k\geq 1,$$

here the function φ has the propert that its gradients w.r.t. the flat metric and the intersection form of M_{AL} yield the unity and the Euler vector fields.

The first few of them are given by

$$\theta_{1,1} = (v^2 + \log v^1)v^1 + (e^{v^2} - v^1), \quad \theta_{2,1} = v^1 e^{v^2} + \frac{1}{2}(v^1)^2,$$
$$\theta_{1,2} = (v^2 + \log v^1)\theta_{2,1} + \frac{1}{4}(e^{2v^2} - 4v^1 e^{v^2} - (v^1)^2),$$
$$\theta_{2,2} = \frac{1}{2}v^1 e^{2v^2} + (v^1)^2 e^{v^2} + \frac{1}{6}(v^1)^3,$$
$$\theta_{0,1} = v^1 v^2, \quad \theta_{0,2} = \frac{1}{2}(v^2 + 1)(v^1)^2 + (v^2 - 1)e^{v^2} v^1,$$
$$\theta_{0,-1} = -\frac{v^1}{(v^1 - e^{v^2})^2}, \quad \theta_{0,-2} = \frac{2\theta_{2,1}}{(v^1 - e^{v^2})^4}.$$

The Principal Hierarchy of M_{AL} consists of the following Hamiltonian systems of hydrodynamic type:

$$\frac{\partial v^\alpha}{\partial t^{\beta,k}} = \eta^{\alpha\varepsilon}\frac{\partial}{\partial x}\left(\frac{\partial\theta_{\beta,k+1}(v)}{\partial v^\varepsilon}\right) = \left\{v^\alpha(x), H^{[0]}_{\beta,k}\right\}_0, \quad \alpha,\beta = 1,2,\ k\geq 0,$$
$$\frac{\partial v^\alpha}{\partial t^{0,\ell}} = \eta^{\alpha\varepsilon}\frac{\partial}{\partial x}\left(\frac{\partial\theta_{0,\ell+1}(v)}{\partial v^\varepsilon}\right) = \left\{v^\alpha(x), H^{[0]}_{0,\ell}\right\}_0, \quad \alpha = 1,2,\ \ell\in\mathbb{Z}.$$

Here the Poisson bracket $\{\cdot\,,\cdot\}_0$ is given by the Hamiltonian operator

$$\mathcal{P}^{[0]}_0 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix},$$

and the Hamiltonians are given by

$$H^{[0]}_{\beta,k} = \int \theta_{\beta,k+1}(v(x))\,\mathrm{d}x, \quad H^{[0]}_{0,\ell} = \int \theta_{0,\ell+1}(v(x))\,\mathrm{d}x.$$

These flows are mutually commutative, and they satisfy the following bihamitonian recursion relations:

$$\begin{aligned}
\left\{v^\alpha(x), H^{[0]}_{2,p-1}\right\}_1 &= (p+1)\left\{v^\alpha(x), H^{[0]}_{2,p}\right\}_0,\\
\left\{v^\alpha(x), H^{[0]}_{1,p-1}\right\}_1 &= p\left\{v^\alpha(x), H^{[0]}_{1,p}\right\}_0 + 2\left\{v^\alpha(x), H^{[0]}_{2,p-1}\right\}_0,\\
\left\{v^\alpha(x), H^{[0]}_{0,p-1}\right\}_1 &= p\left\{v^\alpha(x), H^{[0]}_{0,p}\right\}_0 + \{v^\alpha(x), H^{[0]}_{2,p-1}\}_0,\\
\left\{v^\alpha(x), H^{[0]}_{0,-p-1}\right\}_1 &= -p\left\{v^\alpha(x), H^{[0]}_{0,-p}\right\}_0,
\end{aligned}$$

where the Poisson bracket $\{\cdot\,,\cdot\}_1$ is given by the Hamiltonian operator

$$\mathcal{P}^{[0]}_1 = \begin{pmatrix} 2v^1 e^{v^2}\partial_x + (v^1 e^{v^2})_x & (v^1 + e^{v^2})\partial_x \\ (v^1 + e^{v^2})_x + (v^1 + e^{v^2})\partial_x & 2\partial_x \end{pmatrix}.$$

The functions $\theta_{\alpha,k}$, $\alpha = 1, 2, k \geq 0$ and $\theta_{0,\ell}$, $\ell \in \mathbb{Z}$ can be represented in terms of the superpotential

$$\lambda(p) = p + v^1 + v^1 e^{v^2} \left(p - e^{v^2}\right)^{-1}$$

of M_{AL}. To this end, we need to expand $\log(\lambda(p))$ in two ways. The first way is to assume $|v^1| < |p - e^{v^2}| < |e^{v^2}|$ and expand $\log(\lambda(p))$ as follows:

$$\log^+(\lambda(p)) = v^2 + \log\left(1 + \frac{p - e^{v^2}}{e^{v^2}}\right) + \log\left(1 + \frac{v^1}{p - e^{v^2}}\right)$$

The second way is to assume $|e^{v^2}| < |p - e^{v^2}| < |v^1|$ and and expand $\log(\lambda)$ as follows:

$$\log^-(\lambda(p)) = \log v^1 + \log\left(1 + \frac{p - e^{v^2}}{v^1}\right) + \log\left(1 + \frac{e^{v^2}}{p - e^{v^2}}\right).$$

Proposition

We have the following formulae:

$$\theta_{0,-k} = (-1)^{k-1} k! \, \mathrm{Res}_{p=0} \, \lambda(p)^{-k} \frac{\mathrm{d}p}{p},$$

$$\theta_{0,k} = \frac{1}{k!} \mathrm{Res}_{p=e^{v^2}} \, \lambda^k(p) \left(\log^+(\lambda(p)) - H_k\right) \frac{\mathrm{d}p}{p},$$

$$\theta_{1,k} = \frac{1}{k!} \mathrm{Res}_{p=e^{v^2}} \, \lambda^k(p) \left(\log^+(\lambda(p) + \log^-(\lambda(p)) - 2H_k\right) \frac{\mathrm{d}p}{p},$$

$$\theta_{2,k} = -\frac{1}{(k+1)!} \mathrm{Res}_{p=\infty} \, \lambda^{k+1}(p) \frac{\mathrm{d}p}{p},$$

here $k \geq 1$ *and* $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

The Principal Hierarchy possesses a tau cover

$$\frac{\partial f^{[0]}}{\partial t^{\alpha,k}} = f^{[0]}_{\alpha,k}, \quad \frac{\partial f^{[0]}_{\alpha,k}}{\partial t^{\beta,\ell}} = \Omega^{[0]}_{\alpha,k;\beta,\ell}, \quad \frac{\partial v^{\gamma}}{\partial t^{\alpha,k}} = \eta^{\gamma\varepsilon}\partial_x\Omega^{[0]}_{\varepsilon,0;\alpha,k},$$

where $\gamma = 1, 2$, the indices $(\alpha, k), (\beta, \ell)$ belong to the set

$$\mathcal{I} = \{(\xi, q) \mid \xi = 1, 2,\ q \geq 0\} \cup \{(0, q) \mid q \in \mathbb{Z}\},$$

and the functions $\Omega_{\alpha,k;\beta,\ell}$ can be defined in terms of the gradients of the functions $\theta_{\alpha,p}$. These functions satisfy the relations

$$\partial_x\Omega^{[0]}_{\alpha,k;\beta,\ell} = \frac{\partial\theta_{\alpha,k}}{\partial t^{\beta,\ell}} = \frac{\partial\theta_{\beta,\ell}}{\partial t^{\alpha,k}}, \quad \Omega^{[0]}_{\alpha,k;\beta,\ell} = \Omega^{[0]}_{\beta,\ell;\alpha,k}, \quad \Omega^{[0]}_{0,0;\alpha,k} = \theta_{\alpha,k}.$$

For a solution $(f^{[0]}, f^{[0]}_{\alpha,k}, v^{\gamma})$ of the tau cover, we call $\tau^{[0]} = e^{f^{[0]}}$ the tau function for the solution $v^1(t), v^2(t)$ of the Principal Hierarchy.

The tau cover of the Principal Hierarchy possesses an infinite number of Virasoro symmetries, they can be represented in terms of the Virasoro operators of the form

$$L_m = a_m^{\alpha,p;\beta,q}\varepsilon^2\frac{\partial^2}{\partial t^{\alpha,p}\partial t^{\beta,q}} + b_{m;\beta,q}^{\alpha,p}t^{\beta,q}\frac{\partial}{\partial t^{\alpha,p}} + \frac{1}{\varepsilon^2}c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q} + \kappa\delta_{m,0}, \quad m \geq -1.$$

These operators satisfy the Virasoro commutation relations

$$[L_m, L_n] = (m-n)L_{m+n}, \quad m, n \geq -1.$$

The actions of these Virasoro symmetries on the tau cover of the Principal Hierarchy of M_{AL} are given by

$$\frac{\partial f^{[0]}}{\partial s_m} = a_m^{\alpha,p;\beta,q}f^{[0]}_{\alpha,p}f^{[0]}_{\beta,q} + b_{m;\beta,q}^{\alpha,p}t^{\beta,q}f^{[0]}_{\alpha,p} + c_{m;\alpha,p;\beta,q}t^{\alpha,p}t^{\beta,q}.$$

The topological deformation of the Principal Hierarchy of M_{AL} is constructed via a quasi-Miura transformation of the form

$$v^\alpha \to w^\alpha = v^\alpha + \frac{\partial^2}{\partial t^{1,0}\partial t^{\alpha,0}}\left(\sum_{k\geq 1}\varepsilon^k \mathcal{F}_k(v, v_x, \dots, v^{(m_k)})\right), \quad \alpha = 1, \dots, n$$

which is induced from the a deformation of the tau function

$$\log\tau^{[0]} \to \log\tau = \varepsilon^{-2}\log\tau^{[0]} + \sum_{k\geq 1}\varepsilon^{k-2}\mathcal{F}_k(v, v_x, \dots, v^{(m_k)})$$

of the Principal Hierarchy, we require that the action of the Virasoro symmetries of the Principal Hierarchy on the deformed tau function are given by

$$\frac{\partial\tau}{\partial s_m} = L_m\tau, \quad m \geq -1.$$

For a general semisimple generalized Frobenius manifold with non-flat unity, the Principal Hierarchy and its topological deformation were constructed by Si-Qi Liu, Haonan Qu & Z.

3. Deformation of the Principal Hierarchy of M_{AL}

We write the positive and negative flows of the Ablowitz-Ladik hierarchy in the form

$$\frac{\partial L}{\partial t^{2,p}} = \frac{1}{\varepsilon(p+1)!}[(L^{p+1})_+, L], \quad p \geq 0,$$
$$\frac{\partial L}{\partial t^{0,-p}} = \frac{(-1)^{p-1}(p-1)!}{\varepsilon}[(M^p)_-, L], \quad p \geq 1,$$

where the operators L, M are given by

$$L = (1 - Q\Lambda^{-1})^{-1}(\Lambda - P), \quad M = (\Lambda - P)^{-1}(1 - Q\Lambda^{-1}).$$

Introduce the variables

$$w^1 = Q - P, \quad w^2 = \log Q.$$

Then the positive and negative flows are tau-symmetric deformations of the flows $\frac{\partial}{\partial t^{2,p}}$ $(p \geq 0)$ and $\frac{\partial}{\partial t^{0,-p}}$ $(p \geq 1)$ of the Principal Hierarchy of M_{AL}.

The above flows can be represented as bihamiltonian systems of the form:

$$\frac{\partial}{\partial t^{2,p}}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} \frac{\delta H_{2,p}}{\delta w^1} \\ \frac{\delta H_{2,p}}{\delta w^2} \end{pmatrix} = \frac{1}{p+1}\mathcal{P}_1 \begin{pmatrix} \frac{\delta H_{2,p-1}}{\delta w^1} \\ \frac{\delta H_{2,p-1}}{\delta w^2} \end{pmatrix}, \quad p \geq 0,$$

$$\frac{\partial}{\partial t^{0,-p}}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \mathcal{P}_0 \begin{pmatrix} \frac{\delta H_{0,-p}}{\delta w^1} \\ \frac{\delta H_{0,-p}}{\delta w^2} \end{pmatrix} = -\frac{1}{k}\mathcal{P}_1 \begin{pmatrix} \frac{\delta H_{0,-p-1}}{\delta w^1} \\ \frac{\delta H_{0,-p-1}}{\delta w^2} \end{pmatrix}, \quad p \geq 1,$$

where Hamiltonian operators are given by

$$\mathcal{P}_0 = \varepsilon^{-1}\begin{pmatrix} 0 & \Lambda - 1 \\ 1 - \Lambda^{-1} & 0 \end{pmatrix},$$

$$\mathcal{P}_1 = \varepsilon^{-1}\begin{pmatrix} -e^{w^2}\Lambda^{-1}w^1 + w^1\Lambda e^{w^2} & w^1(\Lambda - 1) + e^{w^2}(1 - \Lambda^{-1}) \\ (1 - \Lambda^{-1})w^1 + (\Lambda - 1)e^{w^2} & \Lambda - \Lambda^{-1} \end{pmatrix},$$

The Hamiltonians

$$H_{2,p} = \int h_{2,p+1}\,\mathrm{d}x, \quad H_{0,-q} = \int h_{0,-q+1}\,\mathrm{d}x, \quad p \geq -1,\ q \geq 1$$

have the densities

$$h_{2,p} = \frac{1}{(p+1)!}\,\mathrm{Res}\,L^{p+1}, \quad p \geq 0,$$

$$h_{0,-q} = \begin{cases} \frac{\varepsilon\partial_x}{1-\Lambda^{-1}}\left(w^2 - \log(e^{w^2} - w^1)\right), & q = 0, \\ (-1)^q (q-1)!\,\mathrm{Res}\,M^q, & q \geq 1, \end{cases}$$

which satisfy the tau-symmetry condition

$$\frac{\partial h_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q}}{\partial t^{\alpha,p}}.$$

Now let us consider the deformation of the flows $\frac{\partial}{\partial t^{0,p}}$ $(p \geq 0)$ and $\frac{\partial}{\partial t^{1,p}}$ $(p \geq 0)$ of the Principal Hierarchy of M_{AL}. Since the bihamiltonian structure $(\mathcal{P}_0, \mathcal{P}_1)$ is a deformation of that of the Principal Hierarchy of M_{AL}, there exist bihamiltonian conserved quantities

$$H_{0,p} = \int h_{0,p+1}(w, w_x, \dots)\, \mathrm{d}x, \quad H_{1,p} = \int h_{1,p+1}(w, w_x, \dots)\, \mathrm{d}x, \quad p \geq -1$$

of the bihamiltonian structure $(\mathcal{P}_0, \mathcal{P}_1)$ with densities of the form

$$h_{\alpha,p}(w, w_x, \dots) = \theta_{\alpha,p}(w) + \varepsilon h^{[1]}_{\alpha,p}(w, w_x) + \varepsilon^2 h^{[2]}_{\alpha,p}(w, w_x, w_{xx}) + \dots, \quad \alpha = 0, 1,\ p \geq 0.$$

We thus obtain a deformation of the Principal Hierarchy of M_{AL} (the extended Ablowitz-Ladik hierarchy):

$$\frac{\partial w^\alpha}{\partial t^{\beta,q}} = \mathcal{P}_0^{\alpha\gamma} \frac{\delta H_{\beta,p}}{\delta w^\gamma}, \quad \alpha = 1, 2,\ (\beta, q) \in \mathcal{I},$$

with

$$\mathcal{I} = \{(\xi, q) \mid \xi = 1, 2,\ q \geq 0\} \cup \{(0, q) \mid q \in \mathbb{Z}\}.$$

Proposition

The densities $h_{0,p}(w, w_x, \dots), h_{1,p}(w, w_x, \dots)$ *of the Hamiltonians* $H_{0,p-1}$ *and* $H_{1,p-1}$ $(p \geq 0)$ *can be chosen to satisfy the tau-symmetry conditions*

$$\frac{\partial h_{\alpha,p}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q}}{\partial t^{\alpha,p}}, \quad (\alpha, p), (\beta, q) \in \mathcal{I}.$$

This deformation of the Principal Hierarchy has a tau structure defined by

$$\frac{1}{\varepsilon}(\Lambda - 1)\Omega_{\alpha,p;\beta,q} = \frac{\partial h_{\alpha,p}}{\partial t^{\beta,q}}, \quad (\alpha, p), (\beta, q) \in \mathcal{I}.$$

$$\Omega_{\alpha,p;\beta,q}(w, w_x, \dots)|_{w^\alpha = w^\alpha(t)} = \epsilon^2 \frac{\partial \log \tau(t)}{\partial t^{\alpha,p} \partial t^{\beta,q}}, \quad (\alpha, p), (\beta, q) \in \mathcal{I}.$$

From this definition of the tau function, we have relations

$$h_{\alpha,p} = \epsilon(\Lambda - 1)\frac{\partial \log \tau}{\partial t^{\alpha,p}},$$

We have the explicit expressions of the flows $\frac{\partial}{\partial t^{1,0}}$, $\frac{\partial}{\partial t^{2,0}}$, $\frac{\partial}{\partial t^{0,-1}}$, $\frac{\partial}{\partial t^{0,0}}$ in terms of the unknown functions P, Q:

$$\varepsilon \frac{\partial P}{\partial t^{2,0}} = P(Q^+ - Q), \quad \varepsilon \frac{\partial Q}{\partial t^{2,0}} = Q(Q^+ - Q^- - P + P^-),$$

$$\varepsilon \frac{\partial P}{\partial t^{0,-1}} = \frac{Q^+}{P^+} - \frac{Q}{P^-}, \quad \varepsilon \frac{\partial Q}{\partial t^{0,-1}} = \frac{Q}{P} - \frac{Q}{P^-},$$

$$e^{\varepsilon\left(\frac{\partial}{\partial t^{1,0}} - \frac{\partial}{\partial x}\right)} P = \frac{P^-(Q^+ - P)}{Q - P^-}, \quad e^{\varepsilon\left(\frac{\partial}{\partial t^{1,0}} - \frac{\partial}{\partial x}\right)} Q = \frac{Q(Q^+ - P)}{Q - P^-}.$$

$$\frac{\partial P}{\partial t^{0,0}} = P_x, \quad \frac{\partial Q}{\partial t^{0,0}} = Q_x.$$

where $P^\pm = \Lambda^{\pm 1} P$, $Q^\pm = \Lambda^{\pm 1} Q$.

We note that the exponential of the flow $\frac{\partial}{\partial t^{1,0}} - \frac{\partial}{\partial x}$ yields an auto-Bäcklund transformation of the extended Ablowitz-Ladik hierarchy.

4. A super extension of the extended Ablowitz-Ladik hierarchy

In order to show that the extended Ablowitz-Ladik hierarchy possesses an infinite set of Virasoro symmetries which act linearly on its tau function, we need to construct a super tau-cover of this integrable hierarchy. To this end, we first recall some notations associated with the infinite jet space of a super manifold.

Let M be a smooth manifold of dimension n, and $\hat{M} = \prod(T^*M)$ be the super manifold of dimension $(n|n)$ obtained from the cotangent bundle of M by reversing the parity of its fibers.

Let $(U; u^1, \dots, u^n)$ be a system of local coordinates of M, then the dual coordinates $\theta_1, \dots, \theta_n$ on the fibers of $\hat{U} \subset \hat{M}$ satisfy the relations

$$\theta_i\theta_j + \theta_j\theta_i = 0, \quad i, j = 1, \dots, n.$$

Let $J^\infty(\hat{M})$ be the infinite jet space of $\hat{M}$, and $\hat{\mathcal{A}}$ be space of differential polynomials on $\hat{M}$. Locally they can be represented in the forms

$$\hat{\mathcal{A}} = C^\infty(\hat{U})[[u^{i,s}, \theta_i^t \mid i = 1, \dots n; s \geq 1, t \geq 0]],$$

where $u^{i,0} = u^i$, $\theta_i^0 = \theta_i$. By using the global vector field

$$\partial_x = \sum_{i=1}^{n} \sum_{s \geq 0} \left(u^{i,s+1} \frac{\partial}{\partial u^{i,s}} + \theta_i^{s+1} \frac{\partial}{\partial \theta_i^s} \right)$$

on $J^\infty(\hat{M})$ we define the space of local functionals as the quotient space

$$\hat{\mathcal{F}} = \hat{\mathcal{A}} / \partial_x \hat{\mathcal{A}},$$

and denote by $\int$ the projection operator.

There are two gradations on $\hat{\mathcal{A}}$ which are defined by

$$\deg_x u^{i,s} = \deg_x \theta_i^s = s; \quad \deg_\theta u^{i,s} = 0,\ \deg_\theta \theta_i^s = 1.$$

We call them the differential gradation and super gradation of $\hat{\mathcal{A}}$ repsectively, which induce two gradations on $\hat{\mathcal{F}}$, and we denote the corresponding homogeneous spaces by

$$\hat{\mathcal{F}}_d = \{f \in \hat{\mathcal{F}} \,|\, \deg_x f = d\}, \quad \hat{\mathcal{F}}^p = \{f \in \hat{\mathcal{F}} \,|\, \deg_\theta f = p\}, \quad \hat{\mathcal{F}}_d^p = \hat{\mathcal{F}}_d \cap \hat{\mathcal{F}}^p.$$

We equip the space of local functionals a graded Lie algebra structure by using the following Schouten-Nijenhuis bracket:

$$[F, G] = \int \left(\frac{\delta F}{\delta \theta_i} \frac{\delta G}{\delta u^i} + (-1)^p \frac{\delta F}{\delta u^i} \frac{\delta G}{\delta \theta_i} \right), \quad F \in \hat{\mathcal{F}}^p,\ G \in \hat{\mathcal{F}}^q.$$

For any local functional $X = \int X^i\theta_i \in \hat{\mathcal{F}}^1$, we can associate with it an evolutionary PDEs of the form

$$\frac{\partial u^i}{\partial t} = X^i, \quad i = 1, \dots, n,$$

here we need to make the replacement $u^{i,s} \mapsto \partial_x^s u^i$.

We call $X \in \hat{\mathcal{F}}^1$ a Hamiltonian evolutionary PDE if there exist $I \in \hat{\mathcal{F}}^2$ and $H \in \hat{\mathcal{F}}^0$ such that

$$X = [H, I], \quad [I, I] = 0.$$

Here I and H are called the Hamiltonian structure and the Hamiltonian of X respectively.

We can represent I and H in the form

$$I = \frac{1}{2}\int \sum_{s\geq 0} P_s^{ij}\theta_i\theta_j^s, \quad H = \int h(u, u_x, \dots),$$

then the Hamiltonian evolutionary PDEs can be represented as

$$\frac{\partial u^i}{\partial t} = \mathcal{P}^{ij}\frac{\delta H}{\delta u^j}, \quad \mathcal{P}^{ij} = \sum_{s\geq 0} P_s^{ij}\partial_x^s.$$

The evolutionary PDE X is called a bihamiltonian system if there exist $I_0, I_1 \in \hat{\mathcal{F}}^2$ and $H, G \in \hat{\mathcal{F}}^0$ such that

$$X = [H, I_0] = [G, I_1], \quad [I_0, I_0] = [I_1, I_1] = [I_0, I_1] = 0.$$

The Ablowitz-Ladik hierarchy corresponds to local functionals on the super manifold with local coordinates

$$u^1 = P, \quad u^2 = Q, \quad \theta_1, \quad \theta_2.$$

The local functionals correspond to the Hamiltonian operators $\mathcal{P}_0, \mathcal{P}_1$ are given by

$$I_0 = \frac{1}{2\varepsilon} \int \begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix} \begin{pmatrix} Q\Lambda^{-1} - \Lambda Q & (1-\Lambda)Q \\ Q(\Lambda^{-1} - 1) & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

$$I_1 = \frac{1}{2\varepsilon} \int \begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix} \begin{pmatrix} 0 & P(\Lambda - 1)Q \\ Q(1 - \Lambda^{-1})P & Q(\Lambda - \Lambda^{-1})Q \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

For each local functional $F \in \hat{\mathcal{F}}^p$ there is associated a derivation $D_F \in \mathrm{Der}(\hat{\mathcal{A}})$ which is defined by

$$D_F = \sum_{i=1}^{n} \sum_{s \geq 0} \partial_x^s \left(\frac{\delta F}{\delta \theta_i} \right) \frac{\partial}{\partial u^{i,s}} + (-1)^p \partial_x^s \left(\frac{\delta F}{\delta u^i} \right) \frac{\partial}{\partial \theta_i^s}.$$

These derivations satisfy the following relations:

$$[F, G] = \int D_F(\tilde{G}), \quad \forall\, F \in \hat{\mathcal{F}}^p,\ G = \int \tilde{G} \in \hat{\mathcal{F}}^q,$$

$$(-1)^{p-1} D_{[F,G]} = D_F \circ D_G - (-1)^{q-1}(-1)^{p-1} D_G \circ D_F.$$

The derivations $D_{I_0}, D_{I_1} \in \mathrm{Der}(\hat{\mathcal{A}})$ given by the bihamiltonian structure (I_0, I_1) yield two odd flows

$$\frac{\partial u^\alpha}{\partial \tau_i} = \frac{\delta I_i}{\delta \theta_\alpha} = D_{I_i}(u^\alpha), \quad \frac{\partial \theta_\alpha}{\partial \tau_i} = \frac{\delta I_i}{\delta u^\alpha} = D_{I_i}(\theta_\alpha), \quad i = 0, 1.$$

They commute with each other, and satisfy the recursion relation

$$\frac{\partial u^\alpha}{\partial \tau_1} = \left(\mathcal{P}_1 \circ \mathcal{P}_0^{-1}\right)^\alpha_\gamma \frac{\partial u^\gamma}{\partial \tau_0} = R^\alpha_\gamma \frac{\partial u^\gamma}{\partial \tau_0},$$

We introduce the following family of odd variables:

$$\{\sigma^s_{\alpha,k} \mid \alpha = 1, 2; k \in \mathbb{Z}, s \geq 0\}.$$

We will also denote $\sigma^0_{\alpha,k}$ by $\sigma_{\alpha,k}$, and we require that $\sigma_{\alpha,0} = \theta_\alpha$. Let us replace the vector field ∂_x on $J^\infty(\hat{M})$ by

$$\partial_x = \sum_{s\geq 0}\left(P^{(s+1)}\frac{\partial}{\partial P^{(s)}} + Q^{(s+1)}\frac{\partial}{\partial Q^{(s)}}\right) + \sum_{i=1}^{2}\sum_{s\geq 0,\, k\in\mathbb{Z}} \sigma^{s+1}_{i,k}\frac{\partial}{\partial \sigma^s_{i,k}},$$

then the odd variables $\sigma^s_{\alpha,k}$ are required to satisfy the recursion relations

$$\mathcal{P}^{\alpha\beta}_0\sigma_{\beta,k+1} = \mathcal{P}^{\alpha\beta}_1\sigma_{\beta,k}.$$

Proposition

We have a super extension of the extended AL hierarchy consisting of the flows of itself and

1) *The odd flows correspond to the local and nonlocal Hamiltonian structures:*

$$\varepsilon\frac{\partial P}{\partial \tau_k} = P(\Lambda-1)Q\sigma_{2,k-1}, \quad \varepsilon\frac{\partial Q}{\partial \tau_k} = Q(\Lambda^{-1}-1)\sigma_{1,k},$$

$$\varepsilon\frac{\partial \sigma_{1,k+m}}{\partial \tau_k} = \sum_{i=0}^{m-1}\sigma_{1,k+i}(1-\Lambda)Q\sigma_{2,k+m-1-i}, \quad m\geq 1,$$

$$\frac{\partial \sigma_{1,k}}{\partial \tau_k} = 0, \quad \frac{\partial \sigma_{1,k}}{\partial \tau_{k+m}} = -\frac{\partial \sigma_{1,k+m}}{\partial \tau_k}, \quad m\geq 1,$$

$$\varepsilon\frac{\partial\left(Q\sigma_{2,k+m}\right)}{\partial \tau_k} = -Q\sum_{i=0}^{m}\sigma^{-}_{1,k+m-i}\sigma_{1,k+i}, \quad m\geq 0,$$

$$\frac{\partial\left(Q\sigma_{2,k}\right)}{\partial \tau_{k+1}} = 0, \quad \varepsilon\frac{\partial\left(Q\sigma_{2,k}\right)}{\partial \tau_{k+m}} = Q\sum_{i=1}^{m-1}\sigma^{-}_{1,k+m-i}\sigma_{1,k+i}, \quad m\geq 2.$$

Here $k\in\mathbb{Z}$*, and* $\sigma^{-}_{\alpha,k} = \Lambda^{-1}\sigma_{\alpha,k}$.

Proposition (continued)

2) *The evolutions of the odd unknown functions along the time variables $t^{\alpha,p}$ defined by*

$$\frac{\partial Q(\sigma_{1,k}+\sigma_{2,k})}{\partial t^{\beta,q}}=\frac{\partial \Omega_{2,0;\beta,q}}{\partial \tau_k},\quad \frac{\partial (P\sigma_{1,k}+Q\sigma_{2,k})}{\partial t^{\beta,q}}=\frac{\partial h_{\beta,q}}{\partial \tau_{k+1}}.$$

3) *We also have the super tau-cover of the extended AL hierarchy by adding the following flows:*

$$\varepsilon\frac{\partial f_{\alpha,p}}{\partial t^{\beta,q}}=\Omega_{\alpha,p;\beta,q},\quad \varepsilon\frac{\partial f_{\alpha,p}}{\partial \tau_n}=\psi^n_{\alpha,p},$$
$$\frac{\partial \psi^n_{\alpha,p}}{\partial t^{\beta,q}}=\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial \tau_n},\quad \frac{\partial \psi^n_{\alpha,p}}{\partial \tau_m}=\Delta^{n,m}_{\alpha,p},\quad (\alpha,p)\in\mathcal{I},\, n\in\mathbb{Z},$$

All these flows are well defined and commute with each other.

5. Virasoro symmetries of the extended Ablowitz-Ladik hierarchy

We have the following theorem.

Theorem (Liu, Wang, Z. 2024)

The extended Ablowitz-Ladik hierarchy has a set of Virasoro symmetries which act linearly on its tau function, i.e.,

$$\frac{\partial \tau}{\partial s_m} = L_m \tau, \quad m \geq -1,$$

where the Virasoro operators L_m have the form

$$L_m = a_m^{\alpha,p;\beta,q} \varepsilon^2 \frac{\partial^2}{\partial t^{\alpha,p} \partial t^{\beta,q}} + b_{m;\beta,q}^{\alpha,p} t^{\beta,q} \frac{\partial}{\partial t^{\alpha,p}} + \frac{1}{\varepsilon^2} c_{m;\alpha,p;\beta,q} t^{\alpha,p} t^{\beta,q} + \kappa \delta_{m,0}, \quad m \geq -1.$$

These operators satisfy the Virasoro commutation relations

$$[L_m, L_n] = (m-n) L_{m+n}, \quad m, n \geq -1.$$

The explicit form of the first Virasoro operators are given by

$$L_{-1} = \sum_{k\geq 1}\left(t^{1,k}\frac{\partial}{\partial t^{1,k-1}} + t^{2,k}\frac{\partial}{\partial t^{2,k-1}}\right) + \sum_{p\in\mathbb{Z}} t^{0,p}\frac{\partial}{\partial t^{0,p-1}} + \frac{1}{\varepsilon^2}t^{1,0}t^{2,0},$$

$$L_0 = \sum_{k\geq 1} k\left(t^{1,k}\frac{\partial}{\partial t^{1,k}} + t^{2,k-1}\frac{\partial}{\partial t^{2,k-1}}\right) + \sum_{p\in\mathbb{Z}} pt^{0,p}\frac{\partial}{\partial t^{0,p}}$$
$$+ \sum_{k\geq 1}\left(2t^{1,k} + t^{0,k}\right)\frac{\partial}{\partial t^{2,k-1}} + \frac{1}{\varepsilon^2}(t^{1,0})^2 + \frac{1}{\varepsilon^2}\sum_{k\geq 0}(-1)^k t^{0,-k}t^{1,k}.$$

The Virasoro symmetries induce the following flows on the tau cover of the extended Ablowitz-Ladik hierarchy:

$$\frac{\partial f_{\gamma,k}}{\partial s_m} = a_m^{\alpha,p;\beta,q}\left(\varepsilon\frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\gamma,k}} + f_{\alpha,p}\Omega_{\gamma,k;\beta,q} + f_{\beta,q}\Omega_{\gamma,k;\alpha,p}\right)$$
$$+ b_{m;\gamma,k}^{\alpha,p} f_{\alpha,p} + \frac{1}{\varepsilon} b_{m;\beta,q}^{\alpha,p} t^{\beta,q}\Omega_{\gamma,k;\alpha,p} + \frac{1}{\varepsilon} c_{m;\alpha,p;\gamma,k} t^{\alpha,p},$$
$$\frac{\partial w^1}{\partial s_m} = (\Lambda - 1)\left(\frac{\partial f_{2,0}}{\partial s_m}\right),$$
$$\frac{\partial\left(w^2 - \log(e^{w^2} - w^1)\right)}{\partial s_m} = (\Lambda - 1)B_-\left(\frac{\partial f_{0,0}}{\partial s_m}\right),$$

where $w^1 = Q - P$, $w^2 = \log Q$, and the operator B_- is given by

$$B_- = \frac{(1 - \Lambda^{-1})}{\varepsilon\partial_x} = 1 - \frac{1}{2}\varepsilon\partial_x + \dots.$$

The theorem is equivalent to the commutation relation

$$\left[\frac{\partial}{\partial s_m}, \frac{\partial}{\partial t^{\alpha,p}}\right] X = 0, \quad m \geq -1, \ (\alpha, p) \in \mathcal{I},$$

here $X = f_{\beta,q}, \ P, \ Q$.

In order to prove it, we modify the Virasoro operators as follows:

$$\tilde{L}_m = L_m + L_m^{\mathsf{odd}}, \quad L_m^{\mathsf{odd}} = \sum_{p \in \mathbb{Z}} p \tau_p \frac{\partial}{\partial \tau_{p+m}},$$

It's easy to check that these modified operators still satisfy the Virasoro commutation relations

$$[\tilde{L}_m, \tilde{L}_n] = (m-n)\tilde{L}_{m+n}, \quad m, n \geq -1.$$

The modified Virasoro operators yield flows on the super tau cover of the extended Ablowitz-Ladik hierarchy

$$\frac{\partial f_{\gamma,k}}{\partial \tilde{s}_m} = a_m^{\alpha,p;\beta,q}\left(\varepsilon \frac{\partial \Omega_{\alpha,p;\beta,q}}{\partial t^{\gamma,k}} + f_{\alpha,p}\Omega_{\gamma,k;\beta,q} + f_{\beta,q}\Omega_{\gamma,k;\alpha,p}\right)$$

$$+ b_{m;\gamma,k}^{\alpha,p} f_{\alpha,p} + \frac{1}{\varepsilon} b_{m;\beta,q}^{\alpha,p} t^{\beta,q}\Omega_{\gamma,k;\alpha,p} + \frac{1}{\varepsilon} c_{m;\alpha,p;\gamma,k} t^{\alpha,p} + \sum_{p\in\mathbb{Z}} (c_0 + p)\tau_p \frac{\partial f_{\gamma,k}}{\partial \tau_{p+m}},$$

$$\frac{\partial w^1}{\partial \tilde{s}_m} = (\Lambda - 1)\left(\frac{\partial f_{2,0}}{\partial \tilde{s}_m}\right), \quad \frac{\partial\left(w^2 - \log(e^{w^2} - w^1)\right)}{\partial \tilde{s}_m} = (\Lambda - 1)B_-\left(\frac{\partial f_{0,0}}{\partial \tilde{s}_m}\right),$$

$$\frac{\partial\left(Q\sigma_{1,n} + Q\sigma_{2,n}\right)}{\partial \tilde{s}_m} = \frac{\partial \psi_{2,0}^n}{\partial \tilde{s}_m}, \quad \frac{\partial\left(P\sigma_{1,n-1} + Q\sigma_{2,n-1}\right)}{\partial \tilde{s}_m} = \Lambda B_-\left(\frac{\partial \psi_{0,0}^n}{\partial \tilde{s}_m}\right),$$

$$\frac{\partial \psi_{\alpha,p}^n}{\partial \tilde{s}_m} = \varepsilon \frac{\partial}{\partial \tau_n}\left(\frac{\partial f_{\alpha,p}}{\partial \tilde{s}_m}\right).$$

Proof of the Theorem:

By using the results on variational bihamiltonian cohmologies (Si-Q Liu, Zhe Wang, Y.Z. 2022, 2023), we can reduce the proof of the theorem to

1) prove that
$$\left[\frac{\partial}{\partial \tilde{s}_i}, \frac{\partial}{\partial \tau_j}\right] K = 0, \quad i = -1,\, 2,\, j = 0,\, 1,$$
for $K = f_{\alpha,p}, \sigma_{1,0}, \sigma_{2,0}, P, Q$.

2) check the validity of the locality condition
$$\left[\frac{\partial}{\partial \tilde{s}_i}, \frac{\partial}{\partial t^{\alpha,p}}\right] K \in \hat{\mathcal{A}}, \quad \text{for } \ i = -1,\, 2,\, (\alpha, p) \in \mathcal{I}$$
where $K = \sigma_{1,0},\, \sigma_{2,0}, P, Q$.

3) By using the super tau cover of the extended Ablowitz-Ladik hierarchy we arrive at
$$\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial t^{\alpha,p}}\right] X = 0, \quad \text{for } \ i = -1, 2,$$
where $X = f_{\beta,q}, P, Q$.

Main Theorem (Liu, Wang, Z. 2024)

The topological deformation of the Principal Hierarchy of the generalized Frobenius manifold M_{AL} coincides with the extended Ablowitz-Ladik hierarchy.

Based on joint work with Si-Qi Liu and Yuewei Wang

Si-Qi Liu and Yuewei Wang, Youjin Zhang, The extended Ablowitz-Ladik hierarchy and a generalized Frobenius manifold, arXiv:2404.08895

Thanks