Large deviations for the log-Gamma polymer partition function

Tom Claeys

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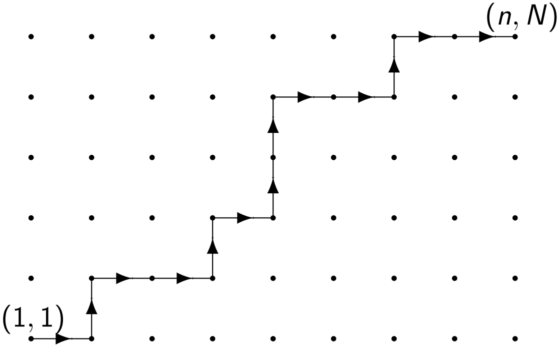
UCLouvain

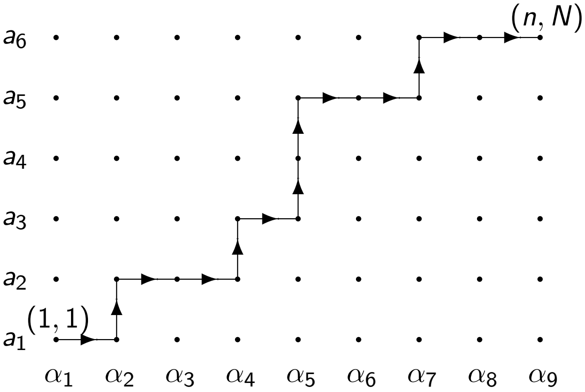
Log-Gamma Polymer

- The **log-Gamma Polymer** is a probability measure on directed up-right lattice paths on a rectangular lattice of size $n \times N$, introduced by *Seppalainen* in 2012.



Log-Gamma Polymer

- To each column $j = 1, \ldots, n$, we assign a parameter α_j, and to each row $k = 1, \ldots, N$, we assign a parameter a_k, and they are such that $\alpha_{j,k} := \alpha_j - a_k > 0$ for all j, k.
- The **homogeneous lattice** corresponds to $\alpha_j = -a_k = \theta > 0$.



Log-Gamma Polymer

- We assign an independent **inverse-Gamma distributed random weight** $d_{i,j} \geq 0$ to each vertex in the lattice,

$$\mathbb{P}(d_{j,k} \leq y) = \frac{1}{\Gamma(\alpha_{j,k})} \int_0^y x^{-\alpha_{j,k}-1} \mathrm{e}^{-1/x} \mathrm{d}x.$$

- The weight of an up-right path π connecting $(1,1)$ with (n,N) is the product of the weights of the vertices in the path, $\prod_{(j,k)\in\pi} d_{j,k}$, and the **partition function** of the model is the **random variable**

$$Z_{n,N}(\vec{\alpha},\vec{a}) := \sum_{\pi:(1,1)\nearrow(n,N)} \prod_{(j,k)\in\pi} d_{j,k},$$

  where the sum is over all up-right paths between $(1,1)$ and (n,N).
- We will focus here on the **homogeneous** $(\alpha_j = -a_k = \theta > 0)$ **square** $(n = N)$ **lattice**, and write $Z_n(\theta)$ for the partition function.

Zero temperature limit

- As $\theta \to 0$, the random variables $u_{i,j}(\theta) := 2\theta \log d_{i,j}(\theta)$ converge in distribution to **independent exponential random variables** $u_{i,j}$, i.e., $\mathbb{P}(u_{i,j} \leq s) = 1 - e^{-u_{i,j}}$.
- It follows that

$$2\theta \log Z_n(\theta) \longrightarrow F_n^{\mathrm{LPP}} := \max_{\pi:(1,1)\to(n,n)} \sum_{(i,j)\in\pi} u_{i,j}$$

  weakly as $\theta \to 0$ with fixed n The right hand side is the maximum additive weight of an up-right path in random exponential environment. This model is known as **last passage percolation** or **corner growth** with exponential weights.
- The distribution of F_n^{LPP} is identical to that of the **largest eigenvalue of an LUE random matrix** *(Johansson '00)*.

Log-Gamma Polymer

- The log-Gamma Polymer fits in the framework of MacDonald processes, **Whittaker processes**, and belongs to the **KPZ universality class**.

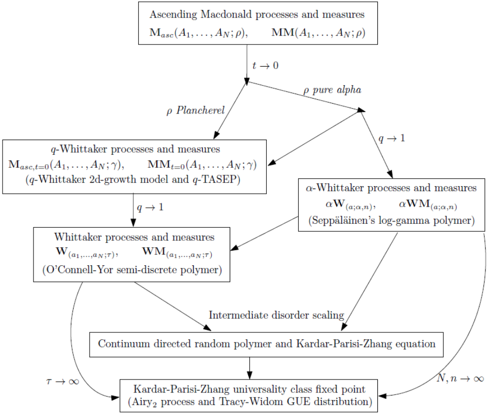


Figure: From *Borodin-Corwin '12*.

An exact Fredholm determinant identity

- The model is **exactly solvable**: the **Laplace transform of the partition function** *(Borodin-Corwin-Remenik '13)* is given by

$$\mathbb{E}(e^{-uZ_n(\theta)}) = \det(I + K_n^{u,\theta})_{L^2(\Sigma)},$$

where $K_n^{u,\theta} : L^2(\Sigma) \to L^2(\Sigma)$ is the integral kernel operator with kernel

$$K_n^{u,\theta}(v,v') = \frac{1}{(2\pi i)^2} \int_{i\mathbb{R}} dw \frac{\pi u^{w-v}}{\sin \pi(v-w)} \frac{W(w)}{W(v)} \frac{1}{w-v'}, \quad W(z) = \frac{\Gamma(\theta - z)^n}{\Gamma(\theta + z)^n},$$

with Σ a circle of radius $r < \theta$ around $-\theta$.

- The determinant is a **Fredholm determinant** given by the **Fredholm series**

$$\det(I+K)_{L^2(\gamma)} = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\gamma^k} \det\left(K(x_i,x_j)\right)_{i,j=1}^{k} dx_1 \cdots dx_k.$$

Known asymptotic results

- **Theorem** *(Borodin-Corwin-Remenik '13, Krishnan-Quastel '18, Barraquand-Corwin-Dimitrov '21)*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log Z_n(\theta) + 2n\psi(\theta)}{(-\psi''(\theta)n)^{1/3}} \le r \right) = F_{\mathrm{TW}}(r), \qquad r \in \mathbb{R},$$

where $F_{\mathrm{TW}}(r)$ is the $\beta = 2$ **Tracy-Widom distribution**

$$F_{\mathrm{TW}}(r) = \det\left(1 - K^{\mathrm{Ai}}\right)_{L^2(r,\infty)}, \; K^{\mathrm{Ai}}(x, x') = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(x') - \mathrm{Ai}(x')\mathrm{Ai}'(x)}{x - x'}.$$

- The **zero-temperature analogue** of this is a well-known result *(Johansson '00)*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{F_n^{\mathrm{LPP}} - 4n}{2^{4/3} n^{1/3}} \le r \right) = F_{\mathrm{TW}}(r), \qquad r \in \mathbb{R},$$

for last passage percolation.

Known asymptotic results

- While the large n behavior of typical fluctuations of $\log Z_n(\theta)$ around $-2n\psi(\theta)$ is well understood, **probabilities of fluctuations of larger order** are not. For instance, $\mathbb{P}\left(\frac{\log Z_n(\theta)+2n\psi(\theta)}{(-\psi''(\theta)n)^{1/3}} \leq r\right)$ as $r \to -\infty$ together with $n \to \infty$?
- At zero temperature, **large deviations** are well understood

$$\lim_{n\to\infty} \frac{-1}{n^2} \log \mathbb{P}\left[F_n^{\mathrm{LPP}} \leq 4ns\right] = \frac{s^2}{2} + \frac{3}{2} - 2s + \log s, \ 0 < s < 1.$$

- Setting $s = 1 + r(2n)^{-2/3}$ with $r \to -\infty$, we find $\log \mathbb{P}\left(\frac{F_n^{\mathrm{LPP}}-4n}{2^{4/3}n^{1/3}} \leq r\right) \sim -\frac{r^3}{12}$, which is indeed the leading order of $\log F_{\mathrm{TW}}(r)$ as $r \to -\infty$.
- Understanding such **tail probabilities and large deviations** is often important in view of applications (random growth phenomena).

Large deviations for the log-Gamma polymer partition function

- Our main result is a **conjecture about the large deviations at non-zero temperature**.
- For $0 < s < -\theta\psi(\theta)$, define $b = b(s;\theta) > 0$ as the unique positive number solving the equation

$$\int_0^1 \left(\theta\psi(\theta + iu\theta b/2) + \theta\psi(\theta - iu\theta b/2) + 2s\right) \frac{du}{\pi\sqrt{1-u^2}} = 0,$$

  where $\psi = \Gamma'/\Gamma$ is the di-Gamma function.
- Define

$$f(s;\theta) = b^2 \int_0^1 \left(\theta\psi(\theta + iu\theta b/2)) + \theta\psi(\theta - iu\theta b/2) + 2s\right) \sqrt{1-u^2}\frac{du}{2\pi},$$

  and

$$F(s;\theta) = \int_0^s f(t;\theta)dt.$$

Large deviations for the log-Gamma polymer partition function

Conjecture (C-Mauersberger '24)

There exists $\theta_0 > 0$ *such that uniformly for* $0 < \theta < \theta_0$ *and* $\epsilon < s \leq -\theta\psi(\theta)$ *for any* $\epsilon > 0$, *we have*

$$\lim_{n\to\infty} \frac{-1}{n^2} \log \mathbb{P}\left[\log Z_n(\theta) \leq \frac{2n}{\theta} s\right] = F(s;\theta).$$

- One can check that

$$\lim_{\theta\to 0} f(s;\theta) = 2 - s - \frac{1}{s}, \quad \lim_{\theta\to 0} F(s;\theta) = \frac{s^2}{2} + \frac{3}{2} - 2s + \log s,$$

  which is precisely the large deviation rate function for last passage percolation.
- We prove this result rigorously, except for one step for which we only have heuristic evidence.

Idea of proofs and heuristics

- Step 1: Characterization of the distribution of $Z_n(\theta)$ in terms of **signed biorthogonal ensembles** *(Cafasso-C '24)* and an alternative Fredholm determinant identity.
- Step 2 (non-rigorous): Approximating the Fredholm determinant by a **step-function Fredholm determinant**.
- Step 3: **Asymptotic analysis** of the step-function Fredholm determinant:
  - conjugation of the operator with two-sided Laplace transform,
  - Riemann-Hilbert characterization using IIKS theory,
  - asymptotic analysis of Riemann-Hilbert problem.

Biorthogonal measures

- A **biorthogonal ensemble** is a **probability measure** on $\mathbb{R}^N$ of the form

$$\frac{1}{Z_N}\det\left(f_m(x_k)\right)_{k,m=1}^N \det\left(g_m(x_k)\right)_{m,k=1}^N \prod_{k=1}^N \mathrm{d}x_k,$$

for certain sets of functions $f_1,\ldots,f_N$ and $g_1,\ldots,g_N$.
  - For $f_m(x)=x^{m-1}$ and $g_m(x)=x^{m-1}w(x)$, we obtain the **orthogonal polynomial ensemble**

$$\frac{1}{Z_N}\prod_{1\leq j<k\leq N}(x_k-x_j)^2\prod_{k=1}^N w(x_k)\mathrm{d}x_k.$$

- We will consider signed measures with this biorthogonal structure which are **not necessarily positive**.

- The relevant biorthogonal measure is a **polynomial ensemble of derivative type**:

$$\mathrm{d}\mu_n(\vec{x}) = \frac{1}{Z_n} \prod_{1\leq j<k\leq n} (x_k - x_j) \det\left(\partial^{m-1} G(x_k)\right)_{m,k=1}^{n} \prod_{k=1}^{n} \mathrm{d}x_k,$$

where

$$G(x;\theta) = \frac{1}{2\pi i}\int_{i\mathbb{R}} \frac{\Gamma(\theta - \frac{\theta}{2n}v)^n}{\Gamma(\theta + \frac{\theta}{2n}v)^n} \frac{e^{-vx}}{(1+\frac{v}{2n})^n} dv.$$

  - As $\theta \to 0$, the biorthogonal ensemble converges to the LUE, with eigenvalues on $[0,1]$ as $n \to \infty$.
  - For $\theta \geq 0$, we have a correlation kernel

$$L_n^{\theta}(x,x') = \frac{1}{(2\pi \mathrm{i})^2}\int_{\Sigma} \mathrm{d}u \int_{i\mathbb{R}} \mathrm{d}v \frac{\Gamma(\theta - \frac{\theta}{2n}v)^n \Gamma(\theta + \frac{\theta}{2n}u)^n}{\Gamma(\theta + \frac{\theta}{2n}v)^n \Gamma(\theta - \frac{\theta}{2n}u)^n} \frac{e^{-vx+ux'}}{v-u}.$$

Biorthogonal measures

- An **average multiplicative statistic** with respect to the biorthogonal measure is a quantity like

$$\mu_n[\sigma] := \int_{\mathbb{R}^n} \prod_{k=1}^{n} (1-\sigma(x_k))\mathrm{d}\mu_n(x_1,\ldots,x_N) = \det(1-\sigma L_n^{\theta})_{L^2(\mathbb{R})}.$$

- The **Laplace transform of the log-Gamma partition function** is an average multiplicative statistic *(C-Cafasso '24)*:

$$\mathbb{E}\left[e^{-e^t Z_n(\theta)}\right] = \mu_N[\sigma_{s,n,\theta}], \quad \sigma_{s,n,\theta}(x) = \frac{1}{1+\mathrm{e}^{-\frac{n}{2\theta}(x-s)}}.$$

- Understanding the **large n asymptotics of** $\mu_n[\sigma_{s,n,\theta}]$ leads to understanding the distribution of $Z_n(\theta)$.

Biorthogonal measures

- One-point function $\frac{1}{n}L_n^\theta(x,x)$

- Test function $\sigma_{s,n,\theta}(x)$

Approximation by a step function Fredholm determinant

- On a macroscopic scale, the test function $\sigma_{s,n,\theta}(x)$ converges to a **step function** $1_{(s,+\infty)}(x)$ as $n \to \infty$.
- Since the **large deviation rate function is a macroscopic quantity**, we expect that it does not change when replacing

$$\mu_n[\sigma_{s,n,\theta}] = \det\left(1 - \sigma_{s,n,\theta} L_n^\theta\right)_{L^2(\mathbb{R})}$$

by

$$\mu_n[1_{(s,+\infty)}] = \det\left(1 - 1_{(s,+\infty)} L_n^\theta\right)_{L^2(\mathbb{R})}.$$

- **Ansatz:** There exists $\theta_0 > 0$ such that

$$\lim_{n\to\infty} \frac{1}{n^2} \log\det\left(1 - \sigma_{s,n,\theta} L_n^\theta\right)_{L^2(\mathbb{R})} = \lim_{n\to\infty} \frac{1}{n^2} \log\det\left(1 - 1_{(s,+\infty)} L_n^\theta\right)_{L^2(\mathbb{R})},$$

for $0 \le \theta \le \theta_0$.
- We have more **heuristic evidence** for this, but no proof.

Analysis of the step function Fredholm determinant

- The step function operator $1_{(s,+\infty)}L_n^\theta$ can be factorized as

$$1_{(s,+\infty)}L_n^\theta = B^{-1}K_n^\theta B,$$

with $K_n^\theta : L^2(i\mathbb{R}\cup\Sigma) \to L^2(i\mathbb{R}\cup\Sigma)$ an integral operator with kernel of **two-integrable form**

$$K_n^\theta(u,u';s) = \frac{f_1(u;s)g_1(u';s) + f_2(u;s)g_2(u';s)}{u-u'}, \quad u,u' \in i\mathbb{R}\cup\Sigma.$$

- Hence,

$$\det\left(1 - 1_{(s,+\infty)}L_n^\theta\right)_{L^2(\mathbb{R})} = \det\left(1 - K_n^\theta\right)_{\det\left(1-1_{(s,+\infty)}L_n^\theta\right)_{L^2(i\mathbb{R}\cup\Sigma)}},$$

and the right hand side can be expressed in terms of a 2×2 **matrix Riemann-Hilbert problem.**

- **RH problem for Y**

  (Y1) $Y : \mathbb{C} \setminus (i\mathbb{R} \cup \Sigma) \to \mathbb{C}^{2\times 2}$ is analytic.

  (Y2) For $z \in i\mathbb{R} \cup \Sigma$, Y satisfies the jump condition

$$Y_+(z) = Y_-(z)J(z),$$

  where (recall $W(z) = \frac{\Gamma(\theta-z)^n}{\Gamma(\theta+z)^n}$)

$$J(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{-sz}W(z) & 1 \end{pmatrix} & \text{for } z \in i\mathbb{R}, \\ \begin{pmatrix} 1 & -e^{sz}W(z)^{-1} \\ 0 & 1 \end{pmatrix} & \text{for } z \in \Sigma. \end{cases}$$

  (Y3) $Y(z) = I + \frac{Y_1(s,\theta)}{z} + \mathcal{O}(z^{-2})$ as $z \to \infty$.

- Differential identity

$$\frac{d}{ds} \log \det(1 - 1_{(s,\infty)} L_n^{\theta})_{L^2(\mathbb{R})} = (Y_1(s,\theta))_{11}\,.$$

Analysis of the step function Fredholm determinant

Large n asymptotic analysis of the RH problem.

- Case 1: $s > -\theta\psi(\theta) + \epsilon$. Small-norm RH problem after choosing suitable jump contours.

Analysis of the step function Fredholm determinant

Large n asymptotic analysis of the RH problem.

- Case 2: $-\theta\psi(\theta) - \epsilon \leq s \leq -\theta\psi(\theta) + \epsilon$. Local Painlevé II parametrix needed near 0.

Analysis of the step function Fredholm determinant

Large n asymptotic analysis of the RH problem.

- Case 3: $\epsilon < s < -\theta\psi(\theta) - \epsilon$. Construction of g-function, global parametrix and Airy parametrices.

Analysis of the step function Fredholm determinant

- RH analysis is of the same type as the 2-**to**-1-**cut transition for Toeplitz determinants** done by *Baik-Deift-Johansson '99* in their study of the length of the longest increasing subsequence of a random permutation.
- Asymptotic analysis leads after computations to the **large deviation rate function** $F(s;\theta)$.
- We can obtain much more detailed asymptotics for the step function Fredholm determinant $\det\left(1 - 1_{(s,+\infty)} L_n^{\theta}\right)_{L^2(\mathbb{R})}$, but it is not clear whether the approximation

$$\log \det \left(1 - \sigma_{s,n,\theta} L_n^{\theta}\right)_{L^2(\mathbb{R})} \approx \det \left(1 - 1_{(s,+\infty)} L_n^{\theta}\right)_{L^2(\mathbb{R})}$$

continues to hold beyond leading order.

Outlook

- Proof of the Ansatz?
- Similar approach should work to find the large deviation rate functions of other models:
  - Non-homogeneous log-Gamma polymer on rectangular lattice (finite temperature version of last passage percolation / LUE with external source).
  - O'Connell-Yor polymer (finite temperature version of corner growth in Brownian environment / GUE with externel source).
  - Mixed polymer (finite temperature version of last passage percolation with sources / sum of GUE and LUE).
- Subleading terms?

Thank you for your attention!