Stability and Instability Phenomena in Fluids

Stefano Pasquali
SISSA

SISSA Junior Math Days 2024

December 2024

- What is a fluid?
- A fluid is any material that cannot sustain a tangential, or shearing, force when at rest and that undergoes a continuous change in shape when subjected to such a stress (e.g. liquids, gases). (source: Encyclopaedia Britannica)

Waves in a small lagoon, Malmö, Sweden.

- Problem: (in)stability of solutions of PDEs describing fluids under small perturbations of initial data.
- A solution u_0 to a nonlinear system is called linearly unstable if the linearization of the equation at this solution has the form

$$\frac{\mathrm{d}}{\mathrm{d}t}u = Au,$$

where u is the perturbation of u_0, A is a *linear operator* whose spectrum contains eigenvalues with positive real part. If all the eigenvalues have negative real part, then the solution is called linearly stable.

What about *nonlinear stability*?

- Consider an autonomous nonlinear dynamical system

$$\dot{x} = f(x(t))$$

where $x(t) \in D \subset \mathbb{R}^d$, D open set containing the origin, $f : D \to \mathbb{R}^d$ cont. vector field on D. Suppose f has an equilibrium at x_e, i.e. $f(x_e) = 0$.

The equilibrium x_e is Lyapunov stable if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|x(0) - x_e\| < \delta$, then $\|x(t) - x_e\| < \epsilon$ for every $t \geq 0$.

Examples of stability phenomena in fluids

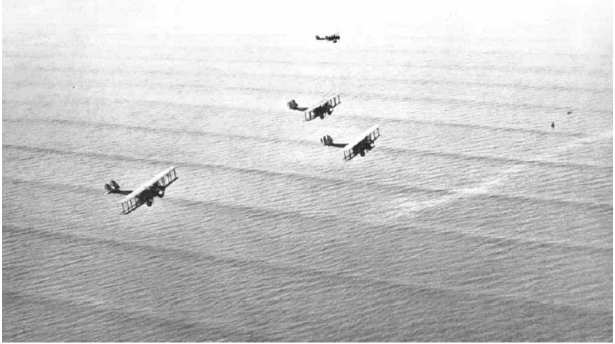


Photo of approximately-periodic swell in shallow water, close to the Panama coast (1933). source:Wikipedia.

Cross swells in front of Île de Ré, France. source:Wikipedia.

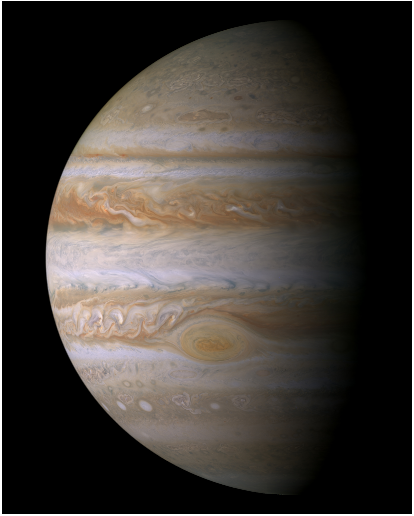


Photo of the planet Jupiter; notice the Great Red Spot in the southern hemisphere. source:NASA.

Many of the PDEs under study, after possibly appropriate change of variables, take the form

$$\mathrm{i}u_t = Lu + N(u)$$

where u belongs to some Banach space, L is a self-adjoint operator, and N is a nonlinear term $\mathcal{O}(|u|^{q+1})$, $q \geq 1$, with $N(0) = 0$.

For initial data of size $\epsilon \ll 1$, nonlinearity is viewed as a perturbation of the linear flow. Classical local in time theory guarantees that the solutions have a linear behaviour (hence the origin is stable) for times of order $\mathcal{O}(\epsilon^{-q})$.

For longer time scales the effect of the nonlinearity becomes non-trivial.

Examples of instability phenomena in fluids

Transitions from laminar to turbulent regimes of the plume of a candle. source:Wikipedia.

Growth of Sobolev norms: consider cubic defocusing nonlinear Schrödinger equation on the 2D torus

$$\begin{aligned} -i\partial_t u + \Delta u &= |u|^2 u \\ u(0,x) &= u_0(x) \end{aligned} \tag{1}$$

where $x \in \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z})^2$, $u : \mathbb{R} \times \mathbb{T}^2 \to \mathbb{C}$.

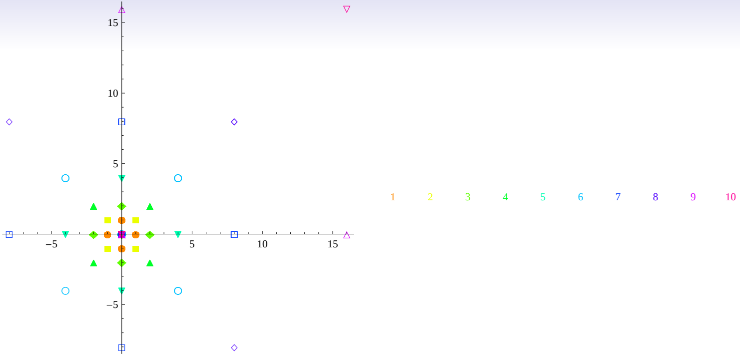
Recall that if $u(t,x) = \sum_{j\in\mathbb{Z}^2} a_j(t) e^{ij\cdot x}$

$$\|u(t)\|_{H^s(\mathbb{T}^2)} = \left(\sum_{j\in\mathbb{Z}^2} |a_j(t)|^2 \langle j \rangle^{2s} \right)^{1/2}, \quad \langle j \rangle := (1+|j|^2)^{1/2};$$

- Eq. (1) is globally well posed in Sobolev spaces $H^s(\mathbb{T}^2)$, $s \geq 1$;
- conservation of energy implies that $\|u(t)\|_{H^1(\mathbb{T}^2)} \leq C\|u_0\|_{H^1(\mathbb{T}^2)}$.

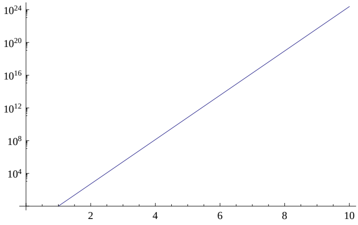
  What about $\|u(t)\|_{H^s(\mathbb{T}^2)}$ for $s > 1$?

- Growth of Sobolev norms is described by transfer of energy from low modes to high modes.



Resonant set of Fourier modes for defocusing cubic NLS on $\mathbb{T}^2$,

following Colliander et al., 2010 (10 generations, i.e. 5120 modes, represented)



Plot of the Growth factor of H^s-norm vs s (in semi-log scale)

Photo of a rogue wave against a ship. source:Georgia Tech.

Our research at SISSA

Aim

Prove rigorous results on *stable and unstable dynamics* in PDEs modelling water waves and geophysical fluids.

- Existence of small amplitude 2D/3D water waves;
- Existence of vortex patches in 2D fluids;
- Energy cascades from low to high frequency modes;
- Modulational instabilities of traveling waves;
- Extreme phenomena formations (e.g. rogue waves).

Difficulties

Many aspects of the models, both from physics

- 2D/3D fluids, gravity, surface tension, vorticity, density,...

and from mathematics

- ansatz on solutions, approximations, structure of linear part (dispersion relation), structure of nonlinearity,...

The above difficulties are *often intertwined*!

- Tools: techniques from nonlinear PDEs (dispersive PDEs, Hamiltonian PDEs,...), Hamiltonian systems,...

Group

- Professors: M. Berti, A. Maspero;
- Researchers/Assistant professors: R. Grande, B. Langella;
- Postdocs: S. Pasquali, E. Roulley, S. Terracina;
- PhD students: T. Barbieri, A.M. Radakovic, M.T. Rotolo, D. Silimbani

+ collaborators, both in Italy (Milan, Naples, Rome, Trieste...) and abroad (France, Germany, Spain, Sweden, UAE, USA,...).

Small amplitude asymmetrical water waves

with M. Groves (Germany), D. Nilsson and E. Wahlén (Sweden)

Aim

Prove existence of *asymmetric small amplitude 3D doubly periodic steady water waves*.

- *Doubly periodic*: there are two lin. indep. wave vectors $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{R}^2$ generating the lattice

$$\Lambda' := \{\mathbf{k} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 : n_j \in \mathbb{Z},\ j = 1, 2\}, \tag{2}$$

and a dual lattice of periods in $\mathbb{R}^2$ such that

$$\Lambda := \{\boldsymbol{\lambda} = m_1\boldsymbol{\lambda}_1 + m_2\boldsymbol{\lambda}_2 : m_j \in \mathbb{Z}, \boldsymbol{\lambda}_j \cdot \mathbf{k}_\ell = 2\pi\delta_{j\ell},\ \ j = 1, 2\}, \tag{3}$$

- *Asymmetric*: generic $\mathbf{k}_1$, $\mathbf{k}_2$ spanning Λ'.

Setting

Incompressible inviscid fluid with constant density occupying a 3D domain with flat bottom, under the action of gravity and surface tension.

- We study small amplitude 3D steady water waves.

  Steady: both the velocity field and free-surface profile are stationary with respect to a uniformly (horizontally) translating frame of reference.

  Notation: $\mathbf{x}' = (x, y) \in \mathbb{R}^2$ horizontal directions, $z \in \mathbb{R}$ vertical direction.

- Fluid domain:

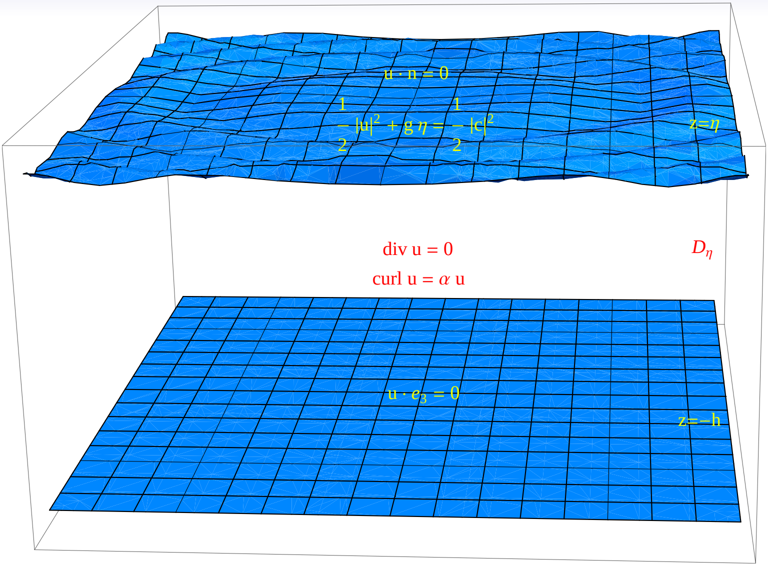
$$D_\eta := \{(\mathbf{x}', z) \in \mathbb{R}^2 \times \mathbb{R} : -h < z < \eta(\mathbf{x}')\}, \tag{4}$$

  free surface given by the graph of an unknown function $\eta : \mathbb{R}^2 \to \mathbb{R}$; $h > 0$ is the depth.

- Beltrami flows: the velocity field $\mathbf{u} : \overline{D_\eta} \to \mathbb{R}^3$ and the vorticity $\operatorname{curl} \mathbf{u}$ are collinear, $\operatorname{curl} \mathbf{u} = \alpha \mathbf{u}$ for some constant $\alpha \in \mathbb{R}$.

  Why Beltrami flows? Physical reasons (e.g. experiments), mathematical reasons (e.g. Arnold theorems on 3D steady Euler Eq.).

- In the literature, the fluid is usually assumed to be irrotational, $\operatorname{curl} \mathbf{u} = \mathbf{0}$.



Water waves system for Beltrami flows, in the pure gravity case.

Equations for steady water waves on strong Beltrami flow

$$\operatorname{div} \mathbf{u} = 0, \quad \text{in} \quad D_\eta, \tag{5}$$
$$\operatorname{curl} \mathbf{u} = \alpha \mathbf{u}, \quad \text{in} \quad D_\eta, \tag{6}$$
$$\mathbf{u} \cdot \mathbf{e}_3 = 0, \quad \text{at} \quad z = -h, \tag{7}$$
$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{at} \quad z = \eta, \tag{8}$$
$$\frac{1}{2}|\mathbf{u}|^2 + g\eta - \beta \left(\frac{\eta_x}{(1+|\nabla \eta|^2)^{1/2}} \right)_x - \beta \left(\frac{\eta_y}{(1+|\nabla \eta|^2)^{1/2}} \right)_y = \frac{1}{2}|\mathbf{c}|^2, \quad \text{at} \quad z = \eta, \tag{9}$$

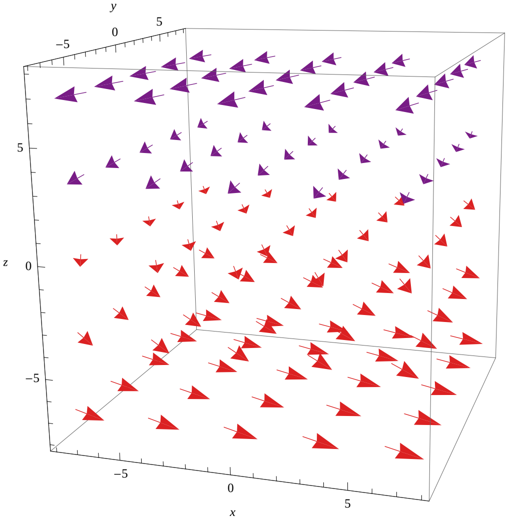
where $\mathbf{e}_3 = (0,0,1)^T$, $\mathbf{n}$ is the outward unit normal vector, g is the gravity constant, β is the surface tension coefficient, $\mathbf{c} = (c_1, c_2)^T$ is the wave velocity, and (8) and (9) are respectively the kinematic and the dynamic boundary conditions at the free surface.

- If we consider a fluid domain with flat boundary, namely

$$D_0 := \{(\mathbf{x}', z) \in \mathbb{R}^2 \times \mathbb{R} : -h < z < 0\}, \tag{10}$$

a "trivial solution" of (5)-(9) is the two-parameter family of laminar flows

$$\mathbf{u}^\star := c_1 \mathbf{u}^{(1)} + c_2 \mathbf{u}^{(2)}, \quad c_1, c_2 \in \mathbb{R}, \tag{11}$$
$$\mathbf{u}^{(1)} := \mathbf{u}^{(1)}[\alpha] := (\cos(\alpha z), -\sin(\alpha z), 0)^T,$$
$$\mathbf{u}^{(2)} := \mathbf{u}^{(2)}[\alpha] := (\sin(\alpha z), \cos(\alpha z), 0)^T.$$



Example of a laminar flow, with $\alpha = 0.1$, $c_1 = 1$, $c_2 = -2$.

Single-equation Formulation

- We consider sol. $(\eta, \mathbf{u})$ of (4)-(9) which are small perturbations of the laminar flow (10)-(11). If $\mathbf{v} := \mathbf{u} - \mathbf{u}^\star$, we write $\mathbf{v}$ by a solenoidal vector potential $\mathbf{A}$ s.t.

$$\mathbf{v} = \operatorname{curl} \mathbf{A}. \tag{12}$$

- *Notation*: $\mathbf{F} = (F_1, F_2, F_3)^T$ three-dimensional vector field,
$\mathbf{F}_\mathrm{h} = (F_1, F_2)^T$,
$\mathbf{F}_{||} = \mathbf{F}_\mathrm{h} + F_3 \nabla \eta|_{z=\eta}$ quantity related to the tangential part of $\mathbf{F}$,
For the two-dimensional vector field $\mathbf{f} = (f_1, f_2)^T$ we denote $\mathbf{f}^\perp = (f_2, -f_1)^T$.
We indicate the evaluation at the free surface with an underscore.

- Hodge-Weyl decomposition for doubly periodic vector fields in $\mathbb{R}^2$:

$$\mathbf{v}_{||} = \gamma + \nabla \Phi + \nabla^\perp \Psi, \quad \nabla = (\partial_x, \partial_y)^T, \quad \nabla^\perp = (\partial_y, -\partial_x)^T, \tag{13}$$

where $\gamma := \langle \mathbf{v}_{||} \rangle$ denotes the mean value of $\mathbf{v}_{||}$ over one periodic cell.

- We write $\underline{\mathbf{u}}^\star \cdot \mathbf{N} = \nabla \cdot \mathbf{S}(\eta)^\perp$, where

$$\mathbf{S}(\eta) := \frac{c_1}{\alpha} \begin{pmatrix} \cos(\alpha\,\eta) - 1 \\ -\sin(\alpha\,\eta) \end{pmatrix} + \frac{c_2}{\alpha} \begin{pmatrix} \sin(\alpha\,\eta) \\ \cos(\alpha\,\eta) - 1 \end{pmatrix},$$

- We choose $\gamma = \mathbf{0}$, writing $\mathbf{c} = \mathbf{c}_0 + \boldsymbol{\mu}$, where $\mathbf{c}_0 \in \mathbb{R}^2$ is a given constant vector, so that
$$\mathbf{u}^\star = (c_{10} + \mu_1)\mathbf{u}^{(1)} + (c_{20} + \mu_2)\mathbf{u}^{(2)}, \quad \mathbf{c}_0 = (c_{10}, c_{20})^T.$$
Hence, the system can be reduced to a single equation:
$$J(\eta, \boldsymbol{\mu}) := \frac{1}{2}|\mathbf{T}(\eta)|^2 - \frac{(-\underline{\mathbf{u}}^\star \cdot \mathbf{N} + \mathbf{T}(\eta) \cdot \nabla\eta)^2}{2(1 + |\nabla\eta|^2)} + \mathbf{T}(\eta) \cdot \underline{\mathbf{u}}^\star_{\mathrm{h}} + g\eta$$
$$- \beta \left(\frac{\eta_x}{(1 + |\nabla\eta|^2)^{1/2}} \right)_x - \beta \left(\frac{\eta_y}{(1 + |\nabla\eta|^2)^{1/2}} \right)_y = 0, \qquad (14)$$
where $\mathbf{T}(\eta) := \mathbf{M}(\eta)(\mathbf{0}, \mathbf{S}(\eta))$, and the operator $\mathbf{M}(\eta)$ is defined as
$$\mathbf{M}(\eta)(\gamma, \mathbf{g}) := -(\mathrm{curl}\,\mathbf{B})_{\|}, \qquad (15)$$
and $\mathbf{B}$ solves a suitable boundary-value problem.

- We want to construct small amplitude solutions of $J(\eta, \boldsymbol{\mu}) = 0$.
We obtain (formally if $\beta = 0$) local solutions of $J(\eta, \boldsymbol{\mu}) = 0$ of the form $\eta = \eta_1 + \eta_2(\eta_1, \boldsymbol{\mu})$, where
$$\eta_1 = A\mathrm{e}^{\mathrm{i}\mathbf{k}_1 \cdot \mathbf{x}'} + B\mathrm{e}^{\mathrm{i}\mathbf{k}_2 \cdot \mathbf{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\mathbf{k}_1 \cdot \mathbf{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\mathbf{k}_2 \cdot \mathbf{x}'} \qquad (16)$$
and $\eta_2 = \mathcal{O}(|(\eta_1, \boldsymbol{\mu})||\eta_1|)$ (with analytic dependence upon η_1 and $\boldsymbol{\mu}$ for $\beta > 0$).

Result

Theorem (Groves-Nilsson-P.-Wahlén, JDE, 2024)

Let us assume that the (generic) conditions (NR) and (T) hold true.

(i) Expanding η_2 as a power series

$$\eta_2(\eta_1, \boldsymbol{\mu}) = \sum_{k+l\geq 2} \eta_{2,k,l}(\eta_1, \boldsymbol{\mu}),$$

where $\eta_{2,k,l}$ is homogeneous of order k, l in η_1, $\boldsymbol{\mu}$ respectively (formally for $\beta = 0$), one can construct $\eta_{2,k,l}$ such that

$$J\left(\eta_1 + \sum_{k+l\leq m} \eta_{2,k,l}(\eta_1, \boldsymbol{\mu}), \boldsymbol{\mu}\right) = \mathcal{O}(|(\eta_1, \boldsymbol{\mu})|^{m+1})$$

for each $m \in \mathbb{N}$.

(ii) Suppose $\beta > 0$. There exist $\varepsilon > 0$ and analytic functions $\mu_i \colon B_\varepsilon(\mathbf{0}, \mathbb{R}^2) \to \mathbb{R}$, $i = 1, 2$ such that $\mu_i(0,0) = 0$ and

$$J(\eta_1 + \eta_2(\eta_1, \mu_1(|A|^2, |B|^2), \mu_2(|A|^2, |B|^2)), \mu_1(|A|^2, |B|^2), \mu_2(|A|^2, |B|^2)) = 0,$$

where η_1 is given by (16), for all $(|A|, |B|) \in B_\varepsilon(\mathbf{0}, \mathbb{R}^2)$.

Ideas (and difficulties) in the proof

- *Norms*: define the Fourier coefficients of a periodic function f on the lattice Λ by

$$\hat{f}_{\mathbf{k}} = |\Omega|^{-1/2} \int_{\Omega} f(\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'} \, \mathrm{d}\mathbf{x}',$$

  where Ω is the parallelogram built with $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$. For $m \geq 0$ we denote by $H^m(\mathbb{R}^2/\Lambda)$ the Sobolev space of periodic functions in variable $\mathbf{x}' \in \mathbb{R}^2/\Lambda$, with the norm

$$\|f\|_{H^m(\mathbb{R}^2/\Lambda)} := \left[\sum_{\mathbf{k}\in\Lambda'} (1 + |\mathbf{k}|)^{2m} |\hat{f}_{\mathbf{k}}|^2 \right]^{1/2} .$$

- *Power series expansion*: write the Taylor expansion of $\mathbf{M}(\eta)$ around $\eta = 0$,

$$\mathbf{M}(\eta) = \sum_{j=0}^{\infty} \mathbf{M}_j(\eta),$$

with $\mathbf{M}_j(\eta)$ homogeneous of degree j in η. Under the non-resonance condition

(NR) the restrictions

$$\begin{cases} |\mathbf{k}| \neq |\alpha|, \\ h\sqrt{\alpha^2 - |\mathbf{k}|^2} \notin \frac{\pi}{2}\mathbb{N}, \quad \text{if } |\mathbf{k}| < |\alpha|, \end{cases}$$

hold for each $\mathbf{k} \in \Lambda'$,

we have

$$\mathbf{M}_0(\gamma, \mathbf{g}) = -\gamma + \frac{1}{D^2}\left(\alpha\,\mathbf{D}^{\perp} + \mathbf{D}\,c(D)\right)\,\mathbf{D}\cdot\mathbf{g}^{\perp},$$

where

$$c(|\mathbf{k}|) := \begin{cases} \sqrt{\alpha^2 - |\mathbf{k}|^2}\,\cot(h\sqrt{\alpha^2 - |\mathbf{k}|^2}), & \text{if } |\mathbf{k}| < |\alpha|, \\ \sqrt{|\mathbf{k}|^2 - \alpha^2}\,\coth(h\sqrt{|\mathbf{k}|^2 - \alpha^2}), & \text{if } |\mathbf{k}| > |\alpha|, \end{cases}$$

- *Dispersion relation*: The linearization at the origin of (14) is given by

$$J_{10}\,\eta := \mathbf{T}_1(\eta)\cdot\mathbf{c}_0 + g\eta - \beta\,\Delta\eta = 0, \tag{17}$$

where $\mathbf{T}_1(\eta) = \mathbf{M}_0(\mathbf{0}, \mathbf{S}_1(\eta))$, and $\mathbf{M}_0$ is the principal part of the operator $\mathbf{M}(\eta)$ in the power series expansion.

If we write the Fourier expansion of η,

$$\eta(\mathbf{x}') = \sum_{\mathbf{k}\in\Lambda'} \hat{\eta}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}'},$$

we have that (17) is equivalent to

$$\rho(\mathbf{k}, \mathbf{c}_0; \beta)\hat{\eta}_{\mathbf{k}} = 0, \quad \mathbf{k}\in\Lambda',$$

where ρ is the dispersion relation

$$\rho(\mathbf{k}, \mathbf{c}; \beta) := \left[g + \beta\,|\mathbf{k}|^2 - \frac{\alpha}{|\mathbf{k}|^2}\,(\mathbf{c}\cdot\mathbf{k})(\mathbf{k}^\perp\cdot\mathbf{c})\right]|\mathbf{k}|^2 t(|\mathbf{k}|) - (\mathbf{c}\cdot\mathbf{k})^2 = 0. \tag{18}$$

$$t(|\mathbf{k}|) := \begin{cases} \dfrac{\tan(h\sqrt{\alpha^2 - |\mathbf{k}|^2})}{\sqrt{\alpha^2 - |\mathbf{k}|^2}}, & \text{if } |\mathbf{k}| < |\alpha|, \\ \dfrac{\tanh(h\sqrt{|\mathbf{k}|^2 - \alpha^2})}{\sqrt{|\mathbf{k}|^2 - \alpha^2}}, & \text{if } |\mathbf{k}| > |\alpha|. \end{cases}$$

We assume the transversality condition

(T) Let $\alpha\in\mathbb{R}$, $g, h > 0$, $\beta \geq 0$ and $\mathbf{c} = \mathbf{c}_0 \in \mathbb{R}^2\setminus\{\mathbf{0}\}$. Then the only solutions $\mathbf{k}\in\Lambda'$ of the dispersion relation (18) are given by $\mathbf{0}$, $\pm\mathbf{k}_1$, $\pm\mathbf{k}_2$.

- Let us investigate $\ker(J_{10})$. Notice that

$$(J_{10}\eta)(\mathbf{x}') = g\hat{\eta}_{\mathbf{0}} + \sum_{\mathbf{k}\in\Lambda'\setminus\{\mathbf{0}\}} \frac{c(|\mathbf{k}|)}{|\mathbf{k}|^2}\rho(\mathbf{k},\mathbf{c}_0,\beta)\hat{\eta}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}'},$$

$$\ker(J_{10}) = \{A\mathrm{e}^{\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}'} + B\mathrm{e}^{\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}'} + \bar{A}\mathrm{e}^{-\mathrm{i}\mathbf{k}_1\cdot\mathbf{x}'} + \bar{B}\mathrm{e}^{-\mathrm{i}\mathbf{k}_2\cdot\mathbf{x}'} : A, B \in \mathbb{C}\},$$

because by (T) we have that $\rho(\mathbf{k},\mathbf{c}_0;\beta) = 0$ if and only if $\mathbf{k} = \mathbf{0}, \pm\mathbf{k_1}, \pm\mathbf{k_2}$.

J_{10} is formally invertible if $\hat{f}_{\pm\mathbf{k}_1} = \hat{f}_{\pm\mathbf{k}_2} = 0$ with formal inverse given by

$$(J_{10}^{-1}f)(\mathbf{x}') = \frac{1}{g}\hat{f}_{\mathbf{0}} + \sum_{\substack{\mathbf{k}\in\Lambda' \\ \mathbf{k}\neq\mathbf{0},\pm\mathbf{k_1},\pm\mathbf{k_2}}} \frac{|\mathbf{k}|^2}{c(|\mathbf{k}|)\rho(\mathbf{k},\mathbf{c}_0;\beta)}\hat{f}_{\mathbf{k}}\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}'}.$$

- with surface tension ($\beta > 0$): $\rho(\mathbf{k},\mathbf{c}_0;\beta) \gtrsim |\mathbf{k}|^3$ for suff. large $|\mathbf{k}|$, so the above series converges in $H^{s+2}(\mathbb{R}^2/\Lambda)$ for $f \in H^s(\mathbb{R}^2/\Lambda)$.
- no surface tension ($\beta = 0$): $\rho(\mathbf{k},\mathbf{c}_0;0)$ is not bounded from below as $|\mathbf{k}| \to \infty$, so the above formula does not define a bounded operator from $H^s(\mathbb{R}^2/\Lambda)$ to $H^{s+1}(\mathbb{R}^2/\Lambda)$ for any s. This is the reason why the result is formal.

Thank you for your attention!

Molo Audace at sunset, Trieste, November 2024.