Mathematical Quantum Statistical Mechanics

(and applications to condensed matter physics)

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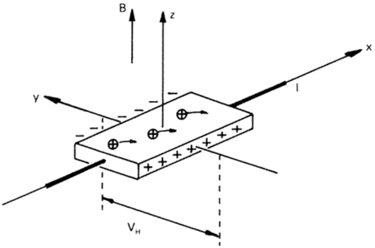
Introduction

- Quantum mechanics is a microscopic theory of nature, explaining many of its key features. Example: the stability of matter.
- It is also very successful in explaining transport phenomena, of relevance in condensed matter physics.

  Famous example: the Quantum Hall Effect, or more recently topological insulators.
- Cond-mat is a constant source of interesting math problems, that motivated the development of various areas of mathematics: functional analysis, spectral theory, probability, noncommutative geometry, etc.
- I am (mostly) interested in the rigorous understanding of the collective behavior of many-body quantum systems, using rigorous field theory and statistical mechanics methods.

The Integer Quantum Hall Effect

- $2d$ insulators exposed to strong magnetic field and in-plane electric field. Extremely low temperatures (about 4 Kelvins)



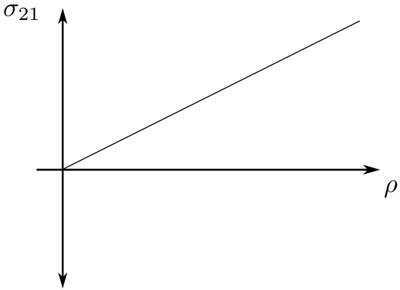
- Linear response (weak E):

$$J_1 = \sigma_{11} E \, , \qquad J_2 = \sigma_{21} E$$

σ_{11} = longitudinal conductivity, $\quad \sigma_{21} = -\sigma_{12}$ = Hall conductivity.

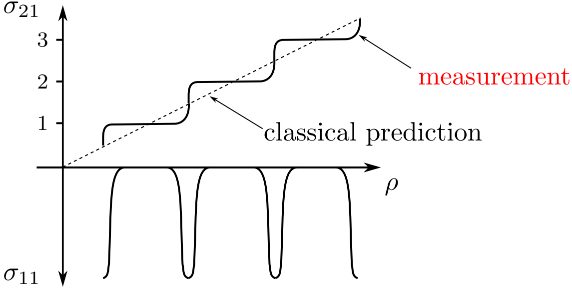
The Integer Quantum Hall Effect

- Classical prediction: linear behavior of transverse conductivity



The Integer Quantum Hall Effect

- von Klitzing et al. '80 Experiment on GaAs-heterostructures (insulators).



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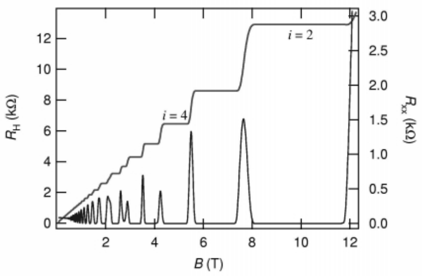

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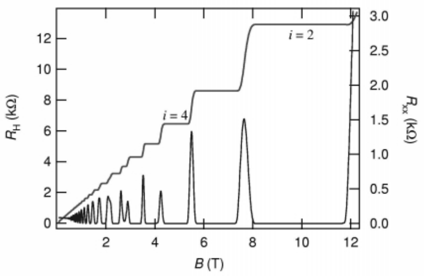

- Integer Quantum Hall effect: σ_{21} is quantized, with 10^{-9} precision!

$$\sigma_{21} = \frac{e^2}{h} \cdot n \,, \qquad n \in \mathbb{Z} \,.$$

Purely quantum phenomenon. First example of topological insulator.

Rigorous results

- This phenomenon is by now rigorously understood, for a wide class of physically realistic models. First example of topological insulator.
  - Thouless-Kohmoto-Nightingale-den Nijs '82: for translation invariant systems, quantization follows from topology.
  - Bellissard-Van Elst-Schulz Baldes '94, Avron-Seiler-Simon '94: translation invariance can be dropped, using noncommutative geometric methods, or index theorems for Fredholm operators.
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- The phenomenon can also be understood via QFT methods:
  - Chern-Simons effective field theories; bulk-edge dualities; nonrenormalization of anomalies; supersymmetry...
- Despite all the effort, the effect of many-body interactions has been only recently rigorously understood. ($\rightsquigarrow$ we'll discuss it later)

Mathematical setting

- Let us start from noninteracting fermions. One-particle on an infinite $2d$ crystalline lattice:

$$\psi \in \ell^2(\mathbb{Z}^2; \mathbb{C}^M) \qquad \text{wave function of the particle}$$

we allow for M internal degrees of freedom (sublattice, spin etc). Normalization: $\|\psi\|_2 = 1$.

- Hamiltonian: $H = H^*$ on $\ell^2(\mathbb{Z}^2; \mathbb{C}^M)$, generates evolution via the Schrödinger equation,

$$i\partial_t \psi_t = H\psi_t \,, \qquad \psi_0 = \psi \,.$$

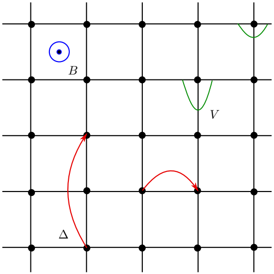
- The zero-temperature equilibrium state for ∞-many fermions is defined by the Fermi projector:

$$\langle O \rangle_\mu := \mathrm{Tr}_{\ell^2}\, O P_\mu \,, \qquad P_\mu := \chi(H \le \mu) \,,$$

for a given chemical potential μ.

Mathematical setting

- Typical class of models to be considered:



where Δ : lattice hopping; V external potential; B: magnetic field.

- Hamiltonian: $H = -\Delta_A + V$, with

$$\Delta_A(x;y) = \Delta(x;y) e^{i \int_{x \to y} d\ell \cdot A(\ell)} , \qquad \int_{\partial(\text{plaquette})} d\ell \cdot A(\ell) = \text{Flux}(B)$$

Insulators

- The beauty of the IQHE is that it does not rely on a specific model. The common feature is the insulating behavior:

  $$|\langle \delta_x, P_\mu \delta_y \rangle| \leq C e^{-c|x-y|} \qquad \text{exponential decay of correlations.}$$

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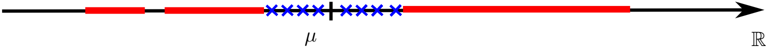
  - $\mu \in \sigma_{\rm pp}(H)$ & μ not eigenvalue of H. This is the case if V is a strong random potential: Anderson localization.



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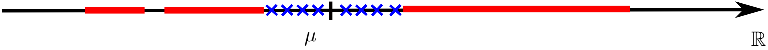

- Not true if $\mu \in \sigma_{\mathrm{ac}}(H)$. For instance, if $H = -\Delta$, $P_\mu(x; y)$ decays as a power law. Physically, these situations describe metals.

Quantization of σ_{12}

- A lenghty and nontrivial computation gives the following result for the Hall conducticity:

$$\sigma_{12} = \lim_{L\to\infty} \frac{i}{L^2} \mathrm{Tr}_{\ell^2(\mathbb{Z}^2)}\, \chi(|x| \leq L) P_\mu [[\hat{x}_1, P_\mu], [\hat{x}_2, P_\mu]]$$

which turns out to be an integer multiple of $1/2\pi$!

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- Simplest possible case. Let:

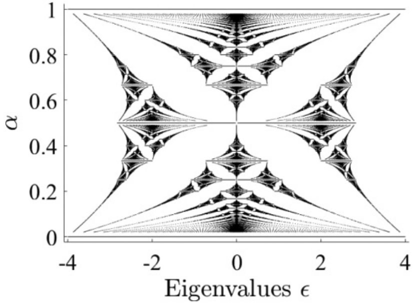
$$P_\mu = \bigoplus_{k\in\mathbb{T}^2} \hat{P}(k)\ , \qquad \hat{P}(k) = |\varphi(k)\rangle\langle\varphi(k)|\ , \qquad \varphi(k) \in \mathbb{C}^M$$

Then:

$$\sigma_{12} = \int_{\mathbb{T}^2} \frac{dk}{(2\pi)^2} \vec{\nabla} \times \langle \varphi(k), i\vec{\nabla}_k \varphi(k)\rangle \in \frac{1}{2\pi}\mathbb{Z}\ .$$

Colored butterflies

- Hofstadter model: Laplacian on $\mathbb{Z}^2$ with constant magnetic field. Spectrum in black, the resolvent in white.



- For rational magnetic field, the spectrum is a finite union of intervals. For irrational fields, the spectrum is a Cantor set! Ten Martini conjecture, proved by Avila & Jitomirskaya; Annals 2005, revisited by Jitomirskaya & Krasovsky; Annals 2021.

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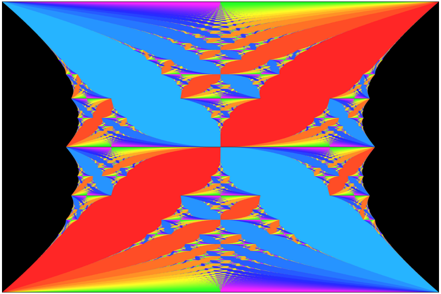

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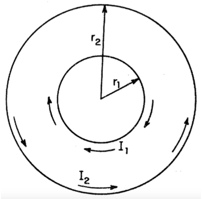
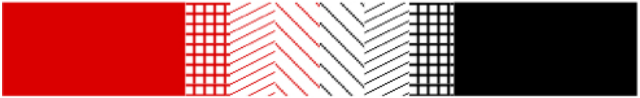
Bulk-edge duality

- Halperin '82. Hall phases must come with robust edge currents.



- Intuition: the transition from the nontrivial Hall phase to the void has to come with a insulator/metal transition:



As long as the system is in the insulating phase, the Hall conductivity is constant.

Edge states in quantum Hall systems

- Let H be a lattice Schrödinger operator on the cylinder Λ_L.
  For simplicity, assume transl. inv.: $H = \bigoplus_{k \in S^1} \hat{H}(k)$.

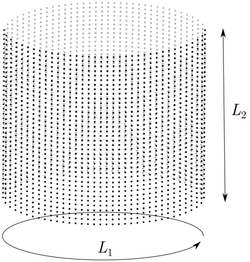


Figure: The lattice Λ_L ($L_1 = L_2$.)

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  $\sigma(H)$ might differ from $\sigma(H_p)$ by the presence of edge states.

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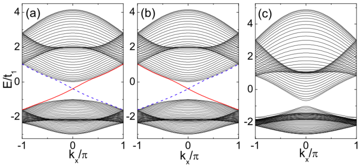


Figure: Spectrum of $\hat{H}(k)$, as function of $k \in S^1$.

- Red curve: eigenvalue branch $\varepsilon(k)$, with eigenstates (edge modes)

$$\varphi_x(k) = e^{ikx_1} \xi_{x_2}(k) \ , \qquad \text{with } \xi_{x_2}(k) \sim e^{-cx_2} \ .$$

The bulk-edge correspondence

- Bulk-edge duality: relation between σ_{12} and the edge states of H.

$$\boxed{\sigma_{12} = \sum_{\omega} \frac{\mathrm{sgn}(v_\omega)}{2\pi}} \qquad (\mathrm{sgn}(v_\omega) = \text{ chirality of edge mode})$$

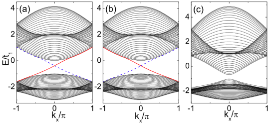


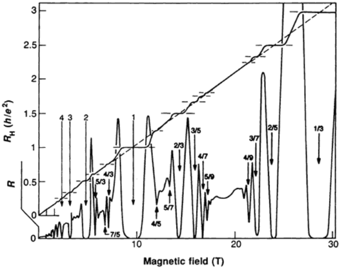
Figure: $(a): \ \sigma_{12} = \frac{1}{2\pi}, \qquad (b): \sigma_{12} = -\frac{1}{2\pi}, \qquad (c): \sigma_{12} = 0.$

- Rigorous results for noninteracting systems:
  - Hatsugai '93: Translation invariant systems.
  - Schulz-Baldes et al. '00: Disordered systems (with bulk gap).
  - Graf et al. '02: Mobility gap regime.
  - Graf-P. '13: Time-reversal invariant systems.

- Boundary states described in terms of effective $(1+1)d$ chiral QFT.

What is missing?

- The previous discussion only focused on non-interacting models. In real systems, electrons unavoidably interact. Taking them into account makes the analysis much harder.
- Still, the noninteracting theory is able to describe many experimentally observed phenomena, to an amazing precision. Why is that?
- A key aspect of the Hall effect not captured by the non-interacting theory is the fractionalization of the Hall conductivity: (FQHE)



Statistical mechanics description

- N-particle fermionic Hilbert space, finite lattice $\Lambda_L = [-L/2, L/2]^2$:

$$\mathfrak{h}_N = \left\{ \psi_N \in \ell^2(\Lambda_L^N) \,\middle|\, \psi_N \text{ is antisymmetric under permutations} \right\}$$

and N-particle Hamiltonian:

$$H_N = \sum_{i=1}^{N} 1^{\otimes(i-1)} \otimes H_i \otimes 1^{\otimes(N-i)} + \lambda \sum_{i<j}^{N} v(\hat{x}_i - \hat{x}_j)$$

- Useful to lift to a setting where the number of particles is not fixed:

$$\mathcal{F} = \bigoplus_{N \geq 0} \mathfrak{h}_N \ , \qquad \mathcal{H} = \bigoplus_{N \geq 0} H_N \ , \qquad \text{(Fock space)}$$

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- The equilibrium state of the system is defined by a density matrix:

$$\rho_{\beta,L,\mu} = \frac{1}{\mathcal{Z}_{\beta,L,\mu}} \bigoplus_{N\geq 0} e^{-\beta(H_N - \mu N)} \;, \qquad \langle \mathcal{O} \rangle_{\beta,L,\mu} := \mathrm{Tr}_{\mathcal{F}} \mathcal{O} \rho_{\beta,L,\mu} \;.$$

Program

- For $\lambda = 0$, the model is noninteracting: quantum analogue of a Gaussian free field. (All correlations computable via Wick rule.)
- For $\lambda \neq 0$, determining whether the system is in an insulating phase or not is much harder. The preliminary question we are interested in is:

$$\text{How fast does } \langle A_X B_Y \rangle - \langle A_X \rangle \langle B_Y \rangle \text{ decay as } \mathrm{dist}(X, Y) \to \infty?$$

Here, $\langle \cdot \rangle \equiv \langle \cdot \rangle_{\beta, L, \mu}$. We are interested in the $\beta, L \to \infty$ limit.

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- In this many-body setting, the Hall conductivity is given by:

$$\sigma_{12} = \lim_{\eta \to 0^+} \lim_{\beta \to \infty} \lim_{L \to \infty} \int_{-\infty}^{0} dt\, e^{\eta t} \langle [e^{i\mathcal{H}t} J_1 e^{-i\mathcal{H}t}, J_2] \rangle_{\beta,L,\mu}$$

for a suitable current operator $J = (J_1, J_2)$. The second task is:

Extend the theory of the QHE to the many-body setting.

QFT & RG

- We study these problems via quantum field theory and renormalization group methods.

- RG has been introduced by Wilson in the 70s for the study of critical phenomena in stat-mech. It is by now a powerful tool in mathematical physics: it allows to rigorously compute physical quantities (e.g. critical exponents) in cases where no exact solutions are available.

- In collaboration with A. Giuliani (Roma 3) and V. Mastropietro (Milan) we used these methods to initiate a rigorous study of interacting topological phases of matter, from a stat-mech viewpoint.

- Recently, with F. Caragiulo (SISSA) and H. P. Singh (SISSA), we extended the techniques to disordered topological insulators, and to study real-time dynamics for edge modes.

In a nutshell

- For small interactions, all these models can be represented as perturbation of Grassmann Gaussian QFTs in $d+1$ dimensions:

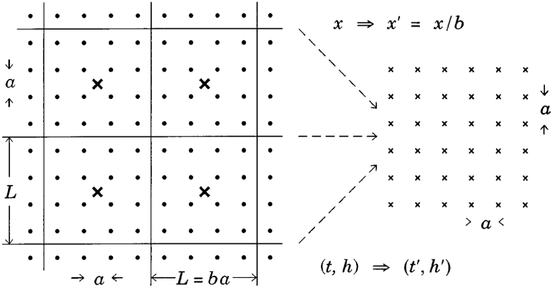
$$\mathcal{Z} = \int \nu(d\psi) e^{V(\psi)} \ ,$$

where $\nu(\cdot)$ is a suitable Grassmann Gaussian measure, and $V(\cdot)$ is a polynomial interaction (*e.g.* quadratic or quartic).

- The free theory enters in the determination of the covariance of $\nu(\cdot)$.
  - Insulators ($\mu \notin \sigma(H)$): the covariance decays exponentially fast. $\mathcal{Z}$ can be evaluated via a convergent determinant expansion.
  - Metals ($\mu \in \sigma(H)$): perturbation theory does not work. $\mathcal{Z}$ can be evaluated via multiscale analysis.

Renormalization group

- Connection between microscopic and macroscopic world, based on iterative coarse graining methods.



Replace microscopic quantities by local averages, end up with new model.

Iterative applications give rise to a “flow of models”.

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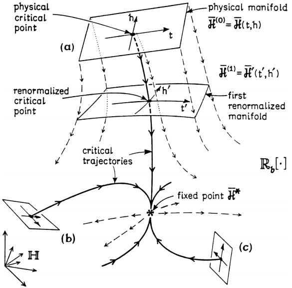

- Dynamical system in the space of models. Different models might share the same fixed point: Universality!
- The fixed point might be much simpler than the original model. Some of its properties (e.g., scaling exponents) might be explicitly computable.

Rigorous results about many-body systems

- Hastings-Michalakis; Comm. Math. Phys. 2015.
  Giuliani, Mastropietro, P.; Comm. Math. Phys. 2016.
  For $|\lambda|$ small enough, the Hall conductivity is universal:

  $$\sigma_{12}(\lambda) = \sigma_{12}(0) \; .$$

  Based on gauge-theoretic methods (Ward identities).

- Mastropietro, P.; Comm. Math. Phys. 2022: universality of the edge conductance:

  $$\sigma_{\mathrm{edge}}(\lambda) = \sigma_{\mathrm{edge}}(0) \; .$$

  Based on construction of scaling limit for edge correlations: Generalized Luttinger model, an integrable QFT.

- The combination of these two results allow to rigorously lift the bulk-edge duality to many-body systems, for small interaction.

Recent works with my group at SISSA

- w/ F. Caragiulo: extension of the methods to disordered systems. Scaling limit of edge correlations, in presence of quasi-periodic potential. Number-theoretic estimates combined with RG tools.

- w/ H. P. Singh: large-scale dynamics of edge currents, starting from the Schrödinger equation. Proof of bosonic behavior of edge currents, from first principles. Anomalous Ward identities and chiral anomalies.

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