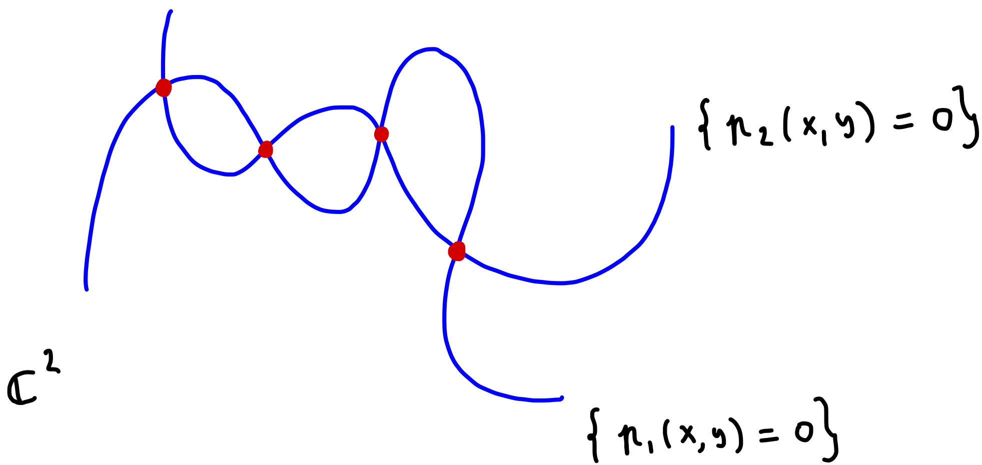
JUNIOR MATH DAYS

SISSA

"CONVEX BODIES AND ALGEBRAIC GEOMETRY"

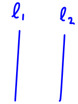
BÉZOUT'S THEOREM



BÉZOUT (1779): for the "generic" choice of $p_1, p_2 \in \mathbb{C}[x,y]_d$

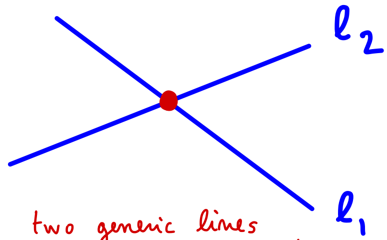
$$\#\{(x,y)\in\mathbb{C}^2 \text{ s.t. } p_1(x,y)=p_2(x,y)=0\} = d^2$$

BABY EXAMPLE: $d = 1$



← non generic situation ... really ?

$$\begin{cases} a_{10} x + a_{01} y + a_{00} = 0 \\ b_{10} x + b_{01} y + b_{00} = 0 \end{cases}$$



two generic lines in the plane intersect in just one point!

$$\det \begin{pmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{pmatrix} \neq 0 \implies \# = 1$$

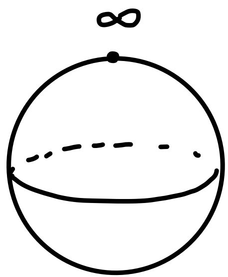
$$\Sigma = \{ \text{coefficients such that } a_{10} b_{01} - a_{01} b_{10} = 0 \}$$

$\Sigma \subset \mathbb{C}^{2 \times 3}$ has measure zero

"generic" $\sim$ "with probability one"

REMARK 1.

sometimes better to count zeroes in $\mathbb{CP}^2$ rather than $\mathbb{C}^2$; and work with homogeneous polynomials.

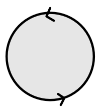
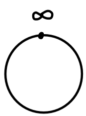
$\mathbb{CP}^1 =$  $\cup \ \{\infty\} =$ 

$\mathbb{C}$

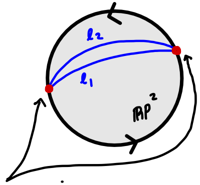
$\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1 = 4\text{-dim}.$

(projective) line at infinity

$4 = 2 \cdot 2$

$\mathbb{RP}^2 =$  $=$ $\cup$  $=$ $\cup$ $\cup \ \{\infty\}$

$\mathbb{R}^2$ $\qquad$ $\mathbb{RP}^1$ $\qquad$ $\mathbb{R}$



NOTE.
two parallel lines still intersect at one point!

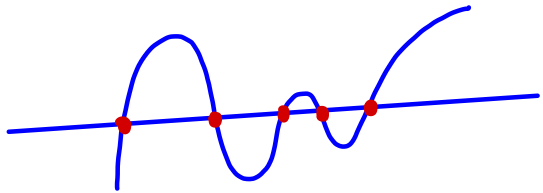
this is the same point!

REMARK 2.

if $\deg p_1 = d_1$ and $\deg p_2 = d_2$

$$\# = d_1 \cdot d_2$$

$$\Rightarrow \quad \deg p_1 = \# \{ p_1 = 0 \} \cap \text{"generic" line}$$

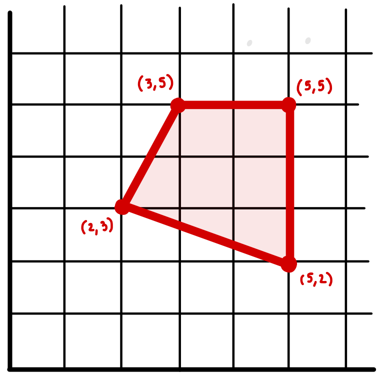


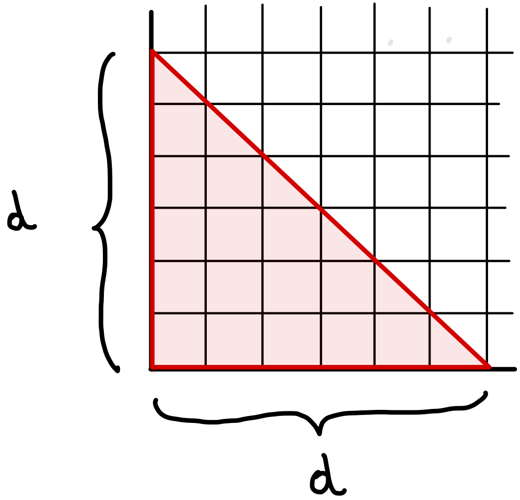
this is a picture over $\mathbb{R}$!

A MORE COMPLICATED EXAMPLE...

$$\begin{cases} a_{2,3}\, x^2 y^3 + a_{5,2}\, x^5 y^2 + a_{5,5}\, x^5 y^5 + a_{3,5}\, x^3 y^5 = 0 \\ b_{2,3}\, x^2 y^3 + b_{5,2}\, x^5 y^2 + b_{5,5}\, x^5 y^5 + b_{3,5}\, x^3 y^5 = 0 \end{cases}$$

$$\begin{bmatrix} a_{2,3} & a_{5,2} & a_{5,5} & a_{3,5} \\ b_{2,3} & b_{5,2} & b_{5,5} & b_{3\,5} \end{bmatrix}$$

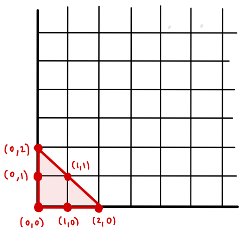




$P =$ convex hull of the exponents (α_1, α_2) with $\alpha_1 + \alpha_2 \leq d$

(case of $\mathbb{C}[x,y]_d$)

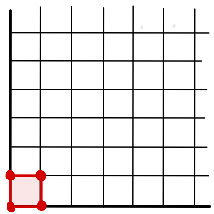
$$\mathrm{vol}(P) = \frac{d^2}{2}$$



$$a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$$

$$a_{00}x^0y^0 + a_{10}x^1y^0 + a_{11}x^1y^1 + a_{20}x^2y^0 + a_{02}x^0y^2$$

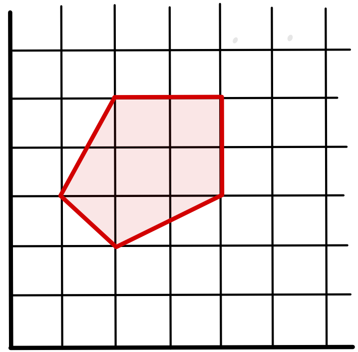
$$\begin{cases} a_{00} + a_{10}x + a_{01}y + a_{11}xy = 0 \\ b_{00} + b_{10}x + b_{01}y + b_{11}xy = 0 \end{cases}$$



$$a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2$$

THEOREM (BERNSTEIN-KHOVANSKII-KOUCHNIRENKO)

Let $A = \{ \alpha = (\alpha_1, \dots, \alpha_n) \} \subset \mathbb{Z}^n$ and consider the system of equations:



$$\begin{cases} \sum_{\alpha \in A} c_{1,\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 \\ \vdots \\ \sum_{\alpha \in A} c_{n,\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0 \end{cases}$$

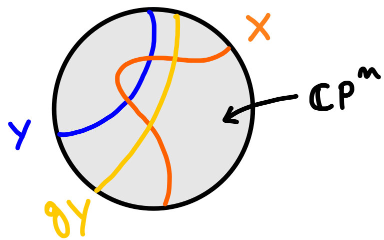
For the generic choice of $[c_{i,\alpha}] \in \mathbb{C}^{n \times |A|}$

$$\#\text{ SOLUTIONS IN } (\mathbb{C}^*)^n = n! \operatorname{vol}(P)$$

THE INTEGRAL GEOMETRY FORMULA

Let $X, Y \subset \mathbb{C}P^n$ be complex submanifolds with $\dim_{\mathbb{C}}(X) = a$, $\dim_{\mathbb{C}}(Y) = b$ and $a+b=n$.

$$\int_{U(n+1)} \# X \cap gY \, dg = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{C}P^a)} \cdot \frac{\mathrm{vol}(Y)}{\mathrm{vol}(\mathbb{C}P^b)}$$



NOTE

$\int_{U(n+1)} dg = 1$

$\mathrm{vol}(X) = 2a$-dim volume

$X \hookrightarrow \mathbb{C}P^n$, inherits a Riemannian structure

Take the Riemannian volume form (actually $\mathrm{vol} = \underbrace{\omega \wedge \cdots \wedge \omega}_{a}$)

$g \cdot [v] \doteq [gv]$

COROLLARY (VOLUME = DEGREE)

$$\int_{U(m+1)} \# X \cap gY \, dg = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{C}P^a)} \cdot \frac{\mathrm{vol}(Y)}{\mathrm{vol}(\mathbb{C}P^b)}$$

$Y = \mathbb{C}P^{m-a}$ $\quad$ $Y = \mathbb{C}P^{m-a}$

$$\int_{U(m+1)} \# X \cap g\mathbb{C}P^{m-a} \, dg = \frac{\mathrm{vol}(X)}{\mathrm{vol}(\mathbb{C}P^a)} \cdot \frac{\mathrm{vol}(\mathbb{C}P^b)}{\mathrm{vol}(\mathbb{C}P^b)}$$

$\# X \cap g\mathbb{C}P^{m-a} = d$ a.e.

$$= \int_{U(m+1)} \deg(X) \, dg$$

$\mathrm{vol}(\mathbb{C}P^a) = \frac{a!}{\pi^a}$

$\int_{U(m+1)} dg = 1$

$$\deg(X) = \mathrm{vol}(X) \frac{a!}{\pi^a}$$

THE GEOMETRY OF VERONESE...

$A = \{ \alpha = (\alpha_1, \dots, \alpha_n) \} \subset \mathbb{Z}^n$ list of "indices"

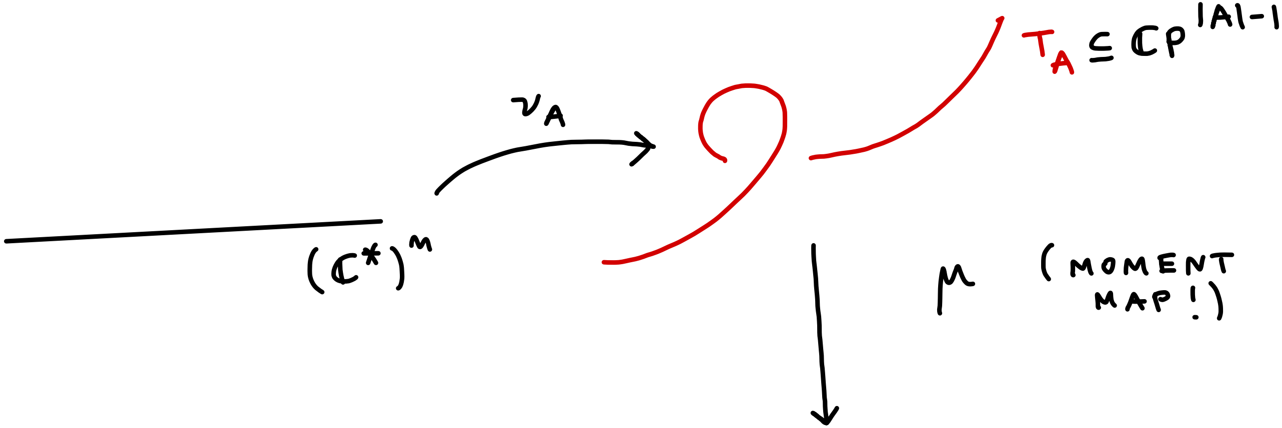
$$\nu_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{CP}^{|A|-1}$$
$$x \longmapsto [\, x^\alpha \,]_{\alpha \in A}$$

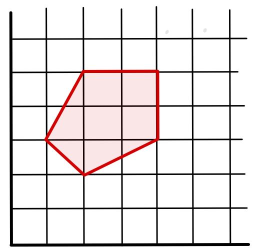
ex $n=1$, $A = \{0, 1, \dots, d\}$ $\quad \nu_A(x) = [1, x, x^2, \dots, x^d]$

ex $n=2$ $A = \{ (\alpha_1, \alpha_2)$ with $0 \le \alpha_1 \le \alpha_2 \le d \}$ $\quad \nu_A =$ VERONESE MAP

$$T_A := \overline{\nu_A (\mathbb{C}^*)^n}$$

MORE GEOMETRY..



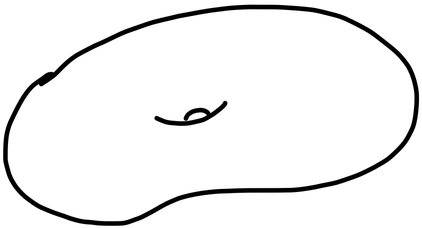


$\mu : T_A \longrightarrow P_A$

s.t. given $W \subset P_A$

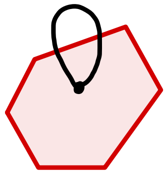
measurable $\Rightarrow$

$$\mathrm{vol}(\mu^{-1}(W)) = \mathrm{vol}(W) \cdot \pi^n$$



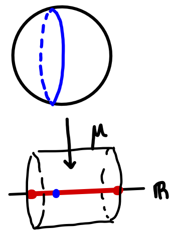
$$\approx$$

$(S^1)^n$



$$\mathrm{vol}(T_A) = \pi^n \cdot \mathrm{vol}(P_A)$$

EXAMPLE

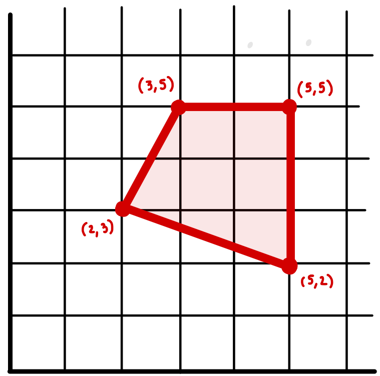


$\mathbb{C}P^1 \simeq S^2$

Let's go back to our original example...

$$\begin{cases} a_{2,3}\, x^{2} y^{3} + a_{5,2}\, x^{5} y^{2} + a_{5,5}\, x^{5} y^{5} + a_{3,5}\, x^{3} y^{5} = 0 \\ b_{2,3}\, x^{2} y^{3} + b_{5,2}\, x^{5} y^{2} + b_{5,5}\, x^{5} y^{5} + b_{3,5}\, x^{3} y^{5} = 0 \end{cases}$$

$$\begin{bmatrix} a_{2,3} & a_{5,2} & a_{5,5} & a_{3,5} \\ b_{2,3} & b_{5,2} & b_{5,5} & b_{3,5} \end{bmatrix}$$



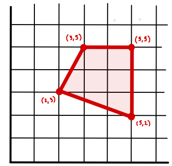
$\#$ SOLN. $= 2!\, \mathrm{vol}(P_A)$

$$\begin{cases} a_{2,3}\, x^2 y^3 + a_{5,2}\, x^5 y^2 + a_{5,5}\, x^5 y^5 + a_{3,5}\, x^3 y^5 = 0 \\ b_{2,3}\, x^2 y^3 + b_{5,2}\, x^5 y^2 + b_{5,5}\, x^5 y^5 + b_{3,5}\, x^3 y^5 = 0 \end{cases}$$

PROOF OF BKK:

$$\# = \mathrm{vol}(P_A)\cdot n!$$

$$\begin{bmatrix} a_{2,3} & a_{5,2} & a_{5,5} & a_{3,5} \\ b_{2,3} & b_{5,2} & b_{5,5} & b_{3,5} \end{bmatrix}$$



1. $$\nu_A : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}P^{|A|-1}$$

$$T_A := \overline{\mathrm{im}(\nu_A)}$$

$$\mathrm{vol}(T_A) = \mathrm{vol}(P_A)\,\pi^n$$

2. $$\int_{U(|A|)} \# T_A \cap g\mathbb{C}P^{n-a}\, dg \overset{(\mathrm{IGF})}{=} \mathrm{vol}(T_A)\cdot \frac{n!}{\pi^n} = \mathrm{vol}(P_A)\, n!$$

3.

$$g \in U(N) \qquad \begin{bmatrix} \mathbb{1} & \Big| & 0 \end{bmatrix} \cdot g = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & & \\ a_{m1} & \cdots & a_{mN} \end{bmatrix} \in \mathbb{C}^{m \cdot N}$$

$$(N = |A|)$$

$$\begin{cases} c_{11} y_1 + \cdots + c_{1N} y_N = 0 \\ \vdots \\ c_{m1} y_1 + \cdots + c_{mN} y_N = 0 \end{cases} = g\,\mathbb{C}P^{N-m}$$

$$T_A \cap g\,\mathbb{C}P^{N-m} = \begin{cases} c_{11} y_1 + \cdots + c_{1N} y_N = 0 \\ \vdots \\ c_{m1} y_1 + \cdots + c_{mN} y_N = 0 \\ (y_1, \dots, y_N) \in \mathcal{V}_A \end{cases}$$

$$\int_{U(|A|)} \# \, T_A \cap g\,\mathbb{C}P^{n-a} \, dg \overset{(\text{IGF})}{=} \operatorname{vol}(T_A) \cdot \frac{n!}{\pi^n} = \operatorname{vol}(P_A)\, n!$$

$$= \begin{cases} \sum_{\alpha \in A} c_{1,\alpha} x_1^{\alpha_1} \cdots x_m^{\alpha_m} = 0 \\ \vdots \\ \sum_{\alpha \in A} c_{m,\alpha} x_1^{\alpha_1} \cdots x_m^{\alpha_m} = 0 \end{cases}$$

$$\Longrightarrow \quad \# = \operatorname{vol}(P_A) \cdot n!$$

$\square$

REMARK

$$\sum_{\alpha \in A_1} c_{1,\alpha} x^{\alpha} = \cdots = \sum_{\alpha \in A_n} c_{n,\alpha} x^{\alpha}$$

$$\# \text{ SOL.} = MV(P_{A_1}, \dots, P_{A_n})$$

$$MV(K_1, \dots, K_n) = \left[\operatorname{vol}(t_1 K_1 + \dots + t_n K_n) \right]_{t_1 \cdots t_n}$$

LET'S GET REAL!

THM (SHUB-SMALE)

$$\mathbb{E}\,\#\left\{\begin{array}{l}\text{real solutions}\\ \text{of a system of}\\ n \text{ equations of degree } d\end{array}\right\} = \sqrt{d^n}$$

"PROOF"

$$\nu^{\mathbb{R}} : \mathbb{RP}^n \longrightarrow \mathbb{RP}^N$$

$$\mathbb{E}\,\# = \int_{O(N+1)} \nu(\mathbb{RP}^n) \cap g\,\mathbb{RP}^{N-n}\, dg = \frac{\mathrm{vol}(\nu(\mathbb{RP}^n))}{\mathrm{vol}(\mathbb{RP}^n)} = \sqrt{d^n}$$

REFERENCES

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GELFAND, KAPRANOV, ZELEVINSKY: "DISCRIMINANT, RESULTANTS..."