A Game of Scales

Concentration and oscillations in asymptotic variational problems

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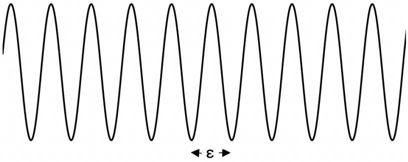
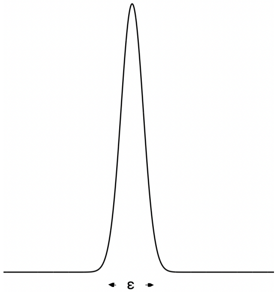
The Distributions revolution (L. Schwartz)

(and Young's measures, and currents, and varifolds, . . .)

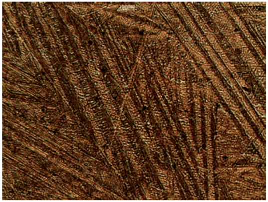
. . . all of a sudden, all functions became infinitely differentiable and all sequences converging. . .



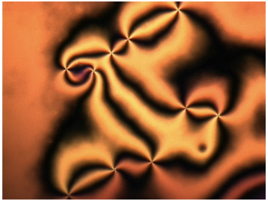


Oscillations and concentration became a source of interest and not an obstruction to regularity.

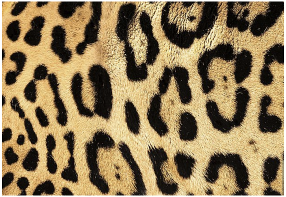
Application-driven suggestions



oscillations in alloys



singularities in liquid crystals



pattern formation

Effects in the Calculus of Variations

The classical questions of the Calculus of Variations regarded existence and regularity for a single minimum problem.

With the freedom of having non-strongly converging minimizing sequences, it became interesting to study sequences of problems.

The variational formulation
(In a classical context) we introduce some (small, positive) parameter ε, and consider problems

$$\min\Big\{\int_\Omega f_\varepsilon(x,u,\nabla u)\,dx : u\in X_\varepsilon\Big\}$$

As we let $\varepsilon\to 0$ do minimizers (or almost minimizers) develop oscillations/concentration at some *scale*?

What are the relevant features of f_ε that drive the scale?

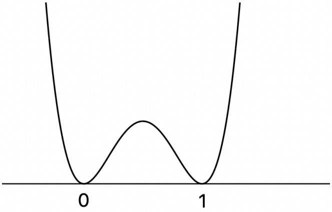
A classic: gradient theory of phase transitions

Van der Waals/Cahn-Hilliard theory of phase transitions
($u =$ density of a fluid)

Variational principle: optimal configurations minimize the free energy

$$\int_\Omega W(u(x))\,dx + \varepsilon^2 \int_\Omega |\nabla u|^2\,dx \quad \text{with given} \quad \int_\Omega u(x)\,dx = m|\Omega|$$

$W =$ double-well potential (we can suppose with two minima at 0 and 1)
and $m \in (0,1)$



Asymptotic analysis

Equivalent energy:

$$F_\varepsilon(u) = \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \varepsilon\int_\Omega |\nabla u|^2\,dx \left(\geq 2\int_\Omega \sqrt{W(u)}|\nabla u|\,dx \right)$$

with $X_\varepsilon = \left\{ u \in H^1(\Omega) : \int_\Omega u(x)\,dx = m|\Omega| \right\}$

Convergence: $u_\varepsilon \to A$ if $u_\varepsilon \to \chi_A$ in measure.

Minimal-interface criterion: the limit A minimizes a *perimeter functional*

$$F_0(A) = c_W \operatorname{Per}(A, \Omega)$$

with $X_0 = \left\{ A \text{ set of finite perimeter in } \Omega : |A| = m|\Omega| \right\}$

Scale: minimizers u_ε make a transition from 0 to 1 in an ε-neighbourhood of $\Omega \cap \partial A$

Classic elliptic homogenization

The simplest example is

$$\min\left\{\int_0^1 a\left(\frac{x}{\varepsilon}\right)|u'(x)|^2\,dx \;-2\int_0^1 g(x)u(x)\,dx : u(0)=0,\; u(1)=z\right\}$$

with a 1-periodic.

The Euler-Lagrange equation is

$$-\left(a\left(\frac{x}{\varepsilon}\right)u'_\varepsilon\right)' = g,$$

which gives (G a primitive of g)

$$u'_\varepsilon = -\frac{G(x)+\mathrm{const}}{a(\frac{x}{\varepsilon})} \rightharpoonup -\frac{G(x)+\mathrm{const}}{\underline{a}},\ \text{with } \frac{1}{\underline{a}} = \int_0^1 \frac{1}{a(s)}\,ds$$

($\underline{a}$ is the *harmonic mean* of a)

Homogenized limit

Using the boundary conditions to determine the constants, we then have that u_ε weakly converges to the minimizer of

$$\min\Big\{\underline{a}\int_0^1 |u'(x)|^2\,dx\ -2\int_0^1 g(x)u(x)\,dx : u(0)=0,\ u(1)=z\Big\}.$$

Scale of u_ε: in this case the *scale* is the period of the energy, which translates in the period of the solutions.

u_ε oscillates at scale ε and optimizes the oscillations according to the coefficient a.

Note that for strongly converging u_ε to u we have

$$\lim_{\varepsilon\to 0}\int_\Omega a\Big(\frac{x}{\varepsilon}\Big)|u_\varepsilon'(x)|^2\,dx = \overline{a}\int_\Omega |u'(x)|^2\,dx,$$

where $\overline{a}=\int_0^1 a(s)ds$ is the *average* of a (and $\overline{a}>\underline{a}$ unless a is constant).

De Giorgi's Γ-convergence

In both examples we have

• there exists $\lim_{\varepsilon \to 0^+} \inf F_\varepsilon \to \min F_0$

• we have convergence of minimizing sequences to minimizers

• the form of F_0 is independent of m, and g and boundary values in the two case.

This fact can be translated in a *convergence of functionals*.

This convergence can be formulated in *topological* terms: $F_\varepsilon : X_\varepsilon \to [-\infty, +\infty]$ Γ**-converge** to $F_0 : X_0 \to [-\infty, +\infty]$, with respect to a convergence $x_\varepsilon \to x_0$ if

(i) (*liminf inequality*) $F_0(x_0) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$ for all $x_\varepsilon \to x_0$;

(ii) (*existence of a recovery sequence*) for all x_0 there exists $\overline{x}_\varepsilon \to x_0$ such that $F_0(x_0) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(\overline{x}_\varepsilon)$

Note:(upon some compactness requirements) The proof that Γ-convergence implies convergence of minima is elementary and is essentially the same of the Weierstrass Theorem. Indeed, Γ-convergence is *equivalent* to convergence of minima and minimizing sequences (and “stability wrt continuous perturbations).

The analytical question

• compute the Γ-limit F_0

• describe the behaviour of minimizing sequences through suitable formulas and a *scale*.

General variational formulas

Phase-transition problems The constant c_W is given by a one-dimensional **optimal-profile problem**

$$c_W = \min\Big\{ \int_{-\infty}^{+\infty} (W(v(t)) + |v'(t)|^2) dt : v(-\infty) = 0,\ v(+\infty) = 1 \Big\}$$

and $u_\varepsilon(x) \sim v(\frac{d(x,\partial A)}{\varepsilon})$ where A is the minimal set and d is the signed distance

Homogenization problems

The function $\underline{a}|z|^2$ can be expressed by a **cell-problem formula**

$$\underline{a}|z|^2 = \min\Big\{ \int_0^1 a(t)|\varphi'(t)|^2 dt : \varphi(t) - zt \text{ 1-periodic } \Big\}$$

(optimization over all periodic perturbations of the function z). If we denote by $\varphi(\cdot, z)$ the minimum, then $u_\varepsilon(x) \sim \varphi(\frac{x}{\varepsilon}, u'(x))$, where u is the solution to the limit problem.

This formula can be used to described the limit also in dimension $d > 1$ e.g. of

$$F_\varepsilon(u) = \int_\Omega \Big\langle A\Big(\frac{x}{\varepsilon}\Big)\nabla u(x), \nabla u(x)\Big\rangle\, dx,$$

with A a $(0,1)^d$-periodic definite positive matrix. The Γ-limit is described by a constant matrix $A_{\rm hom}$ with

$$\langle A_{\rm hom}\xi, \xi\rangle = \min\Big\{ \int_{(0,1)^d} \langle A(y)\nabla\varphi, \nabla\varphi\rangle dy : \varphi(y) - \xi y \text{ 1-periodic } \Big\}$$

Optimization of oscillations in different directions

This formula highlights that optimal oscillations can be different in different direction.
If for example $d = 2$ and $A(y_1, y_2) = a(y_1)\mathrm{Id}$ with a 1-periodic, so that

$$F_\varepsilon(u) = \int_\Omega a\Big(\frac{x_1}{\varepsilon}\Big)|\nabla u|^2\,dx,$$

we have oscillations only in the y_1-direction and $A_{\mathrm{hom}} = \begin{bmatrix} \underline{a} & 0 \\ 0 & \overline{a} \end{bmatrix}$

(and conversely if $A(y) = a(y_2)\mathrm{Id}$), showing that the limit does not depend only on averaged quantities of the coefficients of A.

Consequence: this shows that all (homogeneous) elliptic functionals $\int_\Omega \langle B\nabla u, \nabla u\rangle\,dx$ are Γ-limits of isotropic functionals $\int_\Omega a(\frac{x}{\varepsilon})|\nabla u|^2\,dx$
(Easy: change variables so as to write B as a diagonal matrix...)

Problem (mixtures of N isotropic functionals) If $a = a(x_1, \ldots, x_d)$ is 1-periodic and can take only finitely many values $a_1, a_2, \ldots, a_N$ with given proportions, what are the possible limits of $\int_\Omega a(\frac{x}{\varepsilon})|\nabla u|^2\,dx$?
Difficult. If $N \geq 3$ open.

Oscillations at all periods for vector problems

The cell-problem formula still holds if we consider more general functionals

$$F_\varepsilon(u) = \int_\Omega f\Big(\frac{x}{\varepsilon}, \nabla u\Big)dx,$$

with $f(\cdot,\xi)$ 1-periodic as long as u is **scalar**.

If u is **vector valued** the integrand $f_{\rm hom}(\xi)$ with $\xi \in \mathbb{R}^{m\times d}$ of the limit is given by an **asymptotic formula**

$$f_{\rm hom}(\xi) = \inf_{N\in\mathbb{N}} \frac{1}{N^d} \min\Big\{ \int_{(0,1)^d} f(y,\nabla\varphi)dy : \varphi(y) - \xi y \text{ N-periodic} \Big\};$$

that is, minimizers may develop oscillations at all period N at scale ε (they can be a superposition of εN-periodic functions for all N and hence almost periodic)

Question: under what conditions we have that the minimum is achieved at N finite? Easy answer: $f(y,\cdot)$ convex. Many general answers (especially for special values of ξ; cf. the Cauchy-Born rule)

Oscillations and concentration

Concentration can be driven also by boundary conditions. If these boundary conditions are imposed on periodic varying sets these two effects add up. A classic example is that of Dirichlet boundary conditions on an ε-periodic perforation in $\mathbb{R}^d$ $(d \geq 3)$

$$F_\varepsilon(u) = \int_\Omega |\nabla u|^2 \, dx$$

$X_\varepsilon = \Big\{ u \in H^1(\Omega) : u = 0$ on $\bigcup_k B_{\varepsilon^{d/(d-2)}}(\varepsilon k) \Big\}$,

where $B_r(x)$ denotes the ball of center x and radius r and the sum is performed on all k such that $\overline{B}_{\varepsilon^{d/(d-2)}}(\varepsilon k) \subset \Omega$.

(**“Strange term coming from nowhere”**).The Γ-limit is

$$F_0(u) = \int_\Omega |\nabla u|^2 \, dx + C \int_\Omega |u|^2 \, dx.$$

where C is the *capacity of the unit ball* in $\mathbb{R}^d$

Optimal sequences concentrate on the boundary of the (very) small balls minimizing the capacity (at scale $\varepsilon^{d/(d-2)}$) and at the same time give an average contribution summing up in the last term.

Question: what are the possible limits if we take arbitrary sets where we require that $u = 0$?
(Answer: general “strange terms” integrated with respect to *capacitary measures*. This is a complex theory known as the theory of ‘Relaxed Dirichlet Problems’)

A different type of concentration: point singularities

Ginzburg-Landau functionals. In the vector case (e.g. $d = m = 2$), we can consider problems

$$\min\left\{\int_\Omega |\nabla u|^2\,dx + \frac{1}{\varepsilon^2}\int_\Omega (|u|^2-1)^2\,dx : u = \varphi \text{ on } \partial\Omega\right\}$$

with non-trivial solutions if $\varphi : \partial\Omega \to S^{d-1}$ has *non-zero degree*.

The term $\frac{1}{\varepsilon^2}(|u_\varepsilon|^2-1)^2$ is a penalization term that forces $|u| = 1$ in the limit, but the relevant behaviour of u_ε is that they develop **vortices** around point singularities; i.e., close to a finite set of x_0 they tend to be of the form

$$u_\varepsilon(x) \sim \left(\frac{x-x_0}{|x-x_0|}\right)^n$$

(in complex notation) where n is the degree of the vortex.

Here the relevant converging quantity is the Jacobian of u_ε, whose limit is a finite sum of Dirac deltas at points in Ω with integer coefficients.

Microscopic and macroscopic scales

Recent developments (e.g. in Data Science) are requiring the analysis of (large) sets of points with little geometrical structure or regularity. Variational techniques have been used both directly on discrete sets and on continuum approximations. The simplest standpoint is that of considering

$$F_\varepsilon(u) = \sum_{i \neq j} \Phi_{ij}(u_i, u_j),$$

where $X_\varepsilon = \{\{u_i\}_{i \in \mathcal{I}_\varepsilon}\}$ is a set parameterizing an increasing set of points, and Φ_{ij} is a function measuring some interaction between the points parameterized at i and j. The cardinality of $\mathcal{I}_\varepsilon$ diverges as $\varepsilon \to 0$.

Alternatively, we can consider approximate continuum models

$$F_\varepsilon(u) = \int_X \int_X f_\varepsilon(u(x), u(y))dxdy$$

The study of the asymptotic behaviour of such energies is very complex by their non-locality and superposition of scales

• 'microscopic effects' favouring oscillations of nearby points;

• 'macroscopic effects' favouring oscillations at a scale much larger than the 'miocroscopic one'

• 'incommensurability effects' in the discrete case (e.g. if $\mathcal{I}_\varepsilon$ is parameterized on a lattice);

• 'nonlocal effects', in which microscopic and macroscopic oscillations compete.

An example of a project starting from a recent course

('Another look at Homogenization' (Milan J. Math 2023) by B, Brusca and Donati)

Problem. Compare 'non-local fractional homogenization' with 'local' homogenization.

In terms of functionals, let $\varepsilon \to 0$ and $s_\varepsilon \to 1^-$

$$F_\varepsilon(u) = \frac{1 - s_\varepsilon}{2} \int_0^1 \int_0^1 a\left(\frac{x}{\varepsilon}\right) \frac{|u(x) - u(y)|^2}{|x - y|^{1+2s}} dxdy,$$

Then the limit is $\underline{a} \int_0^1 |u'|^2 \, dx$ (as in the continuum case) if and only if $1 - s << \varepsilon^2$

Open problem: is this threshold optimal; that is, is the limit equal to $\overline{a} \int_0^1 |u'|^2 \, dx$ (averaged limit) if $1 - s >> \varepsilon^2$

Thank you for your attention