Moduli

or,

a leisurely tour through the geometry of some moduli spaces

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Moduli problems (some handwaving)

Moduli functor

$$\mathfrak{M}\colon \mathbf{Sch} \to \mathbf{Set}$$

$$S \mapsto \left\{ \begin{array}{l} \text{Families of "objects"} \\ \text{parametrized by } S \end{array} \right\} \Big/ \text{isomorphism}$$

$\mathfrak{M}$ is *representable* if there exists a scheme M such that

$$\mathfrak{M} \simeq \mathrm{Hom}_{\mathbf{Sch}}(-, M)$$

The object $\mathcal{U} \in \mathfrak{M}(M)$ corresponding to $\mathrm{Id} \in \mathrm{Hom}_{\mathbf{Sch}}(M, M)$ is a universal object:

If $\mathcal{F}$ is a family parametrized by a scheme S, and $f\colon S \to M$ is the corresponding morphism, then $\mathcal{F} \simeq f^*\mathcal{U}$

Coarse moduli

There is also a weaker notion of co-representable functor. There is no universal object, but still given a family F parameterized by S, there is still a "modular morphism" $f : S \to M$; so if $s \in S$, the point $f(s) \in M$ corresponds to the object parametrized by s.

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Example: vector bundles on algebraic curves

X a compact Riemann surface, i.e., a compact complex manifold of dimension 1

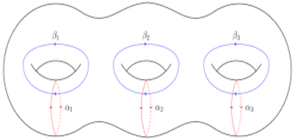
there is a theorem (Kodaira) saying that any compact Riemann surface can be embedded into a complex projective space

another theorem (Chow's lemma) says that any analytic closed subset of a projective space is cut by a set of polynomials (i.e., it is algebraic)

$\rightsquigarrow$ X is a (irreducible, smooth, projective) algebraic curve

$U \subset$ open subset $\rightsquigarrow$ $\mathcal{O}_X(U)$ ring of holomorphic/regular functions on U (this defines a sheaf of rings on X)

A compact Riemann surface has a topological invariant, the genus g: $H^1(X,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. To be on the safe side we shall assume $g \geq 2$.



Rank r vector bundle on X: a sheaf $\mathcal{V}$ of $\mathcal{O}_X$-modules on X such that locally

$$\mathcal{V}(U) \simeq \mathcal{O}_X(U)^{\oplus r}$$

r is the rank of $\mathcal{V}$

One can trade this for a variety with a surjection to X whose sheaf of sections is $\mathcal{V}$

$$\begin{array}{ccc} & & V \\ & \overset{s}{\nearrow} & \downarrow \\ U & \hookrightarrow & X \end{array}$$

$\mathcal{V}$ has a topological invariant, its degree, $\deg \mathcal{V} \in \mathbb{Z}$. Topologically, a vector bundle on a Riemann surface is classified by its rank and degree

Slope of $\mathcal{V}$: $$\mu(\mathcal{V}) = \frac{\deg \mathcal{V}}{r}$$

Stable bundles

Definition

$\mathcal{V}$ is stable if for every proper subbundle $\mathcal{W}$ of $\mathcal{V}$ the inequality

$$\mu(\mathcal{W}) < \mu(\mathcal{V})$$

holds.

A consequence of stability is that the bundle is simple, i.e., it has the minimal amount of automorphisms (homotheties)

There is also a notion of semistability, where the strict inequality is replaced by the condition

$$\mu(\mathcal{W}) \leq \mu(\mathcal{V})$$

Fix (r, d)

Theorem (Narasimhan-Seshadri 1965)

There is a smooth quasi-projective scheme $\mathcal{M}(r, d)$ which co-represents the moduli functor of families of stable table vector bundles on X of rank r and degree d.

If gcd $(r, d) = 1$ the moduli scheme is fine (it represents the functor) and projective.

So in the coprime case there is a universal bundle $\mathcal{U}$ on $X \times \mathcal{M}(r, d)$. Thus if $\mathcal{V}$ is a family of stable vector bundles of rank r and degree d on X parametrized by a scheme S (i.e., a bundle $\mathcal{V}$ on $X \times S$ such that the restriction to every fibre of $X \times S \to S$ is a stable vector bundle on X of rank r and degree d), there is morphism $f \colon S \to \mathcal{M}(r, d)$ such that

$$\mathcal{V} \simeq (\mathrm{id} \times f)^* \mathcal{U}$$

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Higgs bundles

ω_X cotangent (canonical) bundle of X

A Higgs bundle is a pair $\mathcal{E} = (\mathcal{V}, \phi)$, where $\mathcal{V}$ is a vector bundle, and ϕ is a morphism $\mathcal{V} \to \mathcal{V} \otimes \omega_X$

If z is a local coordinate on X, and we locally trivialize $\mathcal{V}$, the Higgs field is given by a 1-form valued matrix

$$\phi(e_\alpha) = \sum_{\beta=1}^{r} \phi_{\alpha\beta}(z)\, e_\beta \otimes dz$$

Same definition of stability, but only with respect to ϕ-invariant subbundles

$$\phi(\mathcal{W}) \subset \mathcal{W} \otimes \omega_X$$

Theorem (Nitsure)

There is a quasi-projective coarse moduli space $\mathcal{M}^H(r,d)$ co-representing the functor of families of stable Higgs bundles of rank r and degree d.
The subset $\mathcal{M}_0^H(r,d)$ corresponding to Higgs vector bundles whose underlying bundle is itself stable is a smooth quasi-projective irreducible variety.

$\mathcal{M}_0^H(r,d)$ contains as an open subset the cotangent bundle to the moduli space of stable bundles $\mathcal{M}(r,d)$ (indeed a point in the cotangent space is in a natural way a Higgs field)

$$T^*_{\mathcal{V}}\mathcal{M}(r,d) \simeq H^1(X, End(\mathcal{V}))^* \simeq \operatorname{Hom}(\mathcal{V}, \mathcal{V} \otimes \omega_X)$$

$\bar{\mathcal{M}}^H(r,d)$ moduli space of semistable Higgs bundles. In spite of the notation it is not projective — but it has a proper morphism to an affine space (the *Hitchin map*)

For every Higgs bundle $\mathcal{E} = (\mathcal{V}, \phi)$ we take the coefficients of the characteristic polynomial of ϕ , which can be expressed in terms of the quantities $\operatorname{tr}\phi^j$. These are global sections of the bundles ω_X^j for $j = 1, \ldots, r = \operatorname{rk}\mathcal{V}$. So we define the affine space

$$\mathbb{A}(X,r,d) = \bigoplus_{j=1}^{r} H^0(X, \omega_X^j)$$

and we have the Hitchin map

$$\mathrm{Hitch}_{X,r,d} \colon \bar{\mathcal{M}}^H(r,d) \to \mathbb{A}(X,r,d)$$

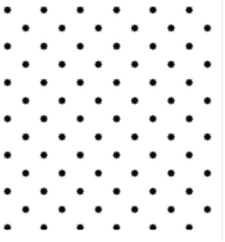
Theorem (Hitchin 1987, Nitsure 1990)

The Hitchin map is proper

Digression I: abelian varieties

Let Λ be a nondegenerate lattice in $\mathbb{C}^g$ (so rank $\Lambda = 2g$)

The quotient $\mathbb{C}^g/\Lambda$ is a g-dimensional complex manifold (called a complex torus), homeomorphic to $(S^1)^{2g}$, as $\mathbb{R}/\mathbb{Z} \simeq S^1$



A lattice in $\mathbb{C}$

If it is algebraic it is called an abelian variety (it is a variety and has a compatible structure of abelian group)

By the Kodaira theorem it is always algebraic when $g = 1$ (it is an elliptic curve)

Digression II: integrable systems

The phase space $\mathcal{P}$ of a mechanical system $\mathcal{S}$ is the cotangent bundle of the configuration space of $\mathcal{S}$, so it has a symplectic form, which in adapted coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ is written

$$\omega = \Sigma_{i=1}^n dp_i \wedge dq_i$$

This defines the Poisson bracket on functions on $\mathcal{P}$:

$$\{f, g\} = \omega^{-1}(df, dg) = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

A first integral of $\mathcal{S}$ is a function I on $\mathcal{P}$ which is constant during the evolution of the system. Two first integrals are in involution if $\{I_1, I_2\} = 0$

Definition

$\mathcal{S}$ is an integrable system if it has n independent first integrals in involution

Theorem (Arnold-Liouville)

If $\mathcal{S}$ is an integrable system, there is an open dense subset of $\mathcal{P}$ which is a fibration over an n-dimensional Euclidean space, and the fibers are (real) n-dimensional tori. The evolution of the system takes place along one such torus, selected by the initial conditions

(The coordinates in the Euclidean space are called actions, and those on fibers, angles)

There is a similar notion in the holomorphic/algebraic setting (algebraically completely integrable system)

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Back to Higgs bundles

Theorem (Hitchin 1987)

The Hitchin map

$$\mathrm{Hitch}_{X,r,d} \colon \bar{\mathcal{M}}^H(r,d) \to \mathbb{A}(X,r,d)$$

is an algebraically completely integrable system

The fibers are abelian varieties

Moduli spaces of curves

Space (of some sort) parameterezing isomorphism classes of algebraic curves of genus g

Riemann already knew that for $g \geq 2$ this space has dimension $3g - 3$ (although the space did not yet exist)

The most modern version of it is the Deligne-Mumford moduli stack of marked stable curves $\bar{\mathcal{M}}_{g,n}$

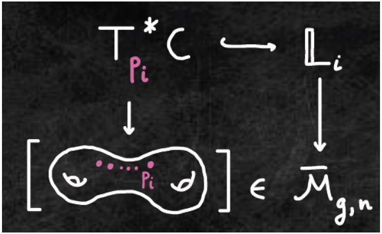
Main properties

- it is smooth (as a stack) of dimension $3g - 3 + n$
- it is proper

$\bar{\mathcal{M}}_{g,n}$ has a fundamental class over which one can integrate

$$\int_{\bar{\mathcal{M}}_{g,n}} \alpha \in \mathbb{Q}$$

Tautological line bundles



$1 \leq i \leq n$

$$\psi_i = c_1(\mathbb{L}_i) \in H^2(\bar{\mathcal{M}}_{g,n}, \mathbb{Q})$$

Intersection numbers

$$\int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \in \mathbb{Q}$$

$(\sum_i a_i = 3g - 3 + n)$

Thanks to Andrea Ricolfi for the figure!

Gromov-Witten potential

$$F(\lambda, t) = \sum_g \sum_n \frac{\lambda^{2g}}{n!} \sum_{a_1,\dots,a_n} \left(\int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \right) t_{a_1} \cdots t_{a_n n}$$

Witten's conjecture (1991) — proved by Kontsevich in 1993:

1. $\mathcal{U} = \frac{\partial^2 F}{\partial t_0^2}$ obeys the Korteweg-de Vries equations. These are an infinite hierarchy of PDEs; the first is the KdV equation

$$u_t + u_{xxx} = 6uu_x$$

describing the propagation of water waves in shallow channels

2. F satifies the string equation

$$\frac{\partial F}{\partial t_0} = \frac{1}{2} t_0^2 + \sum_{i \geq 0} t_{i+1} \frac{\partial F}{\partial t_i}$$

(related to the partition function of topological quantum gravity in 2 dimension)

Thank you for your attention

and

Enjoy the Junior Math Days!