Hyperbolic Conservation Laws

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Example 1: Traffic Flow

Traffic in a highway:

$$\rho(t,x) = \text{density of cars at point } x \in \mathbb{R} \text{ and time } t > 0.$$

Basic assumption: velocity of cars $v(t,x) \geq 0$ depends only on the density $\rho(t,x)$: $v(t,x) \equiv v(\rho(t,x))$.

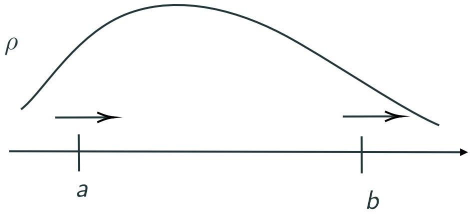


Figure 1: Traffic Flow

Example 1: Traffic Flow

$$\frac{d}{dt}\int_a^b \rho(t,x)dx = \texttt{Flux(t,a)} - \texttt{Flux(t,b)}$$
$$= v(\rho(t,a))\rho(t,a) - v(\rho(t,b))\rho(t,b).$$

If $f(\rho) := \rho \cdot v(\rho)$ we found

$$\int_a^b \partial_t \rho(t,x)dx = f(\rho(t,a)) - f(\rho(t,b))$$
$$= -\int_a^b \partial_x f(\rho(t,x))dx.$$

$\partial_t \rho + \partial_x f(\rho) = 0$ Scalar conservation law

Example 2: Gas Dynamics

Euler, Principes généraux du mouvement des fluides (1757)

$$
\begin{aligned}
&\partial_t \rho + \partial_x(\rho v) = 0 && \text{mass conservation} \\
&\partial_t(\rho v) + \partial_x((\rho v)v + P) = 0 && \text{momentum conservation} \\
&\partial_t E + \partial_x(Ev + Pv) = 0 && \text{energy conservation}
\end{aligned}
$$

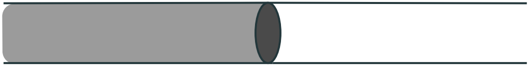
$$P \equiv P(\rho, e), \qquad E = \frac{1}{2}\rho v^2 + \rho e$$

$e =$ internal energy.

Example 2: Gas Dynamics

Adiabatic assumption $\rho e = p/(\gamma - 1)$:

$$\partial_t \rho + \partial_x (\rho v)_x = 0 \qquad \text{mass conservation}$$
$$\partial_t(\rho v) + \partial_x(\rho v^2 + P(\rho)) = 0 \qquad \text{momentum conservation}$$



$(\rho^-,\ v^-)$ $(\rho^+,\ v^+)$

Figure 2: Gas in a pipe

Riemann finds a solution to the initial value problem

$$\rho(0,x) = \begin{cases} \rho^-, & \text{if } x < 0, \\ \rho^+, & \text{if } x > 0 \end{cases} \qquad v(0,x) = \begin{cases} v^-, & \text{if } x < 0, \\ v^+, & \text{if } x > 0 \end{cases}$$

Example 2: Gas Dynamics

General solution has two kind of waves:

- Compression waves
- Shocks

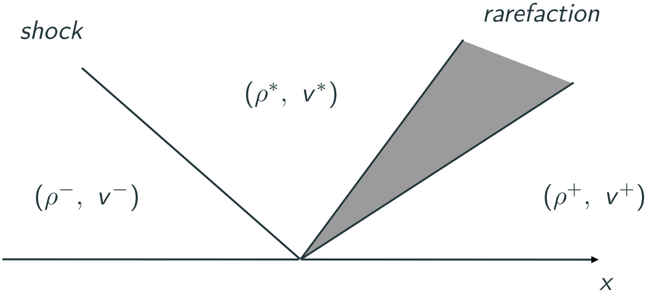


Figure 3: Solution to the Riemann Problem

Shock waves

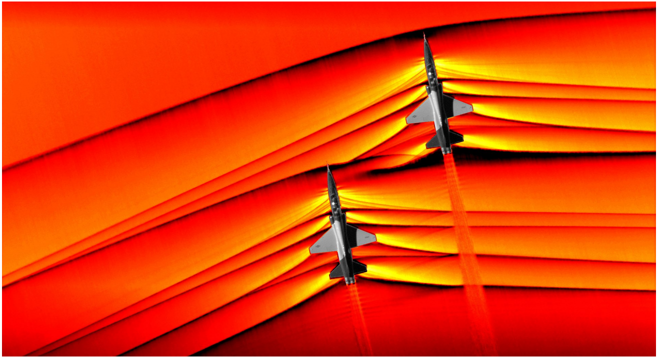


Figure 4: Shock waves interacting between two aircraft

Systems of conservation laws

A system of n conservation laws is

$$\partial_t u + \partial_x f(u) = 0$$

where

$$u \in \mathbb{R}^n, \qquad f(u) \in \mathbb{R}^n.$$

For smooth solutions it is equivalent to

$$\partial_t u + Df(u)\partial_x u = 0.$$

We say that the system is strictly hyperbolic if $Df(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \ldots < \lambda_n(u)$$

System of conservation laws

For gas dynamics:

$$f(\rho, \rho v) = \begin{pmatrix} \rho v \\ \rho v^2 + P(\rho) \end{pmatrix}, \quad P(\rho) = \kappa \rho^\gamma, \quad \gamma > 1$$

Then

$$\lambda_1 = v - \sqrt{\kappa}\rho^{\gamma-1} < v + \sqrt{\kappa}\rho^{\gamma-1} = \lambda_2.$$

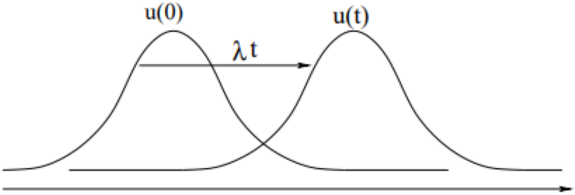
Linear case

The linear and scalar case is the simple transport equation

$$\partial_t u + \lambda \partial_x u = 0$$

If initially $u(0, x) = \bar{u}(x)$, then the unique solution to the Cauchy problem is

$$u(t, x) = \bar{u}(x - \lambda t).$$



Linear systems

The linear case for systems reads

$$\partial_t u + A\partial_x u = 0$$

One can find left and right dual basis of eigenvectors $r_1, \dots r_n$, $\ell_1, \dots, \ell_n$:

$$Ar_i = \lambda_i r_i \qquad \ell_i A = \lambda_i \ell_i$$

and

$$r_i \cdot \ell_j = \delta_{ij}$$

We want to solve the initial value problem

$$u(0,x) = \bar{u}(x)$$

Linear systems

We can *diagonalize* the system by defining components

$$u_i := \ell_i \cdot u \qquad \forall\, i = 1, \dots, n$$

Taking the scalar product

$$0 = \ell_i \cdot \big(\partial_t u + A \partial_x u\big) = \partial_t u_i + \lambda_i \partial_x u_i$$

the system is reduced to n independent equations

$$\begin{cases} \partial_t u_1 + \lambda_1 \partial_x u_1 = 0, \qquad & u_1(0, x) = \bar{u}_1(x) \\ \dots \\ \partial_t u_n + \lambda_n \partial_x u_n = 0, \qquad & u_n(0, x) = \bar{u}_n(x) \end{cases}$$

Therefore

$$u_i(x, t) = \bar{u}_i(x - \lambda_i t) \qquad \forall\, i = 1, \dots, n$$

$$u = \sum_{i=1}^{n} \bar{u}(x - \lambda_i t) \cdot r_i$$

Nonlinear effects: shocks

What about the nonlinear case?

Start with a single conservation law

$$\partial_t u + \partial_x f(u) = 0, \qquad u \in \mathbb{R}.$$

Pick for example $f(u) = u^2/2$, then for smooth solutions

$$\partial_t u + u\partial_x u = 0$$

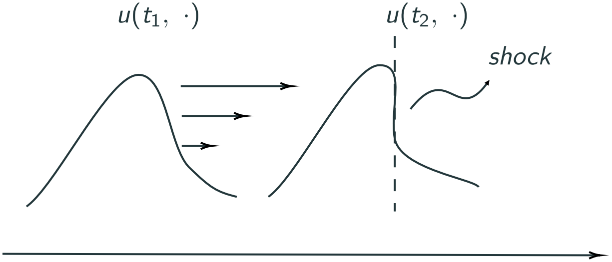
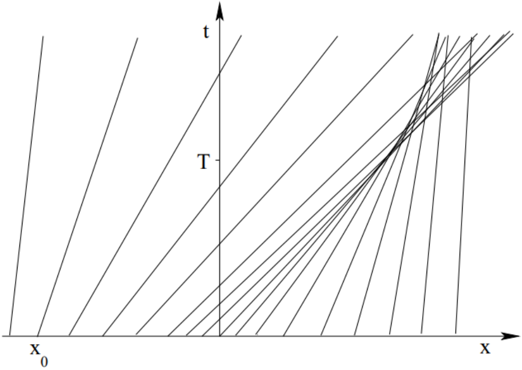


Figure 5: Higher points move with higher speed $\Longrightarrow$ formation of shocks.

Nonlinear effects: shocks

Smooth solutions are constant along characteristics:

$$\frac{d}{dt}u(t, \bar{x} + ut) = \partial_t u + u \cdot \partial_x u = 0.$$



Nonlinear effects: interactions

Left and right dual basis of eigenvectors $r_1(u), \dots r_n(u)$, $\ell_1(u), \dots, \ell_n(u)$:

$$Df(u)r_i(u) = \lambda_i(u)r_i(u) \qquad \ell_i(u)Df(u) = \lambda_i(u)\ell_i(u)$$

and

$$r_i(u) \cdot \ell_j(u) = \delta_{ij}.$$

$$u_x^i := \ell_i \cdot u_x$$

Then

$$u_x = \sum_{i=1}^{n} u_x^i r_i(u) \qquad u_t = -\sum_{i=1}^{n} \lambda_i(u) u_x^i r_i(u)$$

Differentiating the first w.r.t. t and the second w.r.t. x and equating we obtain

$$(u_x^i)_t + (\lambda_i u_x^i)_x = \sum_{j>k} (\lambda_j - \lambda_k)\Big(\ell_i \cdot [r_j, r_k]\Big) u_x^j u_x^k.$$

Nonlinear effects: interactions

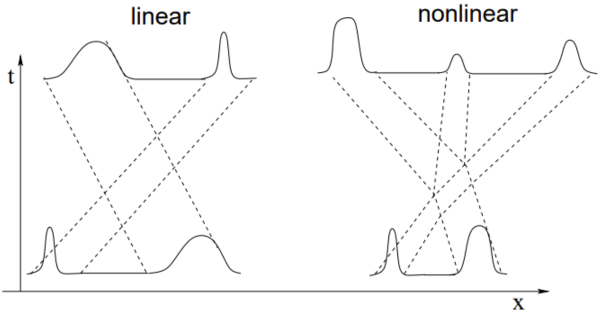


Figure 6: Linear and nonlinear behavior

Vanishing viscosity

What about **existence** of solutions?

Consider the vanishing viscosity approximations

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial^2_{xx} u^\varepsilon$$

Formally we expect the convergence $u^\varepsilon \to u$ to solutions of the system without viscosity:

$$\partial_t u + \partial_x f(u) = 0.$$

Strategy for existence of solutions:

- prove strong compactness of approximate solutions u^ε,
- weak notion of solution

Weak solutions

We have seen that solutions can became discontinuous in finite time, therefore we need a weak concept of solution.

We say that $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ is a **weak solution** to the conservation law if

$$\iint \partial_t \varphi(t,x) u + \partial_x \varphi(t,x) f(u) dt dx = 0 \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$$

or

$$\partial_t u + \partial_x f(u) = 0 \qquad \text{in } \mathcal{D}' \quad \text{(in distributions).}$$

Simplest weak solution is a *single shock* like

$$u(t,x) = \begin{cases} u^-, & \text{if } x < \lambda t \\ u^+ & \text{if } x > \lambda t \end{cases}$$

Rankine-Hugoniot conditions

Using the **divergence theorem** applied to the field $(u, f(u))$ we can show that u is a weak solution if and only if

$$f(u^+) - f(u^-) = \lambda(u^+ - u^-) \qquad \text{Rankine-Hugoniot conditions}$$

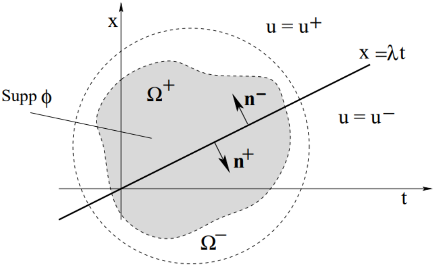


Figure 7: Derivation of R-H conditions

Rankine-Hugoniot conditions

For every smooth φ we find

$$\begin{aligned}
\mathbf{0} &= \iint_{\Omega} \varphi_t u + \varphi_x f(u) dtdx \\
&= \int_{\Omega^+} \mathrm{div}(\varphi u^+, \varphi f(u^+)) dtdx + \int_{\Omega^-} \mathrm{div}(\varphi u^-, \varphi f(u^-)) dtdx \\
&= \int \varphi(t, \lambda t)(u^+, f(u^+)) \cdot \vec{n}_+ dt + \int \varphi(t, \lambda t)(u^-, f(u^-)) \cdot \vec{n}_- dt \\
&= \int \varphi(t, \lambda t)(\boldsymbol{\lambda}(\mathbf{u}^+ - \mathbf{u}^-) - \mathbf{f}(\mathbf{u}^+) - \mathbf{f}(\mathbf{u}^-)) dt
\end{aligned}$$

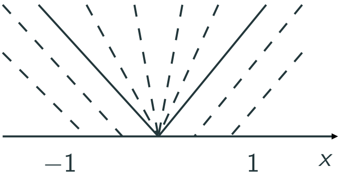
$$\vec{n}_+ = (\lambda, -1), \qquad \vec{n}_- = (-\lambda, 1)$$

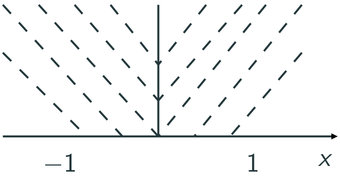
Non uniqueness of weak solutions

Weak solutions are not unique. Consider Burgers equation

$$\partial_t u + \partial_x u^2/2 = 0, \qquad u_0(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

$$u_1(t,x) = \begin{cases} -1 & \text{if } x \leq -t, \\ \frac{x}{t} & \text{if } -t < x < t, \\ 1 & \text{if } x \geq t. \end{cases} \qquad u_2(t,x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0 \end{cases}$$





travelling waves

The building blocks of solutions to

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \partial^2_{xx} u^\varepsilon \tag{1}$$

are **travelling waves**

$$U^\varepsilon(t,x) \doteq \bar{U}^\varepsilon(x - \lambda t) \qquad \text{'approximate shocks'}$$

U^ε is a solution to (1) if and only if

$$\big(Df(\bar{U}(s)) - \lambda I_{n\times n}\big)\bar{U}'(s) = \bar{U}''(s)$$

In the $n = 1$ case

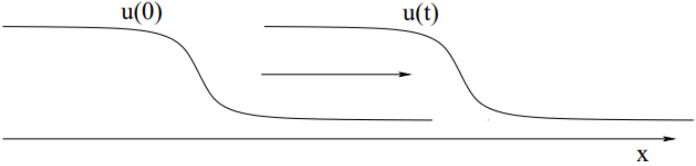
$$\big(f'(\bar{U}(s))) - \lambda\big)\bar{U}'(s) = \bar{U}''(s)$$

Travelling waves

We say that u^-, u^+ are connected by a **viscous travelling wave** if

$$\bar{U}^\varepsilon(-\infty) = u^-, \qquad \bar{U}^\varepsilon(+\infty) = u^+$$

(More precisely, if there is an invariant curve for the flow of the o.d.e. connecting u^-, u^+, directed from u^- to u^+)



Travelling waves

In the scalar case, integrating and setting $\bar{U}(-\infty) = u^-$,

$$\dot{\bar{U}}(s) = f(\bar{U}(s)) - f(u^-) - \lambda(\bar{U}(s) - u^-)$$

u_- is connected to u_+ by a travelling wave if and only if

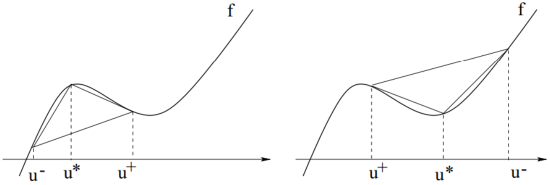
- the right-hand side does not change sign between u_- and u_+;
- is nonnegative when $u_- < u_+$ and nonpositive when $u_- > u_+$
- RH conditions hold

$$0 = \lim_{s \to +\infty} \dot{\bar{U}}(s) = f(u^+) - f(u^-) - \lambda(u^+ - u^-).$$

Similar reasoning for systems $->$ we obtain admissiblity conditions for a general shock.

Admissibility conditions: scalar case

The graph of f must lie below $(u^- < u^+)$ above $(u^+ < u^-)$ the chord.



Analogous condition for systems.

The shock with left and right states u_-, u_+ satisfies the **Liu admissibility condition** provided that its speed is less or equal to the speed of every smaller shock, joining u_- with an intermediate state u_*.

Under this condition it can be proved that **small** BV solutions are unique.

General Cauchy problem

Studying the interactions between travelling waves, one can control the total variation of u^ε and therefore obtain strong compactness.

Theorem (Bianchini and Bressan) There is $\delta > 0$ such that if

$$\|\bar{u}\|_{BV} \leq \delta$$

then there exists a unique global entropy solution to the Cauchy problem

$$\partial_t u + \partial_x f(u) = 0, \qquad u(0,x) = \bar{u}(x).$$

Also, solutions are $\mathbf{L}^1$ stable

$$\|v(t) - u(t)\| \leq C\|v(0) - u(0)\|.$$

A lot of open problems in the field:

- Vanishing viscosity with general diffusion matrices

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x (B(u) \partial_x u), \qquad B(u) \geq 0.$$

- "Big" initial data
- Multidimensional case (i.e. $x \in \mathbb{R}^d$, $d > 1$)
- ...

References

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