Block-Göttsche invariants from GW invariants

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Tuesday 20 June 2017

School and Workshop on Algebraic Geometry and Physics 2017
SISSA, Trieste

Introduction

Goal

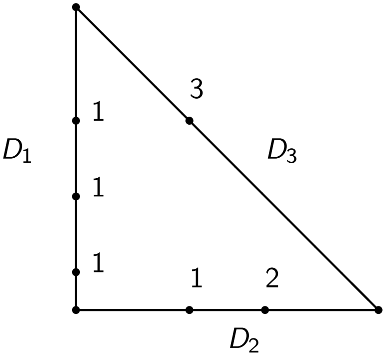
Relate some **virtual** counts of complex algebraic curves in complex toric surfaces to some **refined** counts of tropical curves in $\mathbb{R}^2$.

How is that not off-topic?

- In some cases, refined counts of tropical curves are [Stoppa, Filippini] examples of refined DT invariants of quivers (in the sense of Sven's talk).
- Should think of the complex torus $(\mathbb{C}^*)^2$ in the toric surface as a silly example of character variety and cluster variety.
- Should be part of a more general story relating DT theory and holomorphic curves in complex integrable systems (including moduli spaces of Higgs bundles) (in a way compatible with Dylan's talk).

Curves in toric surfaces

Projective toric surface: $(\mathbb{C}^*)^2$-equivariant compactification of $(\mathbb{C}^*)^2$. Consider curves with prescribed number of intersection points with each toric divisor and prescribed multiplicities of these intersection points.



Curves in toric surfaces

The same information (a toric surface, number of intersection points and multiplicities of intersection points) is contained in a collection $\Delta = \{v_1, \dots, v_r\}$ of vectors in $\mathbb{Z}^2$ summing to zero.

- Rays $\mathbb{R}_{\geq 0} v_i$ define the fan of a toric surface X_Δ.
- Vectors v_i generating the same ray define intersection point with the dual toric divisor.
- Divisibility of vectors v_i in $\mathbb{Z}^2$ defines the multiplicity the corresponding intersection point.

A curve in X_Δ satisfying the intersection and tangency constraints specified by Δ is said to be of type Δ.
Fix n general points $P = \{P_1, \dots, P_n\}$ in $(\mathbb{C}^*)^2$. Denote $g_{\Delta,n} = n + 1 - r$.
Let $N^{\Delta,n} \in \mathbb{N}$ be the number of genus $g_{\Delta,n}$ curves of type Δ passing through P.

Unrefined correspondence theorem

Theorem

[Mikhalkin, Nishinou-Siebert] For every Δ and n as above, and $p = \{p_1, \dots, p_n\}$ a collection of n general points in $\mathbb{R}^2$, we have

$$N^{\Delta,n} = N^{\Delta,p}_{\mathrm{trop}} ,$$

where

$$N^{\Delta,p}_{\mathrm{trop}} := \sum_{(h:\Gamma \to \mathbb{R}^2) \in T_{g_{\Delta,n},p}} m(h) ,$$

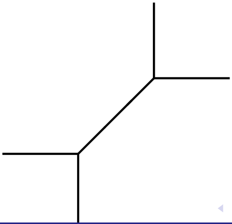
where $T_{g_{\Delta,n},p}$ is the set of genus $g_{\Delta,n}$ tropical curves of type Δ in $\mathbb{R}^2$, and where $m(h)$ is the multiplicity of h.

Tropical curves in $\mathbb{R}^2$

- Graph Γ and a map $h\colon \Gamma \to \mathbb{R}^2$. Edges are mapped to straight lines of rational slopes.
- Vertices V are decorated by a nonnegative integer $g(V)$, the genus of V. Define the genus of the tropical curve h by

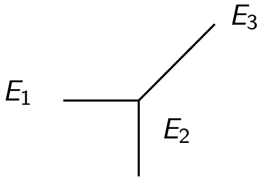
$$g(h) := g_\Gamma + \sum_V g(V)$$

  where g_Γ is the genus of the graph Γ.
- Edges E are decorated by a positive integer $w(E)$, the weight of E.
- Balancing condition at the vertices.
- Type Δ: fix the directions and the weights of the unbounded edges.



Tropical curves in $\mathbb{R}^2$

Fix $p = \{p_1, \dots, p_n\}$ a collection of n general points in $\mathbb{R}^2$.
Let $T_{g_{\Delta,n},p}$ be the set of genus $g_{\Delta,n}$ tropical curves of type Δ in $\mathbb{R}^2$. This set is finite and for every $(h\colon \Gamma \to \mathbb{R}^2) \in T_{g_{\Delta,n},p}$, the graph Γ is trivalent and has vertices of genus zero.



Let u_{E_1}, u_{E_2}, u_{E_3} be the primitive vectors in $\mathbb{Z}^2$ in the directions of the edges E_1, E_2, E_3 and going out of the vertex. Balancing condition:
$w(E_1)u_{E_1} + w(E_2)u_{E_2} + w(E_3)u_{E_3} = 0$.
Multiplicity of a trivalent vertex: $m(V) := w(E_1)w(E_2)|\det(u_{E_1}, u_{E_2})|$.
Multiplicity of h: $m(h) := \prod_V m(V)$.

Unrefined correspondence theorem

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Tropicalization

For $\hbar > 0$, consider

$$\pi_\hbar \colon (\mathbb{C}^*)^2 \to \mathbb{R}^2$$

$$(z_1, z_2) \mapsto (\hbar \log |z_1|, \hbar \log |z_2|) \,.$$

Fix $p = \{p_1, \dots, p_n\}$ collection of n general points in $\mathbb{R}^2$.
Define $P_i(\hbar) = e^{\frac{p_i}{\hbar}} \in (\mathbb{C}^*)^2$, $i = 1, \dots, n$. We have $\pi_\hbar(P_i(\hbar)) = p_i$.
Let $C(\hbar)$ be a family of curves in $(\mathbb{C}^*)^2$ passing through $P_i(\hbar)$.
The image $\pi_\hbar(C(\hbar))$ of $C(\hbar)$ is a complicated shape in $\mathbb{R}^2$ called an amoeba.
For $\hbar \to 0$, the amoeba retracts on a tropical curve.

Refined tropical count

Refined multiplicity of a vertex V of multiplicity $m(V)$:

$$[m(V)]_q := \frac{q^{\frac{m(V)}{2}} - q^{-\frac{m(V)}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} = q^{-\frac{m(V)-1}{2}}(1 + q + \cdots + q^{m(V)-1})$$

Refined multiplicity of a trivalent tropical curve $h\colon \Gamma \to \mathbb{R}^2$,

$$\prod_V [m(V)]_q$$

Define [Block-Göttsche]

$$N_{\mathrm{trop}}^{\Delta,p}(q) := \sum_{(h:\Gamma\to\mathbb{R}^2)\in T_{g_{\Delta,n},p}} \prod_V [m(V)]_q$$

Unrefined limit: $N_{\mathrm{trop}}^{\Delta,p}(q=1) = N_{\mathrm{trop}}^{\Delta,p}$.
Remarkable property [Itenberg-Mikhalkin]: independent of general p.

Question

- Meaning of the refined tropical count $N_{\text{trop}}^{\Delta,p}(q)$ from the point of view of the complex geometry of the toric surface X_Δ?
- Geometric meaning of the extra variable q?
- Göttsche-Shende conjecture: refinement from some topological Euler characteristic to some Hirzebruch genus.
- This talk: different point of view, via Gromov-Witten theory.

Gromov-Witten theory

- Stable maps: map $f\colon C \to X_\Delta$ from a nodal curve C such that $|\mathrm{Aut}(f)|$ is finite.
- We have [Mandel-Ruddat]

$$N^{\Delta,n} := \int_{[\overline{M}^{\log}_{g_{\Delta,n},n,\Delta}]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathrm{pt})$$

- Technical aspects: virtual fundamental class, logarithmic theory [Abramovich-Chen-Gross-Siebert] to interact nicely with the toric divisors.
- Question: how to find a parameter q from this Gromov-Witten theory point of view?

Lambda classes

- Idea: consider curves of genus $g \geq g_{\Delta,n}$.
- Problem: the moduli space $\overline{M}^{\log}_{g_{\Delta,n},n,\Delta}$ has (virtual) dimension $g - g_{\Delta,n}$. It does not make sense to try to count these curves.
- Idea corrected: insert a cohomology class of (complex) degree $g - g_{\Delta,n}$.
- $\pi \colon C \to M$ a family of genus g nodal curves, Hodge bundle E whose fiber at C is $H^0(C, \omega_C)$. It is a rank g vector bundle. Lambda classes are Chern classes of the Hodge bundle:

$$\lambda_j := c_j(E)\,,$$

$j = 0, \ldots, g$.

GW invariants

For every $g \geq g_{\Delta,n}$, define

$$N_g^{\Delta,n} := \int_{[\overline{M}^{\log}_{g,n,\Delta}]^{\mathrm{virt}}} (-1)^{g-g_{\Delta,n}} \lambda_{g-g_{\Delta,n}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathrm{pt}) \in \mathbb{Q}$$

Main result: refined correspondence theorem

Theorem [-]

For every Δ and n, we have

$$\sum_{g \geq g_{\Delta,n}} N_g^{\Delta,n} u^{2g-2+r} = N_{\text{trop}}^{\Delta,p}(q) \left((-i)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \right)^{2g_{\Delta,n}-2+r}$$

of power series in u with rational coefficients, where

$$q = e^{iu} = \sum_{n \geq 0} \frac{(iu)^n}{n!} .$$

Remarks

- Analogue results for K3 and abelian surfaces (involving the Göttsche-Shende refinement).
- Previous talks, several ways to interpret the variable q: number of elements in a finite field, variable keeping track of some cohomological information. In the previous theorem, completely different way to interpret this variable q: write $q = e^{iu}$, expand in power series, get a genus expansion.

Thank you for your attention!