Topological mirror symmetry via p-adic integration

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- Their conjecture can be reformulated in terms of counting points over finite fields. This in turn can be done by computing p-adic volumes.
- We can compare the p-adic volumes of the two moduli spaces, since "singular Hitchin fibers have measure 0".

Moduli space of SL_n Higgs bundles.

- Let C be a smooth projective curve of genus g and $K = K_C$ the canonical bundle.
- A *Higgs bundle* on C is a pair (E, ϕ), where E is a rank n vector bundle on C and $\phi \in H^0(C, \mathrm{End}(E) \otimes K)$.

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Definition

For an integer d coprime to n and a line bundle L of degree d on C define the moduli space of (twisted) SL_n-Higgs bundles as

$$\mathcal{M}^d_{\mathrm{SL}_n}(C) = \{\text{Stable Higgs bundles } (E, \phi), \text{ with } \det E \cong L, \ \mathrm{tr}\phi = 0\} / \sim$$

- $\mathcal{M}^d_{\mathrm{SL}_n}(C)$ is a smooth quasi-projective variety.

Moduli space of PGL_n Higgs bundles

- The n-torsion points $\Gamma = \mathrm{Jac}_C[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ act on $\mathcal{M}^d_{\mathrm{SL}_n}(C)$ by tensoring:
$$\gamma \cdot (E, \phi) = (E \otimes \gamma, \phi), \text{ for } \gamma \in \Gamma.$$

Definition

The moduli space of (twisted) PGL_n Higgs bundles is

$$\mathcal{M}^d_{\mathrm{PGL}_n}(C) = \mathcal{M}^d_{\mathrm{SL}_n}(C)/\Gamma.$$

- Remark: More generally one can construct moduli space of G-Higgs bundles for any reductive G, it is however unclear how to "twist" in general.

Hitchin Fibration

- Given a Higgs bundle $(E, \phi) \in H^0(C, \mathrm{End}(E) \otimes K)$ we can consider its characteristic polynomial $h(\phi) \in \bigoplus_{i=1}^n H^0(C, K^{\otimes i})$. This gives morphisms

$$\mathcal{M}^d_{\mathrm{SL}_n} \xrightarrow{h_{\mathrm{SL}_n}} \mathbb{A} = \bigoplus_{i=2}^n H^0(C, K^{\otimes i}) \xleftarrow{h_{\mathrm{PGL}_n}} \mathcal{M}^d_{\mathrm{PGL}_n}$$

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Theorem (Hitchin, Simpson)

The Hitchin maps h_{SL_n}, h_{PGL_n} are proper and their generic fibers are complex Lagrangian torsors for abelian varieties $\mathcal{P}_{SL_n}$ and $\mathcal{P}_{PGL_n}$ respectively. Furthermore $\mathcal{P}_{SL_n}$ and $\mathcal{P}_{PGL_n}$ are dual abelian varieties.

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- If it weren't for the torsor structure , $\mathcal{M}^d_{\mathrm{SL}_n}$ and $\mathcal{M}^d_{\mathrm{PGL}_n}$ would be mirror partners in the sense of Strominger-Yau-Zaslow.

Twisted SYZ Mirror Symmetry

- The correct duality between the fibrations $h_{\mathrm{SL}_n}, h_{\mathrm{PGL}_n}$ should take the torsor structure into account [Hitchin 2001]:

$$\rightsquigarrow \mathbb{Z}/n\mathbb{Z}\text{-Gerbes } B, \bar{B} \text{ on } \mathcal{M}^d_{\mathrm{SL}_n}, \mathcal{M}^d_{\mathrm{PGL}_n}.$$

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Theorem (Hausel-Thaddeus 2003)

The pairs $(\mathcal{M}^d_{SL_n}, B)$ and $(\mathcal{M}^d_{PGL_n}, \bar{B})$ are SYZ mirror partners i.e. for a generic $a \in \mathbb{A}$ we have isomorphisms of $\mathcal{P}_{SL_n}$ and $\mathcal{P}_{PGL_n}$ torsors

$$h^{-1}_{SL_n}(a) \cong Triv(h^{-1}_{PGL_n}(a), \bar{B})$$
$$h^{-1}_{PGL_n}(a) \cong Triv(h^{-1}_{SL_n}(a), B).$$

- Remark: [Donagi-Pantev, 2012] prove a similar statement for any pair of Langlands dual groups.

Topological Mirror Symmetry

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Definition

For any complex variety X define the E-polynomial by

$$E(X; x, y) = \sum_{p,q,i \geq 0} (-1)^i h^{p,q;i}(X) x^p y^q,$$

where $h^{p,q;i}(X) = \dim_{\mathbb{C}}(Gr_p^{Ho} Gr_{p+q}^{w} H_c^i(X))$ denote the compactly supported mixed Hodge numbers of X.

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- The compactly supported cohomology of $\mathcal{M}^d_{\mathrm{SL}_n}$ and $\mathcal{M}^d_{\mathrm{PGL}_n}$ is pure i.e. $h^{p,q;i} = 0$ unless $i = p + q$.

Conjecture (Hausel-Thaddeus 2003)

There is an equality

$$E(\mathcal{M}^d_{SL_n}; x, y) = E^{\bar{B}}_{st}(\mathcal{M}^d_{PGL_n}; x, y).$$

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- The right hand side takes into account the orbifold structure and can be written as

$$E^{\bar{B}}_{st}(\mathcal{M}^d_{\mathrm{SL}_n}/\Gamma; x, y) = \sum_{\gamma \in \Gamma} (xy)^{\mathcal{F}(\gamma)} E^{\bar{B}_\gamma}((\mathcal{M}^d_{\mathrm{SL}_n})^\gamma/\Gamma; x, y),$$

  where $E^{\bar{B}_\gamma}$ denotes the E-polynomial with coefficients in the local system $\bar{B}_\gamma \to (\mathcal{M}^d_{\mathrm{SL}_n})^\gamma/\Gamma$ and $\mathcal{F}(\gamma)$ the Fermionic shift.
- The Conjecture is true for $n = 2, 3$ [HT 2003].

- The point count analogue of $E(X; x, y)$ is $\#X(\mathbb{F}_q)$. Consequently we define

$$\#^{\bar{B}}_{st}\mathcal{M}^d_{\mathrm{PGL}_n}(\mathbb{F}_q) = \sum_{\gamma\in\Gamma} q^{\mathcal{F}(\gamma)} \sum_{x\in(\mathcal{M}^d_{\mathrm{SL}_n})^{\gamma}/\Gamma(\mathbb{F}_q)} \mathrm{tr}(\mathrm{Fr}, (\bar{B}_{\gamma})_x).$$

Reduction to finite fields

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Essentially by a theorem of Katz, the conjecture then follows from

Theorem (Groechening-W.-Ziegler)

$$\#\mathcal{M}^d_{SL_n}(\mathbb{F}_q) = \#^{\bar{B}}_{st}\mathcal{M}^d_{PGL_n}(\mathbb{F}_q). \quad (1)$$

Reduction to p-adic integration

- Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_F$ and residue field $k_F \cong \mathbb{F}_q$.

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- In particular for any $\mathcal{O}_F$-variety X we can integrate top forms on the manifold $X^\circ = X(\mathcal{O}_F) \cap X^{sm}(F)$.

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Theorem (Weil 1982)

Let X be a smooth variety over $\mathcal{O}_F$ of relative dimension n and ω a gauge form on X. Then

$$\int_{X^\circ} \omega = \frac{\#X(\mathbb{F}_q)}{q^n}.$$

Reduction to p-adic integration

- Through Weil's theorem we can control the LHS of (1) by a p-adic integral.
- The same is also true for the RHS, when we integrate a certain weight function $f_{\bar{B}}$ against the canonical class ω_{can} on $\mathcal{M}^d_{\mathrm{PGL}_n} = \mathcal{M}^d_{\mathrm{SL}_n}/\Gamma$ [Denef-Loeser 2002].

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- The topological mirror symmetry conjecture of Hausel-Thaddeus thus follows from the following

Theorem (Groechening-W.-Ziegler)

$$\int_{\mathcal{M}^{d\circ}_{SL_n}} \omega = \int_{\mathcal{M}^{d\circ}_{PGL_n}} f_{\bar{B}} \omega_{can}.$$

Sketch of Proof

- Enough to compare the integrals fiberwise along F-smooth fibers:

$$\int_{h_{\mathrm{SL}_n}^{-1}(a)^\circ} 1 \stackrel{?}{=} \int_{h_{\mathrm{PGL}_n}^{-1}(a)^\circ} f_{\bar{B}} \quad \text{for } a \in \mathbb{A}^{gen}(F) \cap \mathbb{A}(\mathcal{O}_F).$$

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- The measures restricted to such fibers are translation invariant for the actions of $\mathcal{P}_{\mathrm{SL}_n}(F)$ and $\mathcal{P}_{\mathrm{PGL}_n}(F)$.
- From the isomorphism $h_{\mathrm{SL}_n}^{-1}(a) \cong \mathrm{Triv}(h_{\mathrm{PGL}_n}^{-1}(a), \bar{B})$ we deduce

$$h_{\mathrm{SL}_n}^{-1}(a)(F) \neq \emptyset \Leftrightarrow \bar{B}_{|h_{\mathrm{PGL}_n}^{-1}(a)} \textit{ is trivial }.$$

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- If $h_{\mathrm{SL}_n}^{-1}(a)(F) \neq \emptyset$, then $f_{\bar{B}} \equiv 1$ and

$$\int_{h_{\mathrm{SL}_n}^{-1}(a)^{\circ}} 1 = \int_{h_{\mathrm{PGL}_n}^{-1}(a)^{\circ}} 1,$$

  by using the self duality of the isogeny $\mathcal{P}_{\mathrm{SL}_n} \to \mathcal{P}_{\mathrm{PGL}_n}$.

Consequences

- The Weil pairing on the curve C gives an identification $\Gamma \cong \Gamma^* = \mathrm{Hom}(\Gamma, \mu_n)$. If $\gamma \in \Gamma$ corresponds to the character χ we have
$$E^{\chi}(\mathcal{M}^d_{\mathrm{SL}_n}; x, y) = (xy)^{F(\gamma)} E^{\bar{B}_\gamma}((\mathcal{M}^d_{\mathrm{SL}_n})^{\gamma}/\Gamma; x, y).$$

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- For any $a \in \mathbb{A}(\mathbb{F}_q)$ we have

$$\# h^{-1}_{\mathrm{SL}_n}(a)(\mathbb{F}_q) = \#^{\bar{B}}_{st} h^{-1}_{\mathrm{PGL}_n}(a)(\mathbb{F}_q).$$

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- For any d' comprime to n we have
$$E(\mathcal{M}^d_{\mathrm{SL}_n}; x, y) = E(\mathcal{M}^{d'}_{\mathrm{SL}_n}; x, y), \qquad E(\mathcal{M}^d_{\mathrm{GL}_n}; x, y) = E(\mathcal{M}^{d'}_{\mathrm{GL}_n}; x, y).$$

Thank you!