On the E-polynomial of parabolic Sp_{2n}-character varieties

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Outline

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- $[A : B]$:= group commutator.

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Definition

A *parabolic* $\mathrm{Sp}(2n, \mathbb{C})$*-character variety* of Σ_g is the categorical quotient

$$\mathcal{M}_n^{\xi} := \mathcal{U}_n^{\xi} // T$$

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Example

If ϕ is a primitive $(2^n + 1)$-root of unity, an element ξ whose spectrum $\lambda(\xi)$ is equal to $\left\{ \phi^{\pm 2^i} \right\}_{i=0,\dots,n-1}$ satisfies the genericity condition.

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- $\mathcal{M}_n^{\xi}$ is homeomorphic to a moduli space of G-Higgs bundles on Σ_g with parabolic structure at p_0.
- Hausel et al. conjectured Mirror symmetry for character varieties defined over Langlands dual groups in terms of the *stringy E-polynomials*.

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- (Hausel, Letellier and R-Villegas, 2011): E-polynomials of parabolic $\mathrm{GL}(n, \mathbb{C})$-character varieties.

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- For $Z(G) \leq H \leq \boldsymbol{\mu}_2^n$, the set

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- $\left\{ \widetilde{\mathcal{M}}_{n,H}^\xi \right\}_{Z(G) \leq H \leq \boldsymbol{\mu}_2^n}$ is a stratification of $\mathcal{M}_n^\xi$

Geometry of $\mathcal{M}_n^\xi$

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Proposition

There exists a well defined partition $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, depending on H, such that

$$\mathcal{U}_{n,H}^{\xi} \cong \prod_{i=1}^{l} \mathcal{U}_{\lambda_i}^{\xi_i}$$

where the ξ_i's are generic elements.

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Denote $q := xy$

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$$E\left(\mathcal{M}_n^{\xi}; q\right) = \frac{1}{(q-1)^n} \sum_{\tau} \left(H_{\tau}(q)\right)^{2g-1} C_{\tau}.$$

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- The range of the summation only depends on n.

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Corollary

The E-polynomial of $\mathcal{M}_n^{\xi}$ is palindromic and monic. In particular, $\mathcal{M}_n^{\xi}$ is connected.

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$$\sum_{n\geq 0} \frac{\chi\left(\mathcal{M}_n^\xi\right)}{2^n n!} T^n = \prod_{k\geq 1} \frac{1}{(1-T^k)^3} = 1 + 3T + \cdots .$$

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$$\chi\left(\mathcal{M}_1^{\xi}\right) = E\left(\mathcal{M}_1^{\xi}; 1\right) = \begin{cases} 6 & \textit{if } g = 1 \\ 0 & \textit{if } g \gneq 1 \end{cases}$$

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(μ is the Möbius function of the poset of subgroups of $\boldsymbol{\mu}_2^n$).

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- We reduce to compute $\frac{1}{(q-1)^n} N_n^{\xi}(q)$ for any $n \in \mathbb{N}$ and any generic ξ. Eventually, it turns out that

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Computation of $E\left(\mathcal{M}_n^{\xi}; q\right)$

Frobenius formula:

$$\frac{1}{(q-1)^n} N_n^{\xi}(q) = \frac{1}{(q-1)^n} \sum_{\chi \in \mathrm{Irr}(\mathrm{Sp}_{2n}(\mathbb{F}_q))} \chi(\xi) \left(\frac{|\mathrm{Sp}_{2n}(\mathbb{F}_q)|}{\chi(1)} \right)^{2g-1}.$$

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Here, $\mathbb{F}_q$ is a finite field such that $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ admits ξ.

Final comments

- $E\left(\mathcal{M}_n^{\xi}; q\right)$ does not depend on the choice of the generic ξ, so actually we have computed the E-polynomial of a very large family of Sp_{2n}-character varieties.

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- The order of the abelianization of $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ counts connected components of $\mathcal{M}_n^{\xi}$. This seems to be a more general phenomenon, occuring for character varieties defined over simple algebraic group.

Thank you for your attention.