Introducing Donaldson-Thomas theory with an eye towards character varieties.

Sven Meinhardt (jointly with Ben Davison and Markus Reineke)

SISSA

June 19, 2017

Moduli of quiver representations

Moduli of quiver representations

Quiver $Q = (Q_0, Q_1, s, t)$, dimension vector $\gamma = (\gamma_v)_{v \in Q_0} \in \mathbb{N}^{Q_0}$

Moduli of quiver representations

Quiver $Q = (Q_0, Q_1, s, t)$, dimension vector $\gamma = (\gamma_v)_{v \in Q_0} \in \mathbb{N}^{Q_0}$

$$\mathrm{Mat}_\gamma(Q) := \prod_{Q_1 \ni \alpha : v \to w} \mathrm{Mat}_{\gamma_w \times \gamma_v}(\mathbb{C})$$

with action of $G_\gamma := \prod_{v \in Q_0} \mathrm{GL}_{\gamma_v}(\mathbb{C})$ by simultaneous conjugation.

Moduli of quiver representations

Quiver $Q = (Q_0, Q_1, s, t)$, dimension vector $\gamma = (\gamma_v)_{v \in Q_0} \in \mathbb{N}^{Q_0}$

$$\mathrm{Mat}_\gamma(Q) := \prod_{Q_1 \ni \alpha : v \to w} \mathrm{Mat}_{\gamma_w \times \gamma_v}(\mathbb{C})$$

with action of $G_\gamma := \prod_{v \in Q_0} \mathrm{GL}_{\gamma_v}(\mathbb{C})$ by simultaneous conjugation.

Moduli stack of representations: $\mathfrak{Rep}_\gamma(Q) := \mathrm{Mat}_\gamma(Q)/G_\gamma$

Moduli of quiver representations

Quiver $Q = (Q_0, Q_1, s, t)$, dimension vector $\gamma = (\gamma_v)_{v \in Q_0} \in \mathbb{N}^{Q_0}$

$$\mathrm{Mat}_\gamma(Q) := \prod_{Q_1 \ni \alpha : v \to w} \mathrm{Mat}_{\gamma_w \times \gamma_v}(\mathbb{C})$$

with action of $G_\gamma := \prod_{v \in Q_0} \mathrm{GL}_{\gamma_v}(\mathbb{C})$ by simultaneous conjugation.

Moduli stack of representations: $\mathfrak{Rep}_\gamma(Q) := \mathrm{Mat}_\gamma(Q)/G_\gamma$

Question: What about moduli spaces?

Mumford: Need to single out semistable representations

Mumford: Need to single out semistable representations

Definition

A tuple $\xi = (\xi_v)_{v \in Q_0} \in \mathbb{H}_+^{Q_0}$ is called a stability condition.

Mumford: Need to single out semistable representations

Definition

A tuple $\xi = (\xi_v)_{v \in Q_0} \in \mathbb{H}_+^{Q_0}$ is called a stability condition. Given ξ, we call $M = (M_\alpha)_{\alpha \in Q_1} \in \mathrm{Mat}_\gamma(Q)$ ξ-semistable if

$$\arg(\xi \cdot \dim M') \leq \arg(\xi \cdot \dim M) = \arg\Big(\sum_{v \in Q_0} \xi_v \gamma_v\Big)$$

for all subrepresentations M' of M.

Mumford: Need to single out semistable representations

Definition

A tuple $\xi = (\xi_v)_{v \in Q_0} \in \mathbb{H}_+^{Q_0}$ is called a stability condition. Given ξ, we call $M = (M_\alpha)_{\alpha \in Q_1} \in \mathrm{Mat}_\gamma(Q)$ ξ-semistable if

$$\arg(\xi \cdot \dim M') \leq \arg(\xi \cdot \dim M) = \arg(\sum_{v \in Q_0} \xi_v \, \gamma_v)$$

for all subrepresentations M' of M.
$\mathrm{Mat}_\gamma^{\xi-ss}(Q) \subset \mathrm{Mat}_\gamma(Q)$ is the subset of ξ-semistable representations.

Mumford: Need to single out semistable representations

Definition

A tuple $\xi = (\xi_v)_{v \in Q_0} \in \mathbb{H}_+^{Q_0}$ is called a stability condition. Given ξ, we call $M = (M_\alpha)_{\alpha \in Q_1} \in \mathrm{Mat}_\gamma(Q)$ ξ-semistable if

$$\arg(\xi \cdot \dim M') \leq \arg(\xi \cdot \dim M) = \arg(\sum_{v \in Q_0} \xi_v \, \gamma_v)$$

for all subrepresentations M' of M.
$\mathrm{Mat}_\gamma^{\xi-ss}(Q) \subset \mathrm{Mat}_\gamma(Q)$ is the subset of ξ-semistable representations.
A representation without proper subrepresentations of the same phase $\arg(\xi \cdot \dim M)$ is called stable.

Mumford: Need to single out semistable representations

Definition

A tuple $\xi = (\xi_v)_{v \in Q_0} \in \mathbb{H}_+^{Q_0}$ is called a stability condition. Given ξ, we call $M = (M_\alpha)_{\alpha \in Q_1} \in \mathrm{Mat}_\gamma(Q)$ ξ-semistable if

$$\arg(\xi \cdot \dim M') \leq \arg(\xi \cdot \dim M) = \arg(\sum_{v \in Q_0} \xi_v \gamma_v)$$

for all subrepresentations M' of M.
$\mathrm{Mat}_\gamma^{\xi-ss}(Q) \subset \mathrm{Mat}_\gamma(Q)$ is the subset of ξ-semistable representations.
A representation without proper subrepresentations of the same phase $\arg(\xi \cdot \dim M)$ is called stable.

Fact: Every semistable representation has a "Jordan–Hölder" filtration with stable subquotients of the same phase.

Theorem (Davison–M. '17)

For all ξ and all γ, the subset $\mathrm{Mat}_\gamma^{\xi-ss}(Q) \subset \mathrm{Mat}_\gamma(Q)$ is the open subvariety of semistable points for a suitable linearization of the G_γ-action on $\mathrm{Mat}_\gamma^{\xi-ss}(Q)$.

Theorem (Davison–M. '17)

For all ξ and all γ, the subset $\mathrm{Mat}_\gamma^{\xi-ss}(Q) \subset \mathrm{Mat}_\gamma(Q)$ is the open subvariety of semistable points for a suitable linearization of the G_γ-action on $\mathrm{Mat}_\gamma^{\xi-ss}(Q)$. Moreover

$$\mathcal{M}_\gamma^{\xi-ss}(Q) := \mathrm{Mat}_\gamma^{\xi-ss}(Q) /\!\!/ G_\gamma$$

is a quasiprojective variety parameterizing ξ-semistable representations up to S-equivalence.

Theorem (Davison–M. '17)

For all ξ and all γ, the subset $\mathrm{Mat}_{\gamma}^{\xi-ss}(Q) \subset \mathrm{Mat}_{\gamma}(Q)$ is the open subvariety of semistable points for a suitable linearization of the G_{γ}-action on $\mathrm{Mat}_{\gamma}^{\xi-ss}(Q)$. Moreover

$$\mathcal{M}_{\gamma}^{\xi-ss}(Q) := \mathrm{Mat}_{\gamma}^{\xi-ss}(Q) /\!\!/ G_{\gamma}$$

is a quasiprojective variety parameterizing ξ-semistable representations up to S-equivalence.

Here, $M \sim_S M'$ if M and M' have the same stable subquotients (up to isomorphism) counted with multiplicities.

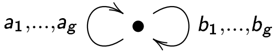
Imposing relations

Typically, the matrices/linear maps M_α have to satisfy extra relations.

Imposing relations

Typically, the matrices/linear maps M_α have to satisfy extra relations.

Example: Fix $g \geq 1$ and consider the quiver

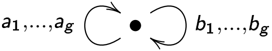


with the relation $\sum_{i=1}^{g}[a_i, b_i] = \sum_{i=1}^{g} a_i b_i - b_i a_i = 0$.

Imposing relations

Typically, the matrices/linear maps M_α have to satisfy extra relations.

Example: Fix $g \geq 1$ and consider the quiver



with the relation $\sum_{i=1}^{g}[a_i, b_i] = \sum_{i=1}^{g} a_i b_i - b_i a_i = 0$.

Hence, $(M_\alpha)_{\alpha \in Q_1}$ satisfying these relations provides a representation of the "preprojective algebra" $\mathbb{C}\langle a_1, \ldots, a_g, b_1, \ldots, b_g \rangle / \sum_{i=1}^{g}[a_i, b_i]$.

In Donaldson–Thomas theory the relations are induced by a "potential" W.

In Donaldson–Thomas theory the relations are induced by a "potential" W.

Definition

A potential W for a quiver Q is a complex linear combination of cycles in Q (up to cyclic order of the cycles).

In Donaldson–Thomas theory the relations are induced by a "potential" W.

Definition

A potential W for a quiver Q is a complex linear combination of cycles in Q (up to cyclic order of the cycles). Given $\alpha \in Q_1$, denote by $\partial W/\partial\alpha$ the non-commutative derivative of W wrt. α obtained by permuting all occurrences of α cyclically to the front and deleting α.

In Donaldson–Thomas theory the relations are induced by a "potential" W.

Definition

A potential W for a quiver Q is a complex linear combination of cycles in Q (up to cyclic order of the cycles). Given $\alpha \in Q_1$, denote by $\partial W/\partial\alpha$ the non-commutative derivative of W wrt. α obtained by permuting all occurrences of α cyclically to the front and deleting α. Given W, denote by $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ the closed subvariety of $\mathrm{Mat}_\gamma^{\xi-ss}(Q)$ of representations satisfying the relations $\partial W/\partial\alpha = 0$ for all $\alpha \in Q_1$.

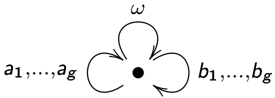
In Donaldson–Thomas theory the relations are induced by a "potential" W.

Definition

A potential W for a quiver Q is a complex linear combination of cycles in Q (up to cyclic order of the cycles). Given $\alpha \in Q_1$, denote by $\partial W/\partial\alpha$ the non-commutative derivative of W wrt. α obtained by permuting all occurrences of α cyclically to the front and deleting α. Given W, denote by $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ the closed subvariety of $\mathrm{Mat}_\gamma^{\xi-ss}(Q)$ of representations satisfying the relations $\partial W/\partial\alpha = 0$ for all $\alpha \in Q_1$.
$\mathfrak{Rep}_\gamma^{\xi-ss}(Q,W) := \mathrm{Mat}_\gamma^{\xi-ss}(Q,W)/G_\gamma$ and
$\mathcal{M}_\gamma^{\xi-ss}(Q,W) := \mathrm{Mat}_\gamma^{\xi-ss}(Q,W)/\!\!/G_\gamma$

Example

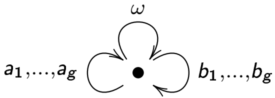
Consider



with potential $W = \omega \sum_{i=1}^{g} [a_i, b_i] = \sum_{i=1}^{g} \omega a_i b_i - \omega b_i a_i$.

Example

Consider

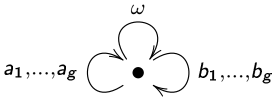


with potential $W = \omega \sum_{i=1}^{g} [a_i, b_i] = \sum_{i=1}^{g} \omega a_i b_i - \omega b_i a_i$.
Hence

$$\begin{aligned} \partial W / \partial \omega &= \sum_{i=1}^{g} [a_i, b_i], \\ \partial W / \partial a_i &= b_i \omega - \omega b_i, \\ \partial W / \partial b_i &= \omega a_i - a_i \omega. \end{aligned}$$

Example

Consider



with potential $W = \omega \sum_{i=1}^{g} [a_i, b_i] = \sum_{i=1}^{g} \omega a_i b_i - \omega b_i a_i$.
Hence

$$\begin{aligned} \partial W / \partial \omega &= \sum_{i=1}^{g} [a_i, b_i], \\ \partial W / \partial a_i &= b_i \omega - \omega b_i, \\ \partial W / \partial b_i &= \omega a_i - a_i \omega. \end{aligned}$$

Thus, $(M_\alpha)_{\alpha \in Q_1}$ satisfying these relations provides a representation $(M_\alpha)_{\alpha \neq \omega}$ of the preprojective algebra $\mathbb{C}\langle a_1, \dots, a_g, b_1, \dots, b_g \rangle / \sum_{i=1}^{g} [a_i, b_i]$ together with an endomorphism M_ω of that representation.

For $g = 1$ we get $\mathcal{M}_{\gamma}^{\xi-ss}(Q, W) = \mathrm{Sym}^{\gamma}(\mathbb{C}^3) = (\mathbb{C}^3)^{\gamma} /\!\!/ S_{\gamma}$ as $(M_{\alpha})_{\alpha \in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1, b_1, ω).

For $g = 1$ we get $\mathcal{M}_\gamma^{\xi-ss}(Q, W) = \mathrm{Sym}^\gamma(\mathbb{C}^3) = (\mathbb{C}^3)^\gamma /\!\!/ S_\gamma$ as $(M_\alpha)_{\alpha \in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1, b_1, ω).

Notice: Representations of the preprojective algebra cannot be described by a quiver with potential.

For $g = 1$ we get $\mathcal{M}_\gamma^{\xi-ss}(Q,W) = \mathrm{Sym}^\gamma(\mathbb{C}^3) = (\mathbb{C}^3)^\gamma /\!\!/ S_\gamma$ as $(M_\alpha)_{\alpha\in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1, b_1, ω).

Notice: Representations of the preprojective algebra cannot be described by a quiver with potential.

Question: Why only relations induced by potentials?

For $g = 1$ we get $\mathcal{M}_\gamma^{\xi-ss}(Q, W) = \mathrm{Sym}^\gamma(\mathbb{C}^3) = (\mathbb{C}^3)^\gamma /\!\!/ S_\gamma$ as $(M_\alpha)_{\alpha \in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1, b_1, ω).

Notice: Representations of the preprojective algebra cannot be described by a quiver with potential.

Question: Why only relations induced by potentials?

1. $\mathrm{Mat}_\gamma^{\xi-ss}(Q, W) = \mathrm{Crit}(\mathrm{Tr}_\gamma(W))$ for some G_γ-invariant function $\mathrm{Tr}_\gamma(W) : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \longrightarrow \mathbb{C}$,

For $g=1$ we get $\mathcal{M}^{\xi-ss}_{\gamma}(Q,W)=\mathrm{Sym}^{\gamma}(\mathbb{C}^3)=(\mathbb{C}^3)^{\gamma}/\!\!/S_{\gamma}$ as $(M_\alpha)_{\alpha\in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1,b_1,ω).

Notice: Representations of the preprojective algebra cannot be described by a quiver with potential.

Question: Why only relations induced by potentials?

1. $\mathrm{Mat}^{\xi-ss}_{\gamma}(Q,W)=\mathrm{Crit}(\mathrm{Tr}_{\gamma}(W))$ for some G_γ-invariant function $\mathrm{Tr}_{\gamma}(W):\mathrm{Mat}^{\xi-ss}_{\gamma}(Q)\longrightarrow\mathbb{C}$,
2. There exists a perverse sheaf/(monodromic) mixed Hodge module of vanishing cycles $\phi_{\mathrm{Tr}_{\gamma}(W)}$ on $\mathrm{Crit}(\mathrm{Tr}_{\gamma}(W))$ measuring the singularities of the fibers of $\mathrm{Tr}_{\gamma}(W)$.

For $g = 1$ we get $\mathcal{M}_\gamma^{\xi-ss}(Q, W) = \mathrm{Sym}^\gamma(\mathbb{C}^3) = (\mathbb{C}^3)^\gamma /\!\!/ S_\gamma$ as $(M_\alpha)_{\alpha\in Q}$ can be considered as a zero-dimensional sheaf on $\mathbb{C}^3$ with coordinates (a_1, b_1, ω).
Notice: Representations of the preprojective algebra cannot be described by a quiver with potential.

Question: Why only relations induced by potentials?

1. $\mathrm{Mat}_\gamma^{\xi-ss}(Q, W) = \mathrm{Crit}(\mathrm{Tr}_\gamma(W))$ for some G_γ-invariant function $\mathrm{Tr}_\gamma(W) : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \longrightarrow \mathbb{C}$,
2. There exists a perverse sheaf/(monodromic) mixed Hodge module of vanishing cycles $\phi_{\mathrm{Tr}_\gamma(W)}$ on $\mathrm{Crit}(\mathrm{Tr}_\gamma(W))$ measuring the singularities of the fibers of $\mathrm{Tr}_\gamma(W)$.

Notice: $\mathrm{Tr}_\gamma(W) = f_\gamma \circ p$ for some function $f_\gamma : \mathcal{M}_\gamma^{\xi-ss}(Q) \longrightarrow \mathbb{C}$, where $p : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q)$ is the quotient map.

Hall algebras - Part I

Fix a "phase" $\vartheta \in (0, \pi)$ and introduce the shorthand
$\Gamma_\vartheta := \{0 \neq \gamma \in \mathbb{N}^{Q_0} \mid \arg(\sum_{v \in Q_0} \xi_v \gamma_v) = \vartheta\} \cup \{0\}$

Hall algebras - Part I

Fix a "phase" $\vartheta \in (0, \pi)$ and introduce the shorthand
$\Gamma_\vartheta := \{0 \neq \gamma \in \mathbb{N}^{Q_0} \mid \arg(\sum_{v \in Q_0} \xi_v \gamma_v) = \vartheta\} \cup \{0\}$
We define the **relative Hall algebra**

$$\overline{\mathcal{H}}(Q, W, \xi, \vartheta) := \bigoplus_{\gamma \in \Gamma_\vartheta} \bigoplus_{i \in \mathbb{Z}} \mathrm{R}^i p_{G_\gamma} \phi_{\mathrm{Tr}_\gamma(W)} \otimes [\text{twist}],$$

where $\mathrm{R}^i p_G$ is the i-th direct G_γ-equivariant image with respect to the perverse t-structure on $\mathcal{M}_\gamma^{\xi-ss}(Q, W)$ (see below),

Hall algebras - Part I

Fix a "phase" $\vartheta \in (0,\pi)$ and introduce the shorthand
$\Gamma_\vartheta := \{0 \neq \gamma \in \mathbb{N}^{Q_0} \mid \arg(\sum_{v \in Q_0} \xi_v \gamma_v) = \vartheta\} \cup \{0\}$
We define the **relative Hall algebra**

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta) := \bigoplus_{\gamma \in \Gamma_\vartheta} \bigoplus_{i \in \mathbb{Z}} \mathrm{R}^i p_{G_\gamma} \phi_{\mathrm{Tr}_\gamma(W)} \otimes [\text{twist}],$$

where $\mathrm{R}^i p_G$ is the i-th direct G_γ-equivariant image with respect to the perverse t-structure on $\mathcal{M}^{\xi-ss}_\gamma(Q,W)$ (see below),
and the **absolute Hall algebra**

$$\mathcal{H}(Q,W,\xi,\vartheta) := \bigoplus_{\gamma \in \Gamma_\vartheta} \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i_{G_\gamma}\left(\mathrm{Mat}^{\xi-ss}_\gamma(Q,W), \phi_{\mathrm{Tr}_\gamma(W)}\right) \otimes [\text{twist}].$$

by taking the G_γ-equivariant cohomology.

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}^{\xi-ss}_\gamma(Q,W)$ and $p : \mathrm{Mat}^{\xi-ss}_\gamma(Q,W) \longrightarrow \mathcal{M}^{\xi-ss}_\gamma(Q,W)$ with

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $p:\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)\longrightarrow\mathcal{M}_\gamma^{\xi-ss}(Q,W)$ with

$EG_\gamma\times_{G_\gamma}\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $EG_\gamma\times_{G_\gamma}\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)\longrightarrow\mathcal{M}_\gamma^{\xi-ss}(Q,W)$

and take cohomology and direct images respectively.

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}^{\xi-ss}_\gamma(Q,W)$ and $p:\mathrm{Mat}^{\xi-ss}_\gamma(Q,W)\longrightarrow\mathcal{M}^{\xi-ss}_\gamma(Q,W)$ with

$$EG_\gamma\times_{G_\gamma}\mathrm{Mat}^{\xi-ss}_\gamma(Q,W) \text{ and } EG_\gamma\times_{G_\gamma}\mathrm{Mat}^{\xi-ss}_\gamma(Q,W)\longrightarrow\mathcal{M}^{\xi-ss}_\gamma(Q,W)$$

and take cohomology and direct images respectively.
In practice, we have $EG_\gamma\times_{G_\gamma}\mathrm{Mat}^{\xi-ss}_\gamma(Q,W)=\varinjlim_n U^{(n)}_\gamma(Q,W,\xi)$ for finite dimensional closed "subvarieties" $U^{(n)}_\gamma(Q,W,\xi)$ such that

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $p : \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ with

$$EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \text{ and } EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$$

and take cohomology and direct images respectively.
In practice, we have $EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) = \varinjlim_n U_\gamma^{(n)}(Q,W,\xi)$ for finite dimensional closed "subvarieties" $U_\gamma^{(n)}(Q,W,\xi)$ such that

1. $\mathrm{H}^i\left(U_\gamma^{(n)}(Q,W,\xi)\right)$ stabilizes for $n \gg 0$,

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $p : \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ with

$$EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \text{ and } EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$$

and take cohomology and direct images respectively.
In practice, we have $EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) = \varinjlim_n U_\gamma^{(n)}(Q,W,\xi)$ for finite dimensional closed "subvarieties" $U_\gamma^{(n)}(Q,W,\xi)$ such that

1. $\mathrm{H}^i\left(U_\gamma^{(n)}(Q,W,\xi)\right)$ stabilizes for $n \gg 0$,
2. $U_\gamma^{(n)}(Q,W,\xi) \longrightarrow \mathfrak{Rep}_\gamma^{\xi-ss}(Q,W)$ is an atlas,

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $p : \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ with

$$EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \text{ and } EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$$

and take cohomology and direct images respectively.
In practice, we have $EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) = \varinjlim_n U_\gamma^{(n)}(Q,W,\xi)$ for finite dimensional closed “subvarieties” $U_\gamma^{(n)}(Q,W,\xi)$ such that

1. $\mathrm{H}^i\left(U_\gamma^{(n)}(Q,W,\xi)\right)$ stabilizes for $n \gg 0$,
2. $U_\gamma^{(n)}(Q,W,\xi) \longrightarrow \mathfrak{Rep}_\gamma^{\xi-ss}(Q,W)$ is an atlas,
3. $U_\gamma^{(n)}(Q,W,\xi) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ is proper.

Equivariant direct images

To compute $\mathrm{H}^i_{G_\gamma}$ and $\mathrm{R}^i p_{G_\gamma}$, we replace $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W)$ and $p : \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ with

$$EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \text{ and } EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$$

and take cohomology and direct images respectively.
In practice, we have $EG_\gamma \times_{G_\gamma} \mathrm{Mat}_\gamma^{\xi-ss}(Q,W) = \varinjlim_n U_\gamma^{(n)}(Q,W,\xi)$ for finite dimensional closed "subvarieties" $U_\gamma^{(n)}(Q,W,\xi)$ such that

1. $\mathrm{H}^i\left(U_\gamma^{(n)}(Q,W,\xi)\right)$ stabilizes for $n \gg 0$,
2. $U_\gamma^{(n)}(Q,W,\xi) \longrightarrow \mathfrak{Rep}_\gamma^{\xi-ss}(Q,W)$ is an atlas,
3. $U_\gamma^{(n)}(Q,W,\xi) \longrightarrow \mathcal{M}_\gamma^{\xi-ss}(Q,W)$ is proper.

Approximation (of p) by proper maps

Notice:

1. The absolute Hall algebra $\mathcal{H}(Q, W, \xi, \vartheta) = \mathcal{H}_*^*(Q, W, \xi, \vartheta)$ is a bi-graded vector space/monodromic mixed Hodge structure.

Notice:

1. The absolute Hall algebra $\mathcal{H}(Q,W,\xi,\vartheta)=\mathcal{H}^*_*(Q,W,\xi,\vartheta)$ is a bi-graded vector space/monodromic mixed Hodge structure.
2. The relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)=\overline{\mathcal{H}}^*_*(Q,W,\xi,\vartheta)$ is a bi-graded perverse sheaf/monodromic mixed Hodge module on $\mathcal{M}^{\xi-ss}_{\vartheta}(Q,W):=\sqcup_{\gamma\in\Gamma_\vartheta}\mathcal{M}^{\xi-ss}_{\gamma}(Q,W)$.

Notice:

1. The absolute Hall algebra $\mathcal{H}(Q, W, \xi, \vartheta) = \mathcal{H}_*^*(Q, W, \xi, \vartheta)$ is a bi-graded vector space/monodromic mixed Hodge structure.
2. The relative Hall algebra $\overline{\mathcal{H}}(Q, W, \xi, \vartheta) = \overline{\mathcal{H}}_*^*(Q, W, \xi, \vartheta)$ is a bi-graded perverse sheaf/monodromic mixed Hodge module on $\mathcal{M}_\vartheta^{\xi-ss}(Q, W) := \sqcup_{\gamma \in \Gamma_\vartheta} \mathcal{M}_\gamma^{\xi-ss}(Q, W)$.

By general arguments there is a "perverse" filtration on $\mathcal{H}_\gamma^*(Q, W, \xi, \vartheta)$ and a spectral sequence with E_2-term

$$\mathrm{H}^i\left(\mathcal{M}_\gamma^{\xi-ss}(Q, W), \overline{\mathcal{H}}_\gamma^j(Q, W, \xi, \vartheta)\right)$$

converging to $\mathfrak{gr}^i \mathcal{H}_\gamma^{i+j}(Q, W, \xi, \vartheta)$.

Notice:

1. The absolute Hall algebra $\mathcal{H}(Q, W, \xi, \vartheta) = \mathcal{H}^*_*(Q, W, \xi, \vartheta)$ is a bi-graded vector space/monodromic mixed Hodge structure.
2. The relative Hall algebra $\overline{\mathcal{H}}(Q, W, \xi, \vartheta) = \overline{\mathcal{H}}^*_*(Q, W, \xi, \vartheta)$ is a bi-graded perverse sheaf/monodromic mixed Hodge module on $\mathcal{M}^{\xi-ss}_{\vartheta}(Q, W) := \sqcup_{\gamma\in\Gamma_\vartheta} \mathcal{M}^{\xi-ss}_{\gamma}(Q, W)$.

By general arguments there is a "perverse" filtration on $\mathcal{H}^*_\gamma(Q, W, \xi, \vartheta)$ and a spectral sequence with E_2-term

$$\mathrm{H}^i\left(\mathcal{M}^{\xi-ss}_{\gamma}(Q, W), \overline{\mathcal{H}}^j_\gamma(Q, W, \xi, \vartheta)\right)$$

converging to $\mathfrak{gr}^i\, \mathcal{H}^{i+j}_\gamma(Q, W, \xi, \vartheta)$.

Proposition (Davison–M. '16)

The spectral sequence collapses at E_2, i.e.

$$\mathfrak{gr}^*\, \mathcal{H}^*_\gamma(Q, W, \xi, \vartheta) \cong \mathrm{H}^*\left(\mathcal{M}^{\xi-ss}_{\gamma}(Q, W), \overline{\mathcal{H}}^*_\gamma(Q, W, \xi, \vartheta)\right).$$

Hall algebras - Part II

Given dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, consider

$$\mathrm{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q) := \left\{(M_\alpha) \in \mathrm{Mat}^{\xi-ss}_{\gamma'+\gamma''}(Q) \mid M_\alpha \text{ upper block triagonal}\right\}$$

Hall algebras - Part II

Given dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, consider

$$\mathrm{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q) := \left\{(M_\alpha) \in \mathrm{Mat}^{\xi-ss}_{\gamma'+\gamma''}(Q) \mid M_\alpha \text{ upper block triagonal}\right\}$$

with its action by the subgroup $G_{\gamma',\gamma''} \subset G_{\gamma'+\gamma''}$ of upper block triagonal invertible matrices.

Hall algebras - Part II

Given dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, consider

$$\mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q) := \left\{ (M_\alpha) \in \mathsf{Mat}^{\xi-ss}_{\gamma'+\gamma''}(Q) \mid M_\alpha \text{ upper block triagonal} \right\}$$

with its action by the subgroup $G_{\gamma',\gamma''} \subset G_{\gamma'+\gamma''}$ of upper block triagonal invertible matrices.

$$\mathfrak{Rep}^{\xi-ss}_{\gamma',\gamma''}(Q) := \mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q)/G_{\gamma',\gamma''}$$

is the stack of short exact sequences.

Hall algebras - Part II

Given dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, consider

$\mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q) := \{(M_\alpha) \in \mathsf{Mat}^{\xi-ss}_{\gamma'+\gamma''}(Q) \mid M_\alpha \text{ upper block triagonal}\}$

with its action by the subgroup $G_{\gamma',\gamma''} \subset G_{\gamma'+\gamma''}$ of upper block triagonal invertible matrices.

$$\mathfrak{Rep}^{\xi-ss}_{\gamma',\gamma''}(Q) := \mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q)/G_{\gamma',\gamma''}$$

is the stack of short exact sequences.
Get equivariant maps

$$\pi_2 : \mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''}(Q) \hookrightarrow \mathsf{Mat}^{\xi-ss}_{\gamma'+\gamma''}(Q)$$

and

$$\pi_1 \times \pi_3 : \mathsf{Mat}^{\xi-ss}_{\gamma',\gamma''} \longrightarrow \mathsf{Mat}^{\xi-ss}_{\gamma'}(Q) \times \mathsf{Mat}^{\xi-ss}_{\gamma''}(Q)$$

inducing a commutative diagram

$$\begin{array}{ccccc}
 & & \mathfrak{Rep}^{\xi-ss}_{\gamma',\gamma''}(Q) & & \\
 & \swarrow^{\pi_1\times\pi_3} & & \searrow^{\pi_2} & \\
\mathfrak{Rep}^{\xi-ss}_{\gamma'}(Q)\times\mathfrak{Rep}^{\xi-ss}_{\gamma''}(Q) & & & & \mathfrak{Rep}^{\xi-ss}_{\gamma'+\gamma''}(Q) \\
\downarrow{\scriptstyle p\times p} & & & & \downarrow{\scriptstyle p} \\
\mathcal{M}^{\xi-ss}_{\gamma'}(Q)\times\mathcal{M}^{\xi-ss}_{\gamma''}(Q) & & \xrightarrow{\quad\oplus\quad} & & \mathcal{M}^{\xi-ss}_{\gamma'+\gamma''}(Q)
\end{array}$$

inducing a commutative diagram

$$\begin{array}{ccc} & \mathfrak{Rep}^{\xi-ss}_{\gamma',\gamma''}(Q) & \\ {}^{\pi_1\times\pi_3}\swarrow & & \searrow^{\pi_2} \\ \mathfrak{Rep}^{\xi-ss}_{\gamma'}(Q)\times\mathfrak{Rep}^{\xi-ss}_{\gamma''}(Q) & & \mathfrak{Rep}^{\xi-ss}_{\gamma'+\gamma''}(Q) \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ \mathcal{M}^{\xi-ss}_{\gamma'}(Q)\times\mathcal{M}^{\xi-ss}_{\gamma''}(Q) & \xrightarrow{\quad\oplus\quad} & \mathcal{M}^{\xi-ss}_{\gamma'+\gamma''}(Q) \end{array}$$

Using adjunction morphisms for pull-back and push-forwards, the Thom–Sebastiani isomorphism and properties of the vanishing cycle functor, we get maps

$$\oplus_*\Big(\overline{\mathcal{H}}_{\gamma'}(Q,W,\xi,\vartheta)\boxtimes\overline{\mathcal{H}}_{\gamma''}(Q,W,\xi,\vartheta)\Big)\longrightarrow\overline{\mathcal{H}}_{\gamma'+\gamma''}(Q,W,\xi,\vartheta)$$

of perverse sheaves/monodromic mixed Hodge modules

$$\oplus_*\Big(\overline{\mathcal{H}}_{\gamma'}(Q,W,\xi,\vartheta)\boxtimes\overline{\mathcal{H}}_{\gamma''}(Q,W,\xi,\vartheta)\Big)\longrightarrow\overline{\mathcal{H}}_{\gamma'+\gamma''}(Q,W,\xi,\vartheta)$$

of perverse sheaves/monodromic mixed Hodge modules
and similarly

$$\mathcal{H}_{\gamma'}(Q,W,\xi,\vartheta)\otimes\mathcal{H}_{\gamma''}(Q,W,\xi,\vartheta)\longrightarrow\mathcal{H}_{\gamma'+\gamma''}(Q,W,\xi,\vartheta).$$

$$\oplus_*\Big(\overline{\mathcal{H}}_{\gamma'}(Q,W,\xi,\vartheta)\boxtimes\overline{\mathcal{H}}_{\gamma''}(Q,W,\xi,\vartheta)\Big)\longrightarrow\overline{\mathcal{H}}_{\gamma'+\gamma''}(Q,W,\xi,\vartheta)$$

of perverse sheaves/monodromic mixed Hodge modules
and similarly

$$\mathcal{H}_{\gamma'}(Q,W,\xi,\vartheta)\otimes\mathcal{H}_{\gamma''}(Q,W,\xi,\vartheta)\longrightarrow\mathcal{H}_{\gamma'+\gamma''}(Q,W,\xi,\vartheta).$$

Summing over $\gamma',\gamma''\in\Gamma_\vartheta$ we get algebras in appropriate symmetric monoidal tensor categories.

Theorem (Davison–M. '16)

1. *The Hall algebras* $\overline{\mathcal{H}}(Q, W, \xi, \vartheta)$ *and* $\mathcal{H}(Q, W, \xi, \vartheta)$ *are associative with unit.*

Theorem (Davison–M. '16)

1. *The Hall algebras* $\overline{\mathcal{H}}(Q, W, \xi, \vartheta)$ *and* $\mathcal{H}(Q, W, \xi, \vartheta)$ *are associative with unit.*
2. *The collapsing spectral sequence is a spectral sequence of algebras inducing an isomorphism of algebras*

$$\mathfrak{gr}^* \mathcal{H}^*(Q, W, \xi, \vartheta) \cong \mathrm{H}^* \left(\mathcal{M}_{\vartheta}^{\xi-ss}(Q, W), \overline{\mathcal{H}}^*(Q, W, \xi, \vartheta) \right).$$

Theorem (Davison–M. '16)

1. *The Hall algebras* $\overline{\mathcal{H}}(Q, W, \xi, \vartheta)$ *and* $\mathcal{H}(Q, W, \xi, \vartheta)$ *are associative with unit.*
2. *The collapsing spectral sequence is a spectral sequence of algebras inducing an isomorphism of algebras*

$$\mathfrak{gr}^* \mathcal{H}^*(Q, W, \xi, \vartheta) \cong \mathrm{H}^* \left(\mathcal{M}_\vartheta^{\xi-ss}(Q, W), \overline{\mathcal{H}}^*(Q, W, \xi, \vartheta) \right).$$

3. *The absolute Hall algebra* $\mathcal{H}(Q, W, \xi, \vartheta)$ *has a compatible (localized) coproduct turning* $\mathcal{H}(Q, W, \xi, \vartheta)$ *into a (localized) bi-algebra and* $\mathfrak{gr}\,\mathcal{H}(Q, W, \xi, \vartheta)$ *into a Hopf algebra.*

Theorem (Davison–M. '16)

1. *The Hall algebras* $\overline{\mathcal{H}}(Q, W, \xi, \vartheta)$ *and* $\mathcal{H}(Q, W, \xi, \vartheta)$ *are associative with unit.*
2. *The collapsing spectral sequence is a spectral sequence of algebras inducing an isomorphism of algebras*

$$\mathfrak{gr}^* \mathcal{H}^*(Q, W, \xi, \vartheta) \cong \mathrm{H}^* \left(\mathcal{M}_\vartheta^{\xi-ss}(Q, W), \overline{\mathcal{H}}^*(Q, W, \xi, \vartheta) \right).$$

3. *The absolute Hall algebra* $\mathcal{H}(Q, W, \xi, \vartheta)$ *has a compatible (localized) coproduct turning* $\mathcal{H}(Q, W, \xi, \vartheta)$ *into a (localized) bi-algebra and* $\mathfrak{gr}\, \mathcal{H}(Q, W, \xi, \vartheta)$ *into a Hopf algebra.*

Question: What can we say about the structures of the Hall algebras?

Genericity

Definition

We call a stability condition ξ generic if for all $\vartheta \in (0, \pi)$ and all $\gamma', \gamma'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha: v \to w} \gamma'_v \gamma''_w$ is symmetric.

Genericity

Definition

We call a stability condition ξ generic if for all $\vartheta \in (0, \pi)$ and all $\gamma', \gamma'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha: v \to w} \gamma'_v \gamma''_w$ is symmetric.

Interpretation: Given representations M' and M'' of dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$,

Genericity

Definition

We call a stability condition ξ generic if for all $\vartheta \in (0, \pi)$ and all $\gamma', \gamma'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha: v \to w} \gamma'_v \gamma''_w$ is symmetric.

Interpretation: Given representations M' and M'' of dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, let $(S_\kappa)_{\kappa \in K}$ be the family of (non-isomorphic) stable factors of M' and M''. For $\kappa, \lambda \in K$ let $A_{\kappa\lambda} := \dim \operatorname{Ext}^1(S_\kappa, S_\lambda)$.

Genericity

Definition

We call a stability condition ξ generic if for all $\vartheta \in (0, \pi)$ and all $\gamma', \gamma'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha: v \to w} \gamma'_v \gamma''_w$ is symmetric.

Interpretation: Given representations M' and M'' of dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, let $(S_\kappa)_{\kappa \in K}$ be the family of (non-isomorphic) stable factors of M' and M''. For $\kappa, \lambda \in K$ let $A_{\kappa\,\lambda} := \dim \operatorname{Ext}^1(S_\kappa, S_\lambda)$. Then, ξ is generic if for all choices of ϑ, M' and M'', the matrix $A = (A_{\kappa\,\lambda})$ is symmetric.

Genericity

Definition

We call a stability condition ξ generic if for all $\vartheta \in (0, \pi)$ and all $\gamma', \gamma'' \in \Gamma_\vartheta$ the bilinear pairing $\sum_{\alpha: v \to w} \gamma'_v \gamma''_w$ is symmetric.

Interpretation: Given representations M' and M'' of dimension vectors $\gamma', \gamma'' \in \Gamma_\vartheta$, let $(S_\kappa)_{\kappa \in K}$ be the family of (non-isomorphic) stable factors of M' and M''. For $\kappa, \lambda \in K$ let $A_{\kappa\,\lambda} := \dim \operatorname{Ext}^1(S_\kappa, S_\lambda)$. Then, ξ is generic if for all choices of ϑ, M' and M'', the matrix $A = (A_{\kappa\,\lambda})$ is symmetric. The quiver with vertex set K and $A_{\kappa\,\lambda}$ arrows from κ to λ is called the (Ext-)quiver of $(S_\kappa)_{\kappa \in K}$. It is symmetric if ξ is generic.

Theorem (Davison–M. '16)

The absolute Hall algebra of the Ext-quiver of $(S_\kappa)_{\kappa\in K}$ with a suitable (formal) potential determines the Hall algebra product on the stalks of the relative Hall-algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ at M' and M''

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'}\otimes\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M''}\longrightarrow\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'\oplus M''}.$$

Theorem (Davison–M. '16)

The absolute Hall algebra of the Ext-quiver of $(S_\kappa)_{\kappa\in K}$ with a suitable (formal) potential determines the Hall algebra product on the stalks of the relative Hall-algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ at M' and M''

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'} \otimes \overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M''} \longrightarrow \overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'\oplus M''}.$$

By a result of Efimov the absolute Hall algebra of a symmetric quiver with zero potential is (graded) commutative. Using this, we can prove

Theorem (Davison–M. '16)

The absolute Hall algebra of the Ext-quiver of $(S_\kappa)_{\kappa\in K}$ with a suitable (formal) potential determines the Hall algebra product on the stalks of the relative Hall-algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ at M' and M''

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'}\otimes\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M''}\longrightarrow\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'\oplus M''}.$$

By a result of Efimov the absolute Hall algebra of a symmetric quiver with zero potential is (graded) commutative. Using this, we can prove

Theorem (Davison–M. '16)

If ξ is a generic stability condition, the relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ is (graded) commutative for all phases $\vartheta\in(0,\pi)$.

Theorem (Davison–M. '16)

The absolute Hall algebra of the Ext-quiver of $(S_\kappa)_{\kappa\in K}$ with a suitable (formal) potential determines the Hall algebra product on the stalks of the relative Hall-algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ at M' and M''

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'}\otimes\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M''}\longrightarrow\overline{\mathcal{H}}(Q,W,\xi,\vartheta)_{M'\oplus M''}.$$

By a result of Efimov the absolute Hall algebra of a symmetric quiver with zero potential is (graded) commutative. Using this, we can prove

Theorem (Davison–M. '16)

If ξ is a generic stability condition, the relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ is (graded) commutative for all phases $\vartheta\in(0,\pi)$.

Question: How does this commutative algebra look like?

Integrality

Theorem (Davison–M. '16)

For a generic stability condition ξ and any phase $\vartheta \in (0,\pi)$ the relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ is a symmetric algebra, i.e.

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta) = \mathrm{Sym}(\mathcal{G})$$

for some (graded) perverse sheaf/monodromic mixed Hodge modules $\mathcal{G}$ on $\mathcal{M}_{\vartheta}^{\xi-ss}(Q,W) = \sqcup_{\gamma\in\Gamma_\vartheta} \mathcal{M}_{\gamma}^{\xi-ss}(Q,W)$.

Integrality

Theorem (Davison–M. '16)

For a generic stability condition ξ and any phase $\vartheta \in (0,\pi)$ the relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ is a symmetric algebra, i.e.

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta) = \mathrm{Sym}(\mathcal{G})$$

for some (graded) perverse sheaf/monodromic mixed Hodge modules $\mathcal{G}$ on $\mathcal{M}^{\xi-ss}_{\vartheta}(Q,W) = \sqcup_{\gamma\in\Gamma_\vartheta} \mathcal{M}^{\xi-ss}_{\gamma}(Q,W)$.

Remark: The absolute Hall algebra $\mathcal{H}(Q,W,\xi,\vartheta)$ is in general not (graded) commutative even for generic ξ. But:

Integrality

Theorem (Davison–M. '16)

For a generic stability condition ξ and any phase $\vartheta \in (0,\pi)$ the relative Hall algebra $\overline{\mathcal{H}}(Q,W,\xi,\vartheta)$ is a symmetric algebra, i.e.

$$\overline{\mathcal{H}}(Q,W,\xi,\vartheta) = \mathrm{Sym}(\mathcal{G})$$

for some (graded) perverse sheaf/monodromic mixed Hodge modules $\mathcal{G}$ on $\mathcal{M}_{\vartheta}^{\xi-ss}(Q,W) = \sqcup_{\gamma\in\Gamma_\vartheta}\mathcal{M}_{\gamma}^{\xi-ss}(Q,W)$.

Remark: The absolute Hall algebra $\mathcal{H}(Q,W,\xi,\vartheta)$ is in general not (graded) commutative even for generic ξ. But:

Corollary

For generic ξ and any ϑ the associated graded algebra $\mathfrak{gr}\,\mathcal{H}(Q,W,\xi,\vartheta)$ wrt. the perverse filtration is a symmetric algebra generated by $\mathrm{H}^*\left(\mathcal{M}_{\vartheta}^{\xi-ss}(Q,W),\mathcal{G}\right)$.

Question: Can we determine $\mathcal{G}$?

Question: Can we determine $\mathcal{G}$?

Recall: $\mathrm{Tr}_\gamma(W) : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \xrightarrow{p} \mathcal{M}_\gamma^{\xi-ss}(Q) \xrightarrow{f_\gamma} \mathbb{C}$ and $\mathrm{Mat}_\gamma^{\xi-ss}(Q, W) = \mathrm{Crit}(\mathrm{Tr}_\gamma(W))$.

Question: Can we determine $\mathcal{G}$?

Recall: $\mathrm{Tr}_\gamma(W) : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \xrightarrow{p} \mathcal{M}_\gamma^{\xi-ss}(Q) \xrightarrow{f_\gamma} \mathbb{C}$ and $\mathrm{Mat}_\gamma^{\xi-ss}(Q,W) = \mathrm{Crit}(\mathrm{Tr}_\gamma(W))$.

Definition

1. For $\gamma \in \mathbb{N}^{Q_0}$ we form the "Donaldson–Thomas sheaf"

$$\mathcal{DT}_\gamma(Q,W,\xi) = \begin{cases} \phi_{f_\gamma}\left(\mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})\right) & \text{if } \mathcal{M}_\gamma^{\xi-st}(Q) \neq \emptyset, \\ 0 & \text{else} \end{cases}$$

Here, $\mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})$ is the intersection complex of $\mathcal{M}_\gamma^{\xi-ss}(Q)$.

Question: Can we determine $\mathcal{G}$?

Recall: $\mathrm{Tr}_\gamma(W) : \mathrm{Mat}_\gamma^{\xi-ss}(Q) \xrightarrow{p} \mathcal{M}_\gamma^{\xi-ss}(Q) \xrightarrow{f_\gamma} \mathbb{C}$ and $\mathrm{Mat}_\gamma^{\xi-ss}(Q, W) = \mathrm{Crit}(\mathrm{Tr}_\gamma(W))$.

Definition

1. For $\gamma \in \mathbb{N}^{Q_0}$ we form the "Donaldson–Thomas sheaf"

$$\mathcal{DT}_\gamma(Q, W, \xi) = \begin{cases} \phi_{f_\gamma}(\mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})) & \text{if } \mathcal{M}_\gamma^{\xi-st}(Q) \neq \emptyset, \\ 0 & \text{else} \end{cases}$$

   Here, $\mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})$ is the intersection complex of $\mathcal{M}_\gamma^{\xi-ss}(Q)$.

2. $\mathcal{DT}_\vartheta(Q, W, \xi) := \bigoplus_{\gamma\in\Gamma_\vartheta} \mathcal{DT}_\gamma(Q, W, \xi)$ a perverse sheaf/monodromic mixed Hodge module on $\mathcal{M}_\vartheta^{\xi-ss}(Q, W)$.

Theorem (Davison–M. '16)

For a generic stability condition ξ and any phase $\vartheta \in (0, \pi)$ we get

$$\mathcal{G} = \mathcal{DT}_{\vartheta}(Q, W, \xi) \otimes \mathrm{H}(B\mathbb{C}^*)_{vir} := \bigoplus_{i \in \mathbb{N}} \mathcal{DT}_{\vartheta}(Q, W, \vartheta) \otimes [twist]^{2i+1}.$$

Theorem (Davison–M. '16)

For a generic stability condition ξ *and any phase* $\vartheta \in (0,\pi)$ *we get*

$$\mathcal{G} = \mathcal{DT}_{\vartheta}(Q,W,\xi) \otimes \mathrm{H}(B\mathbb{C}^*)_{vir} := \bigoplus_{i\in\mathbb{N}} \mathcal{DT}_{\vartheta}(Q,W,\vartheta) \otimes [twist]^{2i+1}.$$

Corollary

For generic ξ *and any* ϑ *the associated graded algebra* $\mathfrak{gr}\,\mathcal{H}(Q,W,\xi,\vartheta)$ *wrt. the perverse filtration is a symmetric algebra generated by* $\mathrm{H}^*\left(\mathcal{M}_{\vartheta}^{\xi-ss}(Q,W), \mathcal{DT}_{\vartheta}(Q,W,\xi)\right) \otimes \mathrm{H}(B\mathbb{C}^*)_{vir}$.

Theorem (Davison–M. '16)

For a generic stability condition ξ *and any phase* $\vartheta \in (0,\pi)$ *we get*

$$\mathcal{G} = \mathcal{DT}_\vartheta(Q,W,\xi) \otimes \mathrm{H}(B\mathbb{C}^*)_{vir} := \bigoplus_{i\in\mathbb{N}} \mathcal{DT}_\vartheta(Q,W,\vartheta) \otimes [twist]^{2i+1}.$$

Corollary

For generic ξ *and any* ϑ *the associated graded algebra* $\mathfrak{gr}\,\mathcal{H}(Q,W,\xi,\vartheta)$ *wrt. the perverse filtration is a symmetric algebra generated by* $\mathrm{H}^*\left(\mathcal{M}_\vartheta^{\xi-ss}(Q,W), \mathcal{DT}_\vartheta(Q,W,\xi)\right) \otimes \mathrm{H}(B\mathbb{C}^*)_{vir}$*.*

Definition

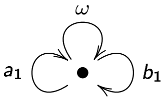
The (alternating) dimension of $\mathrm{H}^*_c\left(\mathcal{M}_\gamma^{\xi-ss}(Q,W), \mathcal{DT}_\gamma(Q,W,\xi)\right)$ is called the Donaldson–Thomas invariant for Q, W, ξ, γ. Its Hodge polynomial is the refined DT invariant.

Examples

1. For $W = 0$, we get $\mathcal{DT}_\gamma(Q, W, \xi) = \mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})$ if $\mathcal{M}_\gamma^{\xi-st}(Q)$ is non-empty and zero else. Thus, the DT invariants compute intersection Euler characteristics and intersection Betti numbers.

Examples

1. For $W = 0$, we get $\mathcal{DT}_\gamma(Q, W, \xi) = \mathcal{IC}_{\mathcal{M}_\gamma^{\xi-ss}(Q)}(\mathbb{Q})$ if $\mathcal{M}_\gamma^{\xi-st}(Q)$ is non-empty and zero else. Thus, the DT invariants compute intersection Euler characteristics and intersection Betti numbers.
2. For



and $W = \omega[a_1, b_1]$, we get $\mathcal{M}_\gamma^{\xi-ss}(Q, W) = \mathrm{Sym}^\gamma(\mathbb{C}^3) = (\mathbb{C}^3)^n /\!\!/ S_n$ and $\mathcal{DT}_\gamma(Q, W, \xi)$ is the constant (perverse) sheaf $\mathbb{Q}[3]$ on the small diagonal $\Delta : \mathbb{C}^3 \hookrightarrow \mathrm{Sym}^\gamma(\mathbb{C}^3)$.

Thank you!