Symplectic resolutions for Higgs moduli spaces

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June 22, 2017

Conventions and notation

Disclaimer

- work over $\mathbb{C}$
- for the duration of the talk, "symplectic = holomorphic symplectic"
- all our moduli spaces will be moduli schemes and not moduli stacks

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Main goal: study the singularities of $\mathcal{M}_H(\Sigma_g, n)$, using results on $\mathcal{M}_B(g, n)$, via the Nonabelian Hodge correspondence

Symplectic singularities

Definition (Beauville, 2000)

Let X be a normal variety. X is a **symplectic singularity** if there exists a symplectic form ω on the smooth locus X^{sm} such that, for every resolution of singularities $\pi : \widetilde{X} \to X$, the form $\pi^*\omega$ extends to a (possibly degenerate) form on $\widetilde{X}$.

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Definition

If X is a symplectic singularity, a resolution $\pi : \widetilde{X} \to X$ is **symplectic** if $\pi^*\omega$ is everywhere non-degenerate.

Examples

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- $\mathfrak{g}$ semisimple complex Lie algebra, $X = \mathcal{N}$ nilpotent cone, symplectically resolved by the flag variety G/B, via the Springer map

Quiver varieties

Given Q quiver with vertex set I, $\alpha \in \mathbb{Z}^I_{\geq 0}$ dimension vector, $\lambda \in \mathbb{C}^I$, $\theta \in \mathbb{Q}^I$, the quiver variety $\mathcal{M}_{Q,\alpha}(\lambda, \theta)$ is the moduli space of θ-polystable representations of dimension α of a certain *deformed* preprojective algebra Π^λ

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Remark. (*) is a combinatorial condition on Q and α

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Remark. Can extend the above to arbitrary degree d and G-Higgs bundles for G reductive.

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Our approach: transfer results on $\mathcal{M}_B(g, n)$ to $\mathcal{M}_H(\Sigma_g, n)$ using NHC, in particular the Isosingularity theorem (for the non-existence part)

Isosingularity theorem

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As a consequence we get

$$\widehat{\mathcal{M}}_B(g,n)_x \cong \widehat{\mathcal{M}}_H(\Sigma_g,n)_{\phi(x)}$$

Sketch of the proof

$g = 1$: explicit geometric description of the moduli space

Theorem (Franco, Garcia-Prada, Newstead, 2012)

$$\mathcal{M}_H(\Sigma_1, n) \cong \mathrm{Sym}^n(T^*\Sigma_1)$$

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$g \geq 2$**:**

Theorem (Namikawa, 2001)

A normal variety is a symplectic singularity if and only if it has admits a symplectic form on the smooth locus and has rational Gorenstein singularities

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Existence of symplectic resolution:

- $(g, n) = (2, 2)$: blow-up of the singular locus; similar to O'Grady's IHS manifolds; no clear moduli theoretic interpretation
- $(g, n) \neq (2, 2)$: by contradiction: using Isosingularity theorem get a symplectic resolution of (an open subet of) the Betti moduli space

Future directions

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- consider $g = 0$, character varieities of punctured Riemann surfaces: representations of *multiplicative* preprojective algebras

Thank you for your attention!