Topological mirror symmetry via *p*-adic integration

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Image: Image:

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- Their conjecture can be reformulated in terms of counting points over finite fields. This in turn can be done by computing *p*-adic volumes.
- We can compare the *p*-adic volumes of the two moduli spaces, since "singular Hitchin fibers have measure 0".

- Let C be a smooth projective curve of genus g and $K = K_C$ the canonical bundle.
- A Higgs bundle on C is a pair (E, φ), where E is a rank n vector bundle on C and φ ∈ H⁰(C, End(E) ⊗ K).

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Definition

For an integer *d* coprime to *n* and a line bundle *L* of degree *d* on *C* define the moduli space of (twisted) SL_n -Higgs bundles as

 $\mathcal{M}^d_{\mathsf{SL}_n}(C) =$

{Stable Higgs bundles (E, ϕ) , with det $E \cong L$, tr $\phi = 0$ } / ~

• $\mathcal{M}^d_{\mathsf{SL}_n}(C)$ is a smooth quasi-projective variety.

Moduli space of PGL_n Higgs bundles

• The *n*-torsion points $\Gamma = \operatorname{Jac}_{C}[n] \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ act on $\mathcal{M}^{d}_{\operatorname{SL}_{n}}(C)$ by tensoring:

$$\gamma \cdot (E, \phi) = (E \otimes \gamma, \phi), \text{ for } \gamma \in \Gamma.$$

Definition

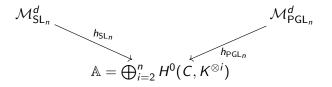
The moduli space of (twisted) PGL_n Higgs bundles is

$$\mathcal{M}^{d}_{\mathsf{PGL}_n}(C) = \mathcal{M}^{d}_{\mathsf{SL}_n}(C)/\Gamma.$$

• Remark: More generally one can construct moduli space of *G*-Higgs bundles for any reductive *G*, it is however unclear how to "twist" in general.

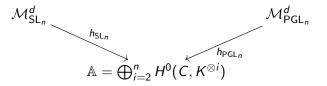
Hitchin Fibration

 Given a Higgs bundle (E, φ) ∈ H⁰(C, End(E) ⊗ K) we can consider its characteristic polynomial h(φ) ∈ ⊕ⁿ_{i=1} H⁰(C, K^{⊗i}). This gives morphisms



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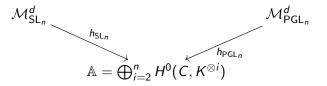


Theorem (Hitchin, Simpson)

The Hitchin maps h_{SL_n} , h_{PGL_n} are proper and their generic fibers are complex Lagrangian torsors for abelian varieties \mathcal{P}_{SL_n} and \mathcal{P}_{PGL_n} respectively. Furthermore \mathcal{P}_{SL_n} and \mathcal{P}_{PGL_n} are dual abelian varieties.

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• If it weren't for the torsor structure , $\mathcal{M}_{SL_n}^d$ and $\mathcal{M}_{PGL_n}^d$ would be mirror partners in the sense of Strominger-Yau-Zaslow.

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Twisted SYZ Mirror Symmetry

• The correct duality between the fibrations h_{SL_n} , h_{PGL_n} should take the torsor structure into account [Hitchin 2001]:

$$\rightsquigarrow \mathbb{Z}/n\mathbb{Z}$$
-Gerbes B, \overline{B} on $\mathcal{M}^d_{\mathsf{SL}_n}, \mathcal{M}^d_{\mathsf{PGL}_n}$.

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Theorem (Hausel-Thaddeus 2003)

The pairs $(\mathcal{M}_{SL_n}^d, B)$ and $(\mathcal{M}_{PGL_n}^d, \overline{B})$ are SYZ mirror partners i.e. for a generic $a \in \mathbb{A}$ we have isomorphisms of \mathcal{P}_{SL_n} and \mathcal{P}_{PGL_n} torsors

$$h_{SL_n}^{-1}(a) \cong Triv(h_{PGL_n}^{-1}(a), \overline{B})$$

$$h_{PGL_n}^{-1}(a) \cong Triv(h_{SL_n}^{-1}(a), B).$$

 Remark: [Donagi-Pantev, 2012] prove a similar statement for any pair of Langlands dual groups.

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Topological Mirror Symmetry

 Because of the lack of properness and the presence of singularities, one cannot hope for the usual symmetry of the Hodge diamond. Instead Hausel-Thaddeus 'argue' that there should be an equality of *stringy Hodge numbers*. Because of the lack of properness and the presence of singularities, one cannot hope for the usual symmetry of the Hodge diamond. Instead Hausel-Thaddeus 'argue' that there should be an equality of stringy Hodge numbers.

Definition

For any complex variety X define the E-polynomial by

$$E(X; x, y) = \sum_{p,q,i \ge 0} (-1)^i h^{p,q;i}(X) x^p y^q,$$

where $h^{p,q;i}(X) = \dim_{\mathbb{C}}(Gr_p^{Ho}Gr_{p+q}^wH_c^i(X))$ denote the compactly supported mixed Hodge numbers of X.

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 The compactly supported cohomology of M^d_{SLn} and M^d_{PGLn} is pure i.e. h^{p,q;i} = 0 unless i = p + q.

Topological Mirror Symmetry

Conjecture (Hausel-Thaddeus 2003)

There is an equality

$$E(\mathcal{M}^{d}_{SL_{n}}; x, y) = E^{\bar{B}}_{st}(\mathcal{M}^{d}_{PGL_{n}}; x, y).$$

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There is an equality

$$E(\mathcal{M}_{SL_n}^d; x, y) = E_{st}^{\bar{B}}(\mathcal{M}_{PGL_n}^d; x, y).$$

• The right hand side takes into account the orbifold structure and can be written as

$$E_{st}^{\bar{B}}(\mathcal{M}_{\mathsf{SL}_n}^d/\Gamma; x, y) = \sum_{\gamma \in \Gamma} (xy)^{\mathcal{F}(\gamma)} E^{\bar{B}_{\gamma}}((\mathcal{M}_{\mathsf{SL}_n}^d)^{\gamma}/\Gamma; x, y),$$

where $E^{\bar{B}_{\gamma}}$ denotes the *E*-polynomial with coefficients in the local system $\bar{B}_{\gamma} \rightarrow (\mathcal{M}_{SL_n}^d)^{\gamma}/\Gamma$ and $\mathcal{F}(\gamma)$ the Fermionic shift.

• The Conjecture is true for n = 2, 3 [HT 2003].

Reduction to finite fields

• The point count analogue of E(X; x, y) is $\#X(\mathbb{F}_q)$. Consequently we define

$$\#^{\bar{B}}_{st}\mathcal{M}^{d}_{\mathsf{PGL}_n}(\mathbb{F}_q) = \sum_{\gamma \in \Gamma} q^{\mathcal{F}(\gamma)} \sum_{x \in (\mathcal{M}^{d}_{\mathsf{SL}_n})^{\gamma} / \Gamma(\mathbb{F}_q)} \mathsf{tr}(\mathsf{Fr}, (\bar{B}_{\gamma})_x).$$

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Essentially by a theorem of Katz, the conjecture then follows from

Theorem (Groechening-W.-Ziegler)

$$\#\mathcal{M}^{d}_{SL_{n}}(\mathbb{F}_{q}) = \#^{\bar{B}}_{st}\mathcal{M}^{d}_{PGL_{n}}(\mathbb{F}_{q}).$$
(1)

Reduction to *p*-adic integration

Let F be a finite extension of Q_p with ring of integers O_F and residue field k_F ≃ F_q.

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- One can integrate differential forms on *p*-adic manifolds in a similar way as on real manifolds.
- In particular for any O_F-variety X we can integrate top forms on the manifold X° = X(O_F) ∩ Xsm(F).

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Theorem (Weil 1982)

Let X be a smooth variety over \mathcal{O}_F of relative dimension n and ω a gauge form on X. Then

$$\int_{X^{\circ}} \omega = \frac{\# X(\mathbb{F}_q)}{q^n}$$

- Through Weil's theorem we can control the LHS of (1) by a *p*-adic integral.
- The same is also true for the RHS, when we integrate a certain weight function $f_{\bar{B}}$ against the canonical class ω_{can} on $\mathcal{M}^{d}_{\mathsf{PGL}_n} = \mathcal{M}^{d}_{\mathsf{SL}_n}/\Gamma$ [Denef-Loeser 2002].

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- The topological mirror symmetry conjecture of Hausel-Thaddeus thus follows from the following

Theorem (Groechening-W.-Ziegler)

$$\int_{\mathcal{M}_{SL_n}^{d\circ}} \omega = \int_{\mathcal{M}_{PGL_n}^{d\circ}} f_{\bar{B}} \omega_{can}.$$

• Enough to compare the integrals fiberwise along *F*-smooth fibers:

$$\int_{h^{-1}_{\mathsf{SL}_n}(a)^\circ} 1 \stackrel{?}{=} \int_{h^{-1}_{\mathsf{PGL}_n}(a)^\circ} f_{\bar{B}} \quad \text{for } a \in \mathbb{A}^{gen}(F) \cap \mathbb{A}(\mathcal{O}_F).$$

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- The measures restricted to such fibers are translation invariant for the actions of $\mathcal{P}_{SL_n}(F)$ and $\mathcal{P}_{PGL_n}(F)$.
- From the isomorphism $h_{SL_n}^{-1}(a) \cong \text{Triv}(h_{PGL_n}^{-1}(a), \overline{B})$ we deduce

$$h_{\operatorname{SL}_n}^{-1}(a)(F) \neq \emptyset \Leftrightarrow \bar{B}_{|h_{\operatorname{PGL}_n}^{-1}(a)}$$
 is trivial.

Sketch of Proof

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• If $h_{\operatorname{SL}_n}^{-1}(a)(F) = \emptyset$, then

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by a character sum argument.

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by a character sum argument.

• If $h_{\mathsf{SL}_n}^{-1}(a)(F) \neq \emptyset$, then $f_{\bar{B}} \equiv 1$ and

$$\int_{h_{\mathsf{SL}_n}^{-1}(\mathbf{a})^\circ} 1 = \int_{h_{\mathsf{PGL}_n}^{-1}(\mathbf{a})^\circ} 1,$$

by using the self duality of the isogeny $\mathcal{P}_{\mathsf{SL}_n} \to \mathcal{P}_{\mathsf{PGL}_n}.$

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Consequences

• The Weil pairing on the curve C gives an identification $\Gamma \cong \Gamma^* = \text{Hom}(\Gamma, \mu_n)$. If $\gamma \in \Gamma$ corresponds to the character χ we have

$$E^{\chi}(\mathcal{M}^{d}_{\mathsf{SL}_n}; x, y) = (xy)^{F(\gamma)} E^{\bar{B}_{\gamma}}((\mathcal{M}^{d}_{\mathsf{SL}_n})^{\gamma} / \Gamma; x, y).$$

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• For any $a \in \mathbb{A}(\mathbb{F}_q)$ we have

$$\#h^{-1}_{\mathsf{SL}_n}(a)(\mathbb{F}_q)=\#^{\bar{B}}_{st}h^{-1}_{\mathsf{PGL}_n}(a)(\mathbb{F}_q).$$

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• For any d' comprime to n we have

$$E(\mathcal{M}^d_{\mathsf{SL}_n}; x, y) = E(\mathcal{M}^{d'}_{\mathsf{SL}_n}; x, y), \quad E(\mathcal{M}^d_{\mathsf{GL}_n}; x, y) = E(\mathcal{M}^{d'}_{\mathsf{GL}_n}; x, y).$$

Thank you!

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Image: A matrix