Kontsevich-Penner model and open intersection numbers

Geometry of Integrable Systems

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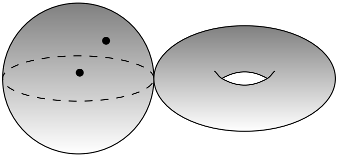
Joint work with Marco Bertola, SISSA/Concordia University

Moduli spaces of Riemann surfaces

$\overline{\mathscr{M}_{g,n}} = \{$stable compact Riemann surfaces of genus g with n marked points$\}/$isomorphism (Deligne and Mumford, 1969).

$\overline{\mathscr{M}_{g,n}}$ is a compact smooth complex orbifold.



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Stability condition: $2g - 2 + n > 0.$

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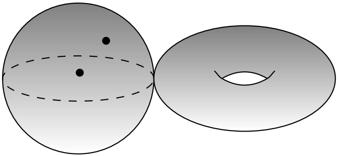

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Free energy and Witten-Kontsevich theorem

Psi classes $\psi_i := c_1(\mathcal{L}_i) \in H^2\left(\overline{\mathscr{M}_{g,n}}, \mathbb{Q}\right)$, $\mathcal{L}_i$ tautological line bdle of cotangent spaces at the $i-$th marked point.

Intersection numbers

$$\left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle = \langle \tau_0^{r_0} \tau_1^{r_1} \cdots \rangle := \int_{\overline{\mathscr{M}_{g,n}}} \psi_1^{d_1} \wedge \cdots \wedge \psi_n^{d_n} \quad (r_j = \sharp\{i\colon d_i = j\})$$

Generating function

$$F(t_0, t_1, \dots) := \left\langle \exp \sum_{j \geq 0} t_j \tau_j \right\rangle = \sum_{r_*} \langle \tau_0^{r_0} \tau_1^{r_1} \cdots \rangle \frac{t_0^{r_0} t_1^{r_1} \cdots}{r_0! r_1! \cdots} = \frac{t_0^3}{6} + \frac{t_1}{24} + \frac{t_0 t_2}{24} + \frac{t_1^2}{24} + \frac{t_0^2 t_3}{48} + \cdots$$

Theorem (E. Witten - M. Kontsevich, 1991)

$\exp F$ is a KdV tau function. In particular $U := \partial_{t_0} F$ satisfies

$$\frac{\partial U}{\partial t_1} = U \frac{\partial U}{\partial t_0} + \frac{1}{12} \frac{\partial^3 U}{\partial t_0^3}$$

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Kontsevich matrix integral

$$Z_n(Y) := \frac{\int_{H_n} \mathrm{d}M \exp \operatorname{Tr}\left(i\frac{M^3}{3} - M^2 Y\right)}{\int_{H_n} \mathrm{d}M \exp \operatorname{Tr}\left(-M^2 Y\right)}$$

$H_n = \mathbb{R}^{n^2} = n \times n$ hermitian matrices, $Y = \operatorname{diag}(y_1, ..., y_n)$.

- $Z_n(Y)$ is a KdV tau function in *Miwa variables*

$$T_k(Y) := -\frac{2^{-\frac{2k+1}{3}}}{(2k+1)!!} \operatorname{Tr} Y^{-(2k+1)}$$

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The Riemann-Hilbert problem

Question

Z_n is genuinely analytic for $\operatorname{Re} y_k > 0$. Does F represent an asymptotic expansion?

Answer: consider RHP in the λ-plane

$$\begin{cases} \Gamma_+^{(n)} = \Gamma_-^{(n)} M_j \\ \Gamma^{(n)}(\lambda) \sim \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1}+i\sigma_1}{\sqrt{2}} (\mathbf{1} + \mathcal{O}(\lambda^{-\frac{1}{2}})) \quad \lambda \to \infty \end{cases}$$

$$M_j := D_-^{-1} e^{-\theta_-} S_j e^{\theta_+} D_+$$

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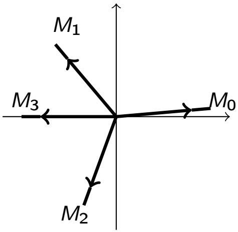

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Kontsevich matrix integral as isomonodromic tau function

The jumps of $\Psi_n := \Gamma_n \mathrm{e}^{-\theta} D^{-1}$ do not depend on $\lambda, \lambda_1, \dots, \lambda_n \Rightarrow$ isomonodromy equations

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$\Rightarrow$ isomonodromic tau function $\tau_n(\lambda_1, \dots, \lambda_n)$ (M. Jimbo, T. Miwa and K. Ueno, 1981)

$$\frac{\partial}{\partial \lambda_j} \log \tau_n(\lambda_1, \dots, \lambda_n) = \mathrm{res}_{\lambda = \lambda_j} \mathrm{d}\lambda \ \mathrm{Tr}\, A_n^2(\lambda; \lambda_1, \dots, \lambda_n)$$

Theorem (M. Bertola - M. Cafasso, 2016)

$\tau_n(\lambda_1, \dots, \lambda_n) = Z_n(Y)$, $\lambda_j = y_j^2$.

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Formulae for closed intersection numbers

Theorem (M. Bertola - B. Dubrovin - D. Yang, 2015)

Let

$$\Theta(\lambda) := \begin{bmatrix} -\frac{1}{2}\sum_{g\geq 1}\frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-6g+4} & -\sum_{g\geq 0}\frac{(6g-1)!!}{24^g g!}\lambda^{-6g} \\ \sum_{g\geq 0}\frac{6g+1}{6g-1}\frac{(6g-1)!!}{24^g g!}\lambda^{-6g+2} & \frac{1}{2}\sum_{g\geq 1}\frac{(6g-5)!!}{24^{g-1}(g-1)!}\lambda^{-6g+4} \end{bmatrix}$$

$$F_n(\lambda_1,\dots,\lambda_n) := \sum_{k_1,\dots,k_n=0}^{\infty}\left\langle \prod_{j=1}^{n}\frac{(2k_j+1)!!}{\lambda_j^{2k_j+1}}\tau_{k_j}\right\rangle$$

Then

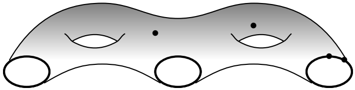
$$F_1(\lambda) = \sum_{g=1}^{\infty}\frac{(6g-3)!!}{24^g g!}\lambda^{-6g+2}$$

$$F_n(\lambda_1,\dots,\lambda_n) = -\frac{1}{n}\sum_{\sigma\in S_n}\frac{\operatorname{Tr}\left(\Theta\left(\lambda_{\sigma(1)}\right)\cdots\Theta\left(\lambda_{\sigma(n)}\right)\right)}{\prod_{j\in\mathbb{Z}/n\mathbb{Z}}\left(\lambda_{\sigma(j)}^2-\lambda_{\sigma(j+1)}^2\right)} - \delta_{n,2}\frac{\lambda_1^2+\lambda_2^2}{\left(\lambda_1^2-\lambda_2^2\right)^2} \qquad n\geq 2$$

Moduli spaces of open Riemann surfaces

$\overline{\mathcal{M}_{g,k,l}}$=moduli spaces of *open* (i.e.*with boundary*) Riemann surfaces (g =doubled genus, $k = \sharp$ bdry markings, $l = \sharp$ int. markings).

Rigorous study initiated by Pandharipande, Solomon and Tessler, 2015.



Main challenges: $\overline{\mathcal{M}_{g,k,l}}$ is a real orbifold with real boundary, possibly nonorientable $\Rightarrow$ difficulties in the definition of intersection numbers.

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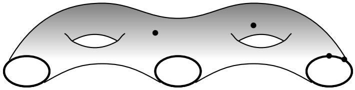

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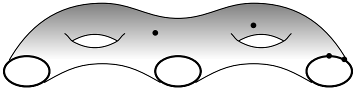

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Open intersection numbers

Write $g = 2h + b - 1$, $h = \sharp$handles, $b = \sharp$bdry components, $\overline{\mathscr{M}_{h,b,k,l}}$ =submoduli of $\overline{\mathscr{M}_{g,k,l}}$ with fixed h, b.

(Refined) open intersection numbers:

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($\neq 0$ only when $2\sum \alpha_i + 2\sum \beta_i = 3g - 3 + k + 2l = 6h - 6 + 3b + k + 2l$).

(Rigorous definitions of the above are yet to come).

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$$F^o(t_0, t_1, \ldots; s_0, s_1, \ldots; N) := \sum_{b \geq 0} N^b \left\langle \exp \sum_{j \geq 0} (t_j \tau_j + s_j \sigma_j) \right\rangle_b$$

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Kontsevich-Penner integral and open intersection numbers

$$\widetilde{Z}(\Lambda; N) := \frac{\int_{H_n} \mathrm{d}\widetilde{M}\, \det\left(\mathbf{1} + \frac{\widetilde{M}}{\Lambda}\right)^{-N} \exp \mathrm{Tr}\left(-\frac{\widetilde{M}^3}{6} - \frac{\Lambda \widetilde{M}}{2}\right)}{\int_{H_n} \mathrm{d}\widetilde{M}\, \exp \mathrm{Tr}\left(-\frac{\Lambda \widetilde{M}}{2}\right)}$$

Conjecture (A. Alexandrov - B. Safnuk - ...)

Identify $t_k = (2k-1)!!\,\mathrm{Tr}\,\Lambda^{-2k-1}$, $s_k = 2^k k!\,\mathrm{Tr}\,\Lambda^{-2k-2}$. Then $F^o(t_0, t_1, ...; s_0, s_1, ...; N)$ gives the expansion of $\widetilde{Z}(\Lambda; N)$ in Miwa variables, for large Λ and $n \to \infty$.

- $\mathbf{N} = \mathbf{1} \Rightarrow$ conjecture is true (A. Alexandrov, 2015), i.e.

  $\left\langle \exp \sum_{j \geq 0} (t_j \tau_j + s_j \sigma_j) \right\rangle$ gives the expansion of $Z_n(\Lambda; 1)$.
- $\mathbf{N} = \mathbf{0} \Rightarrow$ closed case.

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$$\widetilde{Z}(\Lambda; N) := \frac{\int_{H_n} \mathrm{d}\widetilde{M} \det\left(\mathbf{1} + \frac{\widetilde{M}}{\Lambda}\right)^{-N} \exp \operatorname{Tr}\left(-\frac{\widetilde{M}^3}{6} - \frac{\Lambda \widetilde{M}}{2}\right)}{\int_{H_n} \mathrm{d}\widetilde{M} \exp \operatorname{Tr}\left(-\frac{\Lambda \widetilde{M}}{2}\right)}$$

Conjecture (A. Alexandrov - B. Safnuk - ...)

Identify $t_k = (2k-1)!! \operatorname{Tr} \Lambda^{-2k-1}$, $s_k = 2^k k! \operatorname{Tr} \Lambda^{-2k-2}$. Then $F^o(t_0, t_1, ...; s_0, s_1, ...; N)$ gives the expansion of $\widetilde{Z}(\Lambda; N)$ in Miwa variables, for large Λ and $n \to \infty$.

- $\mathbf{N = 1} \Rightarrow$ conjecture is true (A. Alexandrov, 2015), i.e.
  $$\left\langle \exp \sum_{j \geq 0} (t_j \tau_j + s_j \sigma_j) \right\rangle$$ gives the expansion of $Z_n(\Lambda; 1)$.
- $\mathbf{N = 0} \Rightarrow$ closed case.

Kontsevich-Penner integral as isomonodromic tau function

Consider the Kontsevich-Penner integral normalized as

$$Z_n(Y;N) := \frac{\det(\mathrm{i}Y)^N \int_{H_n} \mathrm{d}M \, \exp \mathrm{Tr}\left(\frac{\mathrm{i}}{3}M^3 - YM^2 - N\log(M+\mathrm{i}Y)\right)}{\int_{H_n} \mathrm{d}M \, \exp \mathrm{Tr}(-YM^2)}$$

Theorem (M. Bertola - G. R., 2017)

$Z_n(Y;N)$ is the isomonodromic tau function $\tau_n(Y;N)$ of a suitable $(N+2)\times(N+2)$ Riemann-Hilbert problem depending on parameters $y_1, \ldots, y_n$ and N.

This Riemann-Hilbert problem is obtained by a sequence of *Schlesinger transformations* of the ODE

$$\partial^N(\partial^2 - \lambda)\Psi = 0$$

Limiting isomonodromic problem

We can single out a 3×3 limiting isomonodromic problem ($= n \to \infty$), depending on N (not necessarily nonnegative integer) as well as on isomonodromic times $t_1, t_2, t_3, \ldots$.

$$t_k = t_k(\lambda_1, \lambda_2, \ldots) = \frac{1}{k} \sum_{j \geq 1} \lambda_j^{-\frac{k}{2}} \qquad (\textit{Miwa variables})$$

The Ψ–function has the expansion

$$\Psi(\lambda; t_1, t_2, \ldots; N) = S \cdot G \cdot Y_N(\lambda) \cdot \lambda^{T_0} \cdot \exp \sum_{j \geq 1} \lambda^{\frac{j}{2}} (t_j - \delta_{j,3}) \theta_j$$

$$S := \operatorname{diag}\left(\lambda^{-\frac{1}{2}}, \lambda^{-\frac{1}{2}}, 1\right), \; G := \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \; Y_N(\lambda) = \mathbf{1} + \mathcal{O}\left(\lambda^{-\frac{1}{2}}\right)$$

$$T_0 = \operatorname{diag}\left(\frac{1}{4} + \frac{N}{2}, \frac{1}{4} + \frac{N}{2}, \frac{1}{2} - N\right), \; \theta_j := \operatorname{diag}((-1)^j, 1, 0)$$

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Formulae for open intersection numbers

Applying Bertola-Dubrovin-Yang scheme we can derive explicit and computationally efficient formulae.

E.g. 1-point functions are given by the following polynomials in N

$$\frac{1}{(3k-1)!!}\sum_{k_1+k_2=k}\operatorname*{res}_{\xi=0}\operatorname*{res}_{\eta=0}\left(\frac{1+\xi}{1+\eta}\right)^N\frac{(-1)^{k_1}(2k_1-1)!!(2k_2-1)!!}{\left(\xi^2+\frac{\xi^3}{3}\right)^{k_1+\frac{1}{2}}\left(\eta^2+\frac{\eta^3}{3}\right)^{k_2+\frac{1}{2}}}$$

k even/odd $\iff$ interior/bdry marked point:

$$\langle\tau_1\rangle=\frac{1+12N^2}{24},\ \langle\tau_4\rangle=\frac{1+56N^2+16N^4}{1152},\ \langle\tau_7\rangle=\frac{25+5508N^2+3120N^4+192N^6}{2073600},$$

$$\langle\tau_{10}\rangle=\frac{1225+1030896N^2+848736N^4+102144N^6+2304N^8}{9754214400},\dots$$

$$\langle\tau_{\frac{5}{2}}\rangle=\frac{N+N^3}{12},\ \langle\tau_{\frac{11}{2}}\rangle=\frac{12N+25N^3+3N^5}{2880},\ \langle\tau_{\frac{17}{2}}\rangle=\frac{116N+357N^3+84N^5+3N^7}{725760},$$

$$\langle\tau_{\frac{23}{2}}\rangle=\frac{704N+2764N^3+945N^5+66N^7+N^9}{139345920},\dots$$

THANKS FOR YOUR ATTENTION!!