Poisson cohomology of multidimensional Dubrovin-Novikov Poisson structures and their normal forms.

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with H. Posthuma, S. Shadrin:

1. *Bihamiltonian cohomology of the KdV brackets*, Comm. Math. Phys. (2016)
2. *Bihamiltonian cohomology of scalar Poisson...*, Bull. Lond. Math. Soc. (2016)
3. *Deformations of semisimple Poisson brackets...*, J. Diff. Geom. (2017)

with R. Kramer, S. Shadrin:

4. *Central invariants revisited*, preprint (2016).

with M. Casati, S. Shadrin:

5. *Poisson cohomology of scalar multidimensional...*, J. Geom. Phys. (2017)
6. *Normal forms of dispersive scalar Poisson brackets with two...*, preprint (2017)

I. Deformations of Poisson and bi-Hamiltonian structures

II. $D \geqslant 1$ independent variables: Poisson cohomology

III. $D = 2$ independent variables: classification of Poisson brackets

I. Deformations of Poisson and bi-Hamiltonian structures

The Korteweg - de Vries equation

$$u_t = uu_x + \epsilon^2 u_{xxx}$$

has bi-Hamiltonian formulation

$$u_t(x) = \{u(x), H_1\}_1 = \{u(x), H_0\}_2$$

with respect to two compatible Poisson brackets

$$\{u(x), u(y)\}_1 = \delta'(x-y),$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2}u'(x)\delta(x-y) + \frac{3}{2}\epsilon^2\delta'''(x-y).$$

[Gardner-Zakharov-Faddeev '71, Magri '78]

General problem:
classify dispersive Poisson (or bi-Hamiltonian) structures

$$\{u(x), u(y)\} = \{u(x), u(y)\}^0 + \\ + \sum_{m \geqslant 1} \epsilon^m \sum_{l=0}^{m+1} A_{m,l}(u; u_x, \dots)\delta^{(l)}(x-y)$$

under the action of Miura type transformations

$$u(x) \to u(x) + \epsilon a_1(u; u_x) + \epsilon^2 a_2(u; u_x, u_{xx}) + \dots$$

where $A_{m,l}$, a_i are differential polynomials, and $\{,\}^0$ is of Dubrovin-Novikov (or hydrodynamic) type.

[Dubrovin-Zhang'01]

A Poisson bracket of Dubrovin-Novikov (or hydrodynamic) type is of the form

$$\{u^i(x), u^j(y)\}^0 = g^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u(x))u^k_x(x)\delta(x-y).$$

It is a Poisson structure iff

g^{ij} flat contravariant metric,

Γ^{ij}_k Christoffel symbols of g^{ij}.

[Dubrovin-Novikov'83]

In finite dimensions: the space Λ^* of multivectors on a manifold M is endowed with the Schouten-Nijenhuis bracket

$$[,]:\Lambda^p\times\Lambda^q\to\Lambda^{p+q-1}.$$

On a formal loop space $\mathcal{L}M=\{S^1\to M\}$: one considers the space Λ^*_{loc} of local multivectors of the form (for $M=\mathbb{R}$)

$$\sum_{p_2\cdots p_k\geqslant 0} B_{p_2\dots p_k}(u(x);u_x(x),u_{xx}(x),\dots)\delta^{(p_2)}(x-x_2)\cdots\delta^{(p_k)}(x-x_k)$$

which is closed under a suitably defined Schouten-Nijenhuis bracket

$$[,]:\Lambda^p_{loc}\times\Lambda^q_{loc}\to\Lambda^{p+q-1}_{loc}.$$

Deformations of a single Poisson structure:

Let $P \in \Lambda^2_{loc}$ Poisson of DN type, $[P, P] = 0$.

The Poisson cohomology of P is $H(\Lambda_{loc}, ad_P)$.

Theorem: $H(\Lambda_{loc}, ad_P)$ *is trivial.*

[Dubrovin-Zhang'01, Getzler'00, Degiovanni-Magri-Sciacca'01, Liu-Zhang'09]

$\Rightarrow$ All deformations are trivial.

Remark: Not true for $D > 1$ independent variables.

[C, Casati, Shadrin '15]

Deformations of bi-Hamiltonian structures:

The deformations of a bi-Hamiltonian structure P_1, P_2 of DN type are described by bihamiltonian cohomology

$$BH(\Lambda_{loc}, d_1, d_2) = \frac{\operatorname{Ker} d_1 \cap \operatorname{Ker} d_2}{\operatorname{Im} d_1 d_2}$$

where $d_i = [P_i, \cdot]$.

Infinitesimal deformations ($O(\epsilon^3)$) are classified by $BH^2(\Lambda_{loc})$, i.e., by n functions of a single variable, the central invariants

$$c_i(u) = \frac{1}{3(f^i(u))^2}\left(A^{ii}_{2,3;2} - u^i A^{ii}_{2,3;1} + \sum_{k\neq i} \frac{(A^{ij}_{1,2;2} - u^i A^{ij}_{1,2;1})^2}{f^k(u)(u^k - u^i)}\right).$$

[Liu-Zhang'05, Dubrovin-Liu-Zhang'06]

The problem of existence of deformations:

Given an infinitesimal deformation of a Poisson pencil of DN type, is it possible to extend it to a full dispersive Poisson pencil ?

Theorem

The deformations of any semisimple Poisson pencil of DN type are unobstructed.

[C-Posthuma-Shadrin'15]

Sufficient to show that $BH^3_{\geqslant 5}(\Lambda_{loc}, d_1, d_2)$ vanishes.

Using the methods of homological algebra, in particular the spectral sequences, we have obtained the following results:

1. full bi-Hamiltonian cohomology of KdV. [C-Posthuma-Shadrin'14]

2. full bi-Hamiltonian cohomology of a general scalar bi-Hamiltonian structure. [C-Posthuma-Shadrin'15a]

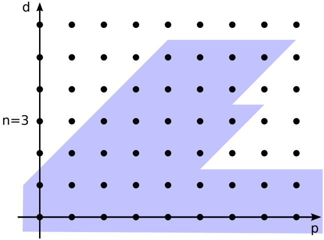
3. Theorem: *For a semi-simple bi-Hamiltonian structure of DN type with n dependent variables, the bi-Hamiltonian cohomology $BH^p_d(\Lambda_{loc}, d_1, d_2)$ vanishes for all degrees (p, d), but for a finite number.*

[C-Posthuma-Shadrin'15b]

For example, in the $n = 3$ case, we claim the bihamiltonian cohomology

$$BH_d^p(\Lambda_{loc}, d_1, d_2)$$

vanishes in all bi-degrees but those highlighted.



In particular, this implies the vanishing of $BH^3_{\geqslant 5}(\Lambda_{loc})$ which in turn implies the vanishing of the obstructions.

It is convenient to use the supervariables formalism.

[Liu-Zhang'12]

Consider the space of formal power series

$$\hat{\mathcal{A}} := C^\infty(\mathbb{R})[[u^1, u^2, \dots; \theta, \theta^1, \dots]]$$

$$f(u; u^1, u^2, \dots; \theta, \theta^1, \dots) \in \hat{\mathcal{A}}$$

in the commuting variables $u^1, u^2, \dots$ and in the anticommuting variables $\theta, \theta^1, \theta^2, \dots$.

- x-derivative: $\partial = \sum_{s \geqslant 0} \left(u^{s+1} \frac{\partial}{\partial u^s} + \theta^{s+1} \frac{\partial}{\partial \theta^s} \right) : \hat{\mathcal{A}} \to \hat{\mathcal{A}}$
- two gradations:

$$\hat{\mathcal{A}}^p_d = \text{homogeneous component with degree } \begin{cases} p \text{ in } \theta, \theta^1, \dots \\ d \text{ in } x\text{-derivatives.} \end{cases}$$

Let $\hat{\mathcal{F}} := \frac{\hat{\mathcal{A}}}{\partial \hat{\mathcal{A}}}$ and denote the projection map $\int : \hat{\mathcal{A}} \to \hat{\mathcal{F}}$.

$$\Lambda^p_{loc} \cong \hat{\mathcal{F}}^p$$

The Schouten-Nijenhuis bracket is

$$[,] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1}$$

$$[P, Q] = \int \left(\delta^\bullet P \delta_\bullet Q + (-1)^p \delta_\bullet P \delta^\bullet Q\right)$$

$$\delta^\bullet = \sum_{s \geqslant 0} (-\partial)^s \frac{\partial}{\partial \theta^s}, \quad \delta_\bullet = \sum_{s \geqslant 0} (-\partial)^s \frac{\partial}{\partial u^s}$$

It is convenient to work in $\hat{\mathcal{A}}$ rather than in $\hat{\mathcal{F}}$.

For any $P \in \hat{\mathcal{F}}^2$, let $d_P = [P, \cdot]$, there exists a map D_P s.t. the diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{A}} & \xrightarrow{D_P} & \hat{\mathcal{A}} \\ \Big\downarrow \int & & \Big\downarrow \int \\ \hat{\mathcal{F}} & \xrightarrow{d_P} & \hat{\mathcal{F}} \end{array}$$

which is given by

$$D_P = \sum_{s \geqslant 0} \left(\partial^s(\delta^\bullet P) \frac{\partial}{\partial u^s} + \partial^s(\delta_\bullet P) \frac{\partial}{\partial \theta^s} \right).$$

The short exact sequence of complexes above gives rise to a long exact sequence in cohomology that allow to recover the cohomology of $\hat{\mathcal{F}}$ from the cohomology of $\hat{\mathcal{A}}$.

In this formalism the proof of triviality theorem becomes very simple!

The differential on $\hat{\mathcal{A}}$ is simply a de Rham operator

$$D_P = \sum_{s \geqslant 0} \theta^{s+1} \frac{\partial}{\partial u^s},$$

therefore the Poincaré lemma follows by standard methods, i.e.,

$$H^{>0}(\hat{\mathcal{A}}, D_P) = 0.$$

Then the short exact sequence

$$0 \to \hat{\mathcal{A}}/\mathbb{R} \to \hat{\mathcal{A}} \to \hat{\mathcal{F}} \to 0$$

in cohomology allows to conclude.

In the bi-Hamiltonian case this allows to reduce to the computations of the standard cohomology of the differential complex

$$(\hat{\mathcal{A}}[\lambda], D_2 - \lambda D_1).$$

We can than use extensively the techniques of spectral sequences.

II. Poisson cohomology for $D \geqslant 1$ independent variables

Multidimensional Poisson brackets of Dubrovin-Novikov type in

N dependent variables: $u = (u^1, \dots, u^N)$
D independent variables: $x = (x^1, \dots, x^D)$

are given by:

$$\{u^i(x), u^j(y)\}^0 = \sum_{\alpha=1}^{D} \big(g^{ij\alpha}(u(x))\partial_{x^\alpha}\delta(x-y) + \\ + b_k^{ij\alpha}(u(x))\partial_{x^\alpha}u^k(x)\delta(x-y)\big).$$

[Dubrovin-Novikov ’83-’84, Mokhov ’88-’08, Ferapontov-Lorenzoni-Savoldi ’15]

We consider dispersive deformations of multidimensional DN brackets of the form

$$\{u^i(x), u^j(y)\} = \{u^i(x), u^j(y)\}^0 + \\ + \sum_{k>0} \epsilon^k \sum_{\substack{k_1,\dots,k_D \geqslant 0 \\ k_1+\dots+k_D \leqslant k+1}} A^{ij}_{k;k_1,\dots,k_D}(u(x)) \partial^{k_1}_{x^1} \cdots \partial^{k_D}_{x^D} \delta(x-y)$$

where $A^{ij}_{k;k_1,\dots,k_D} \in \mathcal{A}$ and $\deg A^{ij}_{k;k_1,\dots,k_D} = k - k_1 \cdots - k_D + 1$.

We consider the the scalar $N = 1$ case

$$\{u(x), u(y)\}^0 = g(u(x))c^\alpha \frac{\partial}{\partial x^\alpha}\delta(x-y) + \frac{1}{2}g'(u(x))c^\alpha \frac{\partial u}{\partial x^\alpha}(x)\delta(x-y)$$

which in flat coordinates reduces to

$$\{u(x), u(y)\}^0 = \sum_{\alpha=1}^{D} c^\alpha \frac{\partial}{\partial x^\alpha}\delta(x-y).$$

Deformation theory is governed by Poisson cohomology groups $H^p(\{,\}^0)$ associated with the Poisson bracket $\{u(x), u(y)\}^0$.

Infinitesimal deformations $\longrightarrow H^2(\{,\}^0)$

Obstructions $\longrightarrow H^3(\{,\}^0)$

Define the ring of polynomials in the anticommuting variables θ^S

$$\Theta = \mathbb{R}[\{\theta^{(s_1,\dots,s_{D-1})}, s_i \geqslant 0\}]$$

and the auxiliary space:

$$H(D) = \frac{\Theta}{\partial_{x_1}\Theta + \cdots + \partial_{x_{D-1}}\Theta}.$$

Theorem

The Poisson cohomology $H^p(\{,\}^0)$ is isomorphic to

$$H^p(\{,\}^0) \simeq H^p(D) \oplus H^{p+1}(D).$$

[C, Casati, Shadrin '15]

For $D = 1$ we recover scalar case of triviality theorem.

For $D = 2$ we have a closed formula for the dimension of $H^p_d(2)$:

d	0	1	2	3	4	5	6	7	8
$\dim H^2_d(\hat{\mathcal{F}})$	0	1	0	2	0	2	1	2	1
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	0	1	0	1	2	1	2

For $D \geqslant 2$ we expect the Poisson cohomology in $p = 2, 3$ to be highly non-trivial.

$D = 3:$

d	0	1	2	3	4	5	6	7	8
$\dim H^2_d(\hat{\mathcal{F}})$	0	2	1	8	3	16	13	26	26
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	1	4	6	14	29	36	72

$D = 4:$

d	0	1	2	3	4	5	6
$\dim H^2_d(\hat{\mathcal{F}})$	0	3	3	20	15	66	73
$\dim H^3_d(\hat{\mathcal{F}})$	0	0	3	11	30	75	183

The situation in $D > 1$ looks much more complicated:

- No triviality theorem,
- Many infinitesimal deformations, also non-homogeneous,
- A priori non-vanishing obstructions.

Deformation theory is non-empty: we find examples of nontrivial deformations of degree 2 for each $D > 2$.

Sketch of proof

1. The Poisson cohomology groups are invariant (up to isomorphism) under linear changes of the independent variables.
2. We can put the Poisson bracket in the special form

$$\{u(x), u(y)\} = \partial_{x^D} \delta(x - y).$$

3. We show that the following sequences are exact:

$$\begin{array}{ccccccccc}
0 & \rightarrow & \hat{\mathcal{A}}/\mathbb{R} & \xrightarrow{\partial_{x^1}} & \hat{\mathcal{A}} & \xrightarrow{\int dx^1} & \hat{\mathcal{F}}_1 & \rightarrow & 0 \\
0 & \rightarrow & \hat{\mathcal{F}}_1/\mathbb{R} & \xrightarrow{\partial_{x^2}} & \hat{\mathcal{F}}_1 & \xrightarrow{\int dx^2} & \hat{\mathcal{F}}_2 & \rightarrow & 0 \\
0 & \rightarrow & \hat{\mathcal{F}}_2/\mathbb{R} & \xrightarrow{\partial_{x^3}} & \hat{\mathcal{F}}_2 & \xrightarrow{\int dx^3} & \hat{\mathcal{F}}_3 & \rightarrow & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \rightarrow & \hat{\mathcal{F}}_{D-1}/\mathbb{R} & \xrightarrow{\partial_{x^D}} & \hat{\mathcal{F}}_{D-1} & \xrightarrow{\int dx^D} & \hat{\mathcal{F}}_D & \rightarrow & 0
\end{array}$$

where

$$\hat{\mathcal{F}}_i = \frac{\hat{\mathcal{A}}}{\partial_{x^1}\hat{\mathcal{A}} + \cdots + \partial_{x^i}\hat{\mathcal{A}}}.$$

4. The differential associated to the Poisson bracket in special form

$$\Delta = \sum_S \theta^{S+\xi_D} \frac{\partial}{\partial u^S},$$

commutes with all the maps, therefore induces exact sequences of complexes.

5. The corresponding long exact sequences in cohomology allow us to compute inductively:

$$H(\hat{\mathcal{F}}_i) = \frac{\Theta}{\partial_{x^1}\Theta + \cdots + \partial_{x^i}\Theta},$$

for $i = 1, \ldots, D-1$.

6. The long exact sequence associated to the last line allows us to conclude.

III. Classification of scalar dispersive Poisson structures in $D = 2$

We classify the dispersive Poisson brackets with one dependent variable u and two independent variables x^1, x^2 of the form

$$\{u(x^1,x^2),u(y^1,y^2)\}^\epsilon = \{u(x^1,x^2),u(y^1,y^2)\}^0 + \\ + \sum_{k>0} \epsilon^k \sum_{\substack{k_1,k_2 \geqslant 0 \\ k_1+k_2 \leqslant k+1}} A_{k;k_1,k_2}(u(x)) \delta^{(k_1)}(x^1-y^1) \delta^{(k_2)}(x^2-y^2)$$

where $A_{k;k_1,k_2} \in \mathcal{A}$ and $\deg A_{k;k_1,k_2} = k - k_1 - k_2 + 1$.

In flat coordinates u and by performing a linear change of the independent coordinates the leading term can be assumed of the form

$$\{u(x^1, x^2), u(y^1, y^2)\}^0 = \delta(x^1 - y^1)\delta^{(1)}(x^2 - y^2).$$

Theorem: *The normal form of Poisson brackets* $\{,\}^\epsilon$ *under Miura transformations is*

$$\{u(x^1,x^2),u(y^1,y^2)\}^{(c)} = \delta(x^1-y^1)\delta^{(1)}(x^2-y^2)+ \\ +\sum_{k\geqslant 1}\epsilon^{2k+1}c_k\delta^{(2k+1)}(x^1-y^1)\delta(x^2-y^2)$$

for a sequence of constants $c=(c_1,c_2,\dots)$.

[C.-Casati-Shadrin '17]

By normal form we mean:

i. for any choice of constants c_k, $\{,\}^{(c)}$ defines a Poisson bracket;

ii. two Poisson brackets of the form $\{,\}^{(c)}$ are Miura equivalent if and only if they are defined by the same constants c_k;

iii. and any Poisson bracket of the form $\{,\}^{\epsilon}$ can be brought to the normal form $\{,\}^{(c)}$ by a Miura transformation.

In $D = 2$ we have the short exact sequences of differential complexes

$$0 \to \hat{\mathcal{A}}/\mathbb{R} \xrightarrow{\partial_x} \hat{\mathcal{A}} \xrightarrow{\int dx} \hat{\mathcal{F}}_1 \to 0,$$

$$0 \to \hat{\mathcal{F}}_1/\mathbb{R} \xrightarrow{\partial_y} \hat{\mathcal{F}}_1 \xrightarrow{\int dy} \hat{\mathcal{F}} \to 0,$$

where the differential is induced an all spaces by

$$\Delta = \sum_{s,t \geqslant 0} \theta^{(s,t+1)} \frac{\partial}{\partial u^{(s,t)}}.$$

On $\hat{\mathcal{F}}$ such differential Δ coincides with $ad_{\{,\}^0}$.

The first short exact sequence induces a long exact sequence in cohomology that gives

$$H(\hat{\mathcal{F}}_1) = \frac{\Theta}{\partial_x \Theta}.$$

The map induced in cohomology by the map ∂_y in the second short exact sequence vanishes, therefore we get the following exact sequence

$$0 \to \left(\frac{\Theta}{\partial_x \Theta}\right)^p_d \xrightarrow{\int dy} H^p_d(\hat{\mathcal{F}}) \to \left(\frac{\Theta}{\partial_x \Theta}\right)^{p+1}_d \to 0,$$

where the third arrow is the Bockstein homomorphism.

We have a splitting map

$$\mathcal{B} : \left(\frac{\Theta}{\partial_x \Theta} \right)_d^{p+1} \to H_d^p(\hat{\mathcal{F}})$$

given by

$$\mathcal{B} = \sum_{i \geqslant 0} u^{(i,0)} \frac{\partial}{\partial \theta^{(i,0)}}.$$

We have therefore an explicit description of the cohomology classes

$$H^p_d(\hat{\mathcal{F}}) = \left(\frac{\Theta}{\partial_x \Theta}\right)^p_d \oplus \mathcal{B}\left(\frac{\Theta}{\partial_x \Theta}\right)^{p+1}_d .$$

In the proof we show that the classes coming from the map $\mathcal{B}$ can never contribute to the Poisson structure.

Thanks for your attention