Some analytical aspects of the Kontsevich matrix model

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Plan of the talk

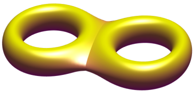
- The Witten's conjecture and the Witten–Kontsevich tau function.
- The Painlevé I hierarchy and the string equation.
- Kontsevich's model.
- The convergence of the Kontsevich model to (some) solutions of the Painlevé I hierarchy.

Collaboration with M. Bertola, arXiv : 1603.06420 (Comm. in Math. Phys. , 2017).

The Deligne-Mumford moduli space of Riemann surfaces

$$\overline{\mathcal{M}}_{g,n} := \left\{ \text{Riemann surfaces with } n \text{ marked points} \right\} \Big/ \sim$$

A point in $\overline{\mathcal{M}}_{g,n}$ is a (possibly singular) Riemann surface with n marked points (modulo isomorphisms).



$\overline{\mathcal{M}}_{g,n}$ is a complex orbifold of dimension $3g-3+n$. We denote with $\mathcal{L}_j$ the tautological line bundle, their fibers over $[\mathcal{C}]$ are given by $T^{\star}_{p_j}\mathcal{C}$, ψ_j will denote the corresponding Chern classes.

Intersection numbers

Intersection numbers are given by the integrals

$$\left\langle \tau_0^{k_0} \tau_1^{k_1} \dots \right\rangle_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\ell_1} \wedge \dots \wedge \psi_n^{\ell_n},$$

where $k_j = \sharp$ occurences of j as an exponent.

Example :

$$< \tau_0^2 \tau_1^3 \tau_2 >= \int \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4^2 \wedge \psi_5^0 \wedge \psi_6^0.$$

The numbers k_i satisfy

$$3g - 3 + n = \sum_{k=0}^{\infty} j k_j,$$

$$n = \sum k_j.$$

Let's define

$$F(T_0, T_1, \dots,) := \sum \langle \tau_0^{k_0} \tau_1^{k_1} \dots \tau_\ell^{k_\ell} \dots \rangle \prod \frac{T_j^{k_j}}{k_j!}$$

Witten' conjecture (Kontsevich theorem) :

Let $\tilde{R}_n[U]$ be the Lenard polynomials defined by the recursion

$$\tilde{R}_0[U] = U, \quad \frac{\partial \tilde{R}_{n+1}}{\partial T_0} = \frac{1}{2n+1}\Big(\frac{\partial U}{\partial T_0} + 2U\frac{\partial}{\partial T_0} + \frac{1}{4}\frac{\partial^3}{\partial T_0^3}\Big)\tilde{R}_n.$$

Theorem : The formal series

$$F(T_0, T_1, \ldots,) := \sum \langle \tau_0^{k_0}\tau_1^{k_1}\ldots\tau_\ell^{k_\ell}\ldots\rangle \prod \frac{T_j^{k_j}}{k_j!}$$

is uniquely determined by the following conditions :

1) $U := \frac{\partial^2 F}{\partial T_0^2}$ is a solution of the Korteweg de-Vries hierarchy

$$\frac{\partial U}{\partial T_i} = \frac{\partial}{\partial T_0}\tilde{R}_i[U], \; i \geq 0.$$

2) F satisfies the string equation

$$\frac{\partial F}{\partial T_0} = \sum_{i\geq 0} T_{i+1}\frac{\partial F}{\partial T_i} + \frac{T_0^2}{2}.$$

In other words, $e^F = \tau$ is a tau function for the KdV hierarchy, uniquely determined by the Virasoro constraints or -equivalently- by its initial value $\ln\tau = \frac{T_0^3}{6}$.

String equation and the Painlevé I hierarchy

$$\begin{cases} \dfrac{\partial^2 F}{\partial T_0^2} = \sum_{i\geq 0} T_{i+1} \dfrac{\partial^2 F}{\partial T_i \partial T_0} + T_0 \\ \dfrac{\partial^2 F}{\partial T_0 \partial T_i} = \tilde{R}_i \left[\dfrac{\partial^2 F}{\partial T_0^2} \right] \end{cases} \Longrightarrow U - T_0 = \sum_{i\geq 0} T_{i+1} \tilde{R}_{i+1}[U].$$

Putting $T_i = 0$ for all $i \neq 0, N$ we get the collection of equations

$$T_N \tilde{R}_N[U] = U - T_0, \quad N \geq 1$$

known as Painlevé I hierarchy.

Remark :

The same equations can be written as

$$\left[L, M\right] = 1, \quad L := \frac{\partial^2}{\partial T_0^2} - U.$$

(Douglas, "String in less than one dimension").

The Kontsevich matrix model

$$Z_n(x;Y) := \frac{\int_{H_n} dM e^{\mathrm{Tr}\left(i\frac{M^3}{3} - YM^2 + ixM\right)}}{\int_{H_n} dM e^{-\mathrm{Tr}\left(YM^2\right)}},$$

(Matrix Airy function)

$$H_n := \left\{M = M^\dagger \in Mat_{n\times n}(\mathbb{C})\right\} \qquad Y := \mathrm{diag}(y_1,\ldots,y_n)$$

(x is added for later convenience).

Theorem (Kontsevich, 1992) :

When $n \to \infty$, the following formal identity holds

$$F(\vec{T}) = \lim_{n\to\infty} \ln Z_n(0;Y), \quad \text{i.e.} \quad \tau_{WK}(\vec{T}) = \lim_{n\to\infty} Z_n(0;Y),$$

under the identification (Miwa's variables)

$$T_j = T_j(Y) := -(2j-1)!! \sum_{\ell=1}^{n} \frac{1}{y_\ell^{2j+1}},$$

for $|Y| \to \infty$. □

A natural question

How do we choose Y in such a way that $Z_n(0;Y)$ converges to a solution of the PI hierarchy ? What are the properties of such solutions ?

The main idea

It's easy to prove that $Z_n(x;Y)$ can be written as a "wronskian" - type determinant

$$Z_n(x;Y) = 2^n \pi^{\frac{n}{2}} \mathrm{e}^{\frac{2}{3}\mathrm{Tr}Y^3 + x\mathrm{Tr}Y} \frac{\det\left[\mathrm{Ai}^{(j-1)}(y_k^2 + x)\right]_{k,j\le n} \prod_{j=1}^{n}(y_j)^{\frac{1}{2}}}{\prod_{j<k}(y_j - y_k)} \quad \mathrm{Re}y_j > 0,$$

and this suggest a link with Darboux transformations...

Let's consider the system

$$\begin{cases} \partial_\lambda \Psi_0(x;\lambda) = \begin{pmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{pmatrix} \Psi_0(x;\lambda), \\ \partial_x \Psi_0(x;\lambda) = \begin{pmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{pmatrix} \Psi_0(x;\lambda), \end{cases}$$

and let's add poles on the points $\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_k = y_k^2$ to get the new system

The main idea II

$$\begin{cases} \partial_\lambda \Psi_n = A\Psi_n, \quad A = i\sigma_+ - i\left(\lambda + \frac{x}{2} - \partial_x a^{(n)}\right)\sigma_- - \sum_{j=1}^{n} \frac{A_j}{\lambda - \lambda_j}, \\ \partial_x \Psi_n = U\Psi_n, \quad U = i\sigma_+ - i\left(\lambda - 2\partial_x a^{(n)}\right)\sigma_-, \\ \partial_{\lambda_k}\Psi_n = -\frac{A_k}{\lambda - \lambda_k}\Psi_n, \quad k = 1, \dots, n. \end{cases} \tag{1}$$

The isomonodromic (Jimbo-Miwa-Ueno) tau function associated to the system above is defined by the equations

$$\partial_{\lambda_k} \ln \tau_n = \mathrm{res}_{\lambda_k} \mathrm{Tr} A^2 d\lambda, \quad \partial_x \ln \tau_n = a^{(n)}$$

and we will prove that

$$\tau_n(x, \{\lambda_k\}) = \mathrm{e}^{\frac{x^3}{12}} Z_n(x, Y).$$

Once this equality is established, one can study the large n limit of the Riemann–Hilbert problem associated to the system (1)...

An extension of the Kontsevich matrix model I :

Remark :

$$\mathrm{Ai}(\lambda) = \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\sqrt{\pi}\lambda^{\frac{1}{4}}}\left(1 + \mathcal{O}(\lambda^{-\frac{3}{2}})\right), \quad \lambda \to \infty,$$

$$\Longrightarrow e^{\frac{2}{3}y^3+xy}\mathrm{Ai}(y^2+x) = \begin{cases} \frac{1}{2\sqrt{\pi}\sqrt{y}}(1+\mathcal{O}(y^{-3})) & \text{for } \mathrm{Re} y > 0, \\ \frac{e^{\frac{4}{3}y^3+2xy}}{2\sqrt{\pi}\sqrt{y}}(1+\mathcal{O}(y^{-3})) & \text{for } \mathrm{Re} y < 0, \end{cases}$$

$$\Longrightarrow Z_n(x;Y) = 2^n\pi^{\frac{n}{2}}e^{\frac{2}{3}\mathrm{Tr}Y^3+x\mathrm{Tr}Y}\frac{\det\left[\mathrm{Ai}^{(j-1)}(y_k^2+x)\right]_{k,j\leq n}\prod_{j=1}^n(y_j)^{\frac{1}{2}}}{\prod_{j<k}(y_j-y_k)},$$

this expression admits a “regular” expansion if $\mathrm{Re}(y_i) > 0$ for all i.

An extension of the Kontsevich matrix model II :

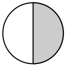
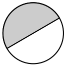
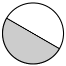
$$Y \longmapsto \mathcal{Y}^{(0)} \sqcup \mathcal{Y}^{(1)} \sqcup \mathcal{Y}^{(2)};$$

$$Z_n(x; \mathcal{Y}^{(0)}, \mathcal{Y}^{(1)}, \mathcal{Y}^{(2)}) = C_n \frac{\mathrm{e}^{\frac{2}{3}\mathrm{Tr}Y^3 + x\mathrm{Tr}Y}}{\prod_{j<k}(y_j - y_k)} \det \begin{bmatrix} \left[\mathbf{Ai}_0^{(k-1)}(y_j^2 + x)\right]_{\substack{y_j \in \mathcal{Y}^{(0)} \\ 1 \leq k \leq n}} \\ \hline \left[\mathbf{Ai}_1^{(k-1)}(y_j^2 + x)\right]_{\substack{y_j \in \mathcal{Y}^{(1)} \\ 1 \leq k \leq n}} \\ \hline \left[\mathbf{Ai}_2^{(k-1)}(y_j^2 + x)\right]_{\substack{y_j \in \mathcal{Y}^{(2)} \\ 1 \leq k \leq n}} \end{bmatrix}.$$

$$\mathbf{Ai}_s(\lambda) := \mathrm{Ai}(\omega^s \lambda)\ , \quad \omega := \mathrm{e}^{\frac{2i\pi}{3}}.$$

This determinant have a regular expansion if

$$\mathcal{Y}^{(a)} \ni y_j \to \infty, \quad y_j \in \mathcal{S}_a.$$

$\mathcal{S}_0$:  ; $\mathcal{S}_1$:  $\mathcal{S}_2$:  .

For what follows let's introduce the parameters $\{\lambda_i, \mu_j\}$ such that $y_i = \sqrt{\lambda_i}$ if $\mathrm{Re}(y_i) > 0$ et $y_j = -\sqrt{\mu_j}$ if $\mathrm{Re}(y_j) \leq 0$.

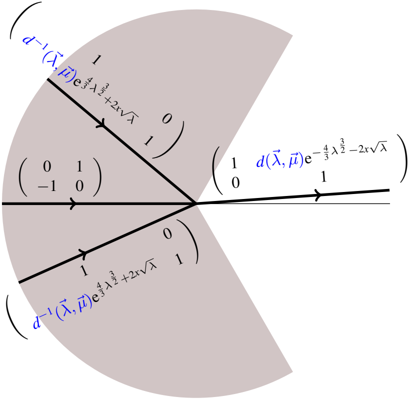
$$d(\vec{\lambda}, \vec{\mu}) := \prod_{j=1} \frac{\sqrt{\mu_j} + \sqrt{\lambda}}{\sqrt{\mu_j} - \sqrt{\lambda}} \prod_{j=1} \frac{\sqrt{\lambda_j} - \sqrt{\lambda}}{\sqrt{\lambda_j} + \sqrt{\lambda}}.$$

A Riemann-Hilbert problem for Z_n

Theorem (M. Bertola, M.C.) :

$Z_n(x; y^{(0)}, y^{(1)}, y^{(2)}) = e^{\frac{x^3}{12}} \tau_n(x, \vec{\lambda}, \vec{\mu})$, τ_n tau function of the Riemann–Hilbert problem with asymptotics

$$\Gamma_n(\lambda) \sim \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left(1 + \frac{a^{(n)}(x; \vec{\lambda}, \vec{\mu})}{\sqrt{\lambda}} \sigma_3 + \mathcal{O}(\lambda^{-1}) \right),$$



The matrix

$$\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) := \Gamma_n e^{(-\frac{2}{3}\lambda^{\frac{3}{2}} - x\sqrt{\lambda})\sigma_3} D^{-1}(\lambda)$$

$$D(\lambda) := \begin{bmatrix} \prod_{j=1}^{n_2}(\sqrt{\lambda_j} + \sqrt{\lambda}) \prod_{j=1}^{n_1}(\sqrt{\mu_j} - \sqrt{\lambda}) & 0 \\ 0 & \prod_{j=1}^{n_2}(\sqrt{\lambda_j} - \sqrt{\lambda}) \prod_{j=1}^{n_1}(\sqrt{\mu_j} + \sqrt{\lambda}) \end{bmatrix}$$

is a solution of the isomonodromic system

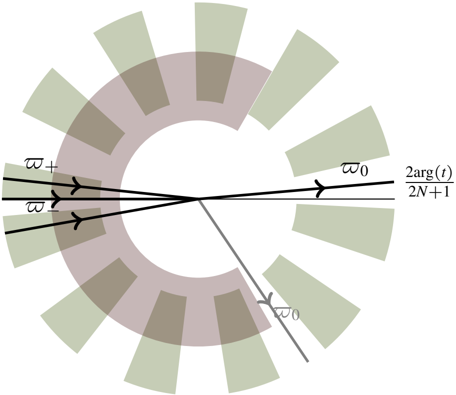
$$\frac{\partial}{\partial \lambda}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = A(\lambda; x, \vec{\lambda}, \vec{\mu})\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu})$$

$$\frac{\partial}{\partial x}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = U(\lambda; x, \vec{\lambda}, \vec{\mu})\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu})$$

$$\frac{\partial}{\partial \lambda_k}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{A_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \lambda_k}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}),$$

$$\frac{\partial}{\partial \mu_k}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{B_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \mu_k}\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}).$$

Riemann-Hilbert for the tronquées solutions of PI_{N-1}



⟵(Example $N = 11$)

$$\vartheta(\lambda; t, x) := t\lambda^{\frac{2N+1}{2}} + \frac{2}{3}\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}},$$

jumps are given by :

$$\mathbf{1} + e^{-2\vartheta(\lambda;t,x)}\sigma_+,\ \lambda \in \varpi_0$$
$$\mathbf{1} + e^{2\vartheta(\lambda;t,x)}\sigma_-,\ \lambda \in \varpi_\pm$$
$$i\sigma_2,\ \lambda \in \mathbb{R}_-.$$

$$\Gamma(\lambda) = \lambda^{-\frac{\sigma_3}{4}} \frac{\mathbf{1} + i\sigma_1}{\sqrt{2}} \left(\mathbf{1} + a\frac{\sigma_3}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1}) \right), \quad \lambda \to \infty.$$

(The matrix $\Psi(\lambda) = \Gamma(\lambda)e^{-\vartheta(\lambda)}$ is the solution of the corresponding RH problem)

Riemann-Hilbert for the tronquées solutions of PI_{N-1}

Let's choose three integers $k_+, k_0, k_- \in \left\{-\left\lfloor \frac{N-1}{2} \right\rfloor, \dots, \left\lfloor \frac{N-1}{2} \right\rfloor\right\}$ with $k_+ > k_-$, $k_+ \geq k_0 \geq k_-$, and

$$\theta_0 \in \left(-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}\right) + \frac{4k_0\pi}{2N+1} - \frac{2\arg(t)}{2N+1},$$

$$\theta_\pm \in \left(-\frac{\pi}{2N+1}, \frac{\pi}{2N+1}\right) + \frac{(4k_\pm \pm 2)\pi}{2N+1} - \frac{2\arg(t)}{2N+1}$$

$$M(\lambda) = \begin{cases} \mathbf{1} + e^{-2\vartheta(\lambda;t,x)}\sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}\mathbb{R}_+ \\ \mathbf{1} + e^{2\vartheta(\lambda;t,x)}\sigma_- & \lambda \in \varpi_\pm := e^{i\theta_\pm}\mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_- \end{cases}$$

$$\vartheta(\lambda;t,x) := t\lambda^{\frac{2N+1}{2}} + \frac{2}{3}\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}}$$

Find a matrix $\Gamma(\lambda)$ such that

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda),\ \lambda \in \varpi_{-,0,+}, \qquad \Gamma_+(\lambda) = \Gamma_-(\lambda)i\sigma_2, \quad \lambda \in \mathbb{R}_-$$

with asymptotics

$$\Gamma(\lambda;\mathbf{t}) = \lambda^{-\frac{\sigma_3}{4}}\frac{\mathbf{1}+i\sigma_1}{\sqrt{2}}\left(\mathbf{1} + a(\mathbf{t})\frac{\sigma_3}{\sqrt{\lambda}} + \mathcal{O}(\lambda^{-1})\right),\ \lambda \to \infty.$$

Which solutions are they ?

- For $N = 2$ there is just one solution, the one studied by Boutroux.
- For $N = 3$ there 4 solutions. One is the *tritronquée* solution U_0 of PI_2 related to the Dubrovin's conjecture on universality Dubrovin (with t fixed) (for $t = 0$ it had been studied by Brezin-Marinari-Parisi and Moore).
- The other three belong to the set of "two parameters solutions" studied by Grava-Kapaev-Klein.
- For N générique, the analog of U_0 had been used by Claeys-Its-Krasovsky to describe "higher order" Tracy–Widom distributions.

How to go from Z_n to PI_{N-1} ?

Jumps for Z_n :

$$M_n(\lambda) = \begin{cases} \mathbf{1} + \mathbf{d}(\lambda) e^{-\frac{4}{3}\lambda^{\frac{3}{2}} - 2x\lambda^{\frac{1}{2}}} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}\mathbb{R}_+ \\ \mathbf{1} + \frac{1}{\mathbf{d}(\lambda)} e^{\frac{4}{3}\lambda^{\frac{3}{2}} + 2x\lambda^{\frac{1}{2}}} \sigma_- & \lambda \in \varpi_\pm := e^{i\theta_\pm}\mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_- \end{cases}$$

Jumps for PI_{N-1} :

$$M(\lambda) = \begin{cases} \mathbf{1} + e^{-2\vartheta(\lambda;t,x)} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}\mathbb{R}_+ \\ \mathbf{1} + e^{2\vartheta(\lambda;t,x)} \sigma_- & \lambda \in \varpi_\pm := e^{i\theta_\pm}\mathbb{R}_+ \\ i\sigma_2 & \lambda \in \mathbb{R}_- \end{cases}$$

$$\vartheta(\lambda; t, x) = t\lambda^{\frac{2N+1}{2}} + \frac{2}{3}\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}}$$

So we need to approximate $e^{-t\lambda^{\frac{2N+1}{2}}}$ using the rational function $d(\lambda)$...

Padé approximants !

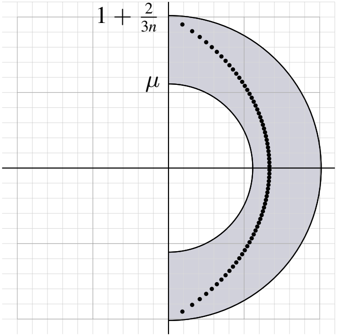
Padé's Approximants

Let P_r be the r-th Padé approximant for e^{-z} :

$$\mathrm{e}^{-z} = \frac{P_r(z)}{P_r(-z)} + \mathcal{O}(z^{2r+1}), \quad z \to 0.$$

The distribution of its zeros is known [SaffVarga78], they are all contained in the region $\mathrm{Re} z > 0$.

$\mu \mathrm{e}^{1+\mu} = 1, \quad \mu \simeq 0,278...$



Let's fix $N \in \mathbb{N}$ and

$$\mathcal{Y}^{(0)} \sqcup \mathcal{Y}^{(1)} \sqcup \mathcal{Y}^{(2)} := \mathcal{Y} = \left\{ y : \quad P_r(2ty^{2N+1}) = 0 \right\}, \quad n = r(2N+1).$$

IMPORTANT : There's an ambiguity on the choice of the location of the zeroes...

- $k_0 =$ ($\#\{\mathcal{Y}_\kappa\}$ in the second quadrant assigned to $\mathcal{Y}^{(2)}$) - ($\#\{\mathcal{Y}_\kappa\}$ in the third quadrant assigned to $\mathcal{Y}^{(1)}$)
- $k_- = -\left\lfloor \frac{N}{2} \right\rfloor$ + ($\#\{\mathcal{Y}_\kappa\}$ in the first quadrant assigned to $\mathcal{Y}^{(1)}$) ;
- $k_+ = \left\lfloor \frac{N}{2} \right\rfloor$ - ($\#\{\mathcal{Y}_\kappa\}$ in the fourth quadrant assigned to $\mathcal{Y}^{(2)}$) ;

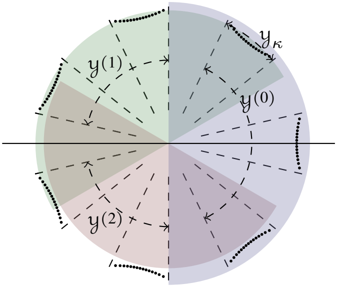


FIGURE: Exemple avec $N = 3$, $k_+ = 1, k_0 = 0, k_- = -1$.

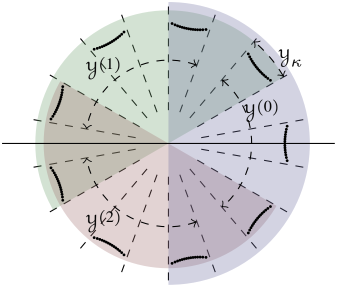


FIGURE: Example with $N = 4$, $k_+ = 1, k_0 = 0, k_- = -1$.

Theorem (M. Bertola, M.C.) :

Let's fix $N \in \mathbb{N}$ and choose $\mathcal{Y} = \{y_1, \ldots, y_n\}$ as above.

$Z_n(x; \mathcal{Y})$ converges, for $n \to \infty$, to the tau function PI_N identified by the corresponding $k_0, k_\pm$.

In particular, $u(x,t) := 2\partial_x^2 \tau(x,t)$ satisfies the equation

$$(2N+1)tR_N[u(x;t)] + u(x;t) + x = 0,$$

with R_j defined by the Lenard's recursion

$$\frac{\partial}{\partial x} R_{N+1}[u] = \left(\frac{1}{4}\frac{\partial^3}{\partial x^3} + u(x)\frac{\partial}{\partial x} + \frac{1}{2}u_x(x)\right) R_N[u], \quad R_0[u] = 1.$$

□

Example :

$$N = 2; \qquad \frac{5}{8}t\left(u'' + 3u^2\right) + u + x = 0,$$

$$N = 3; \qquad \frac{7}{32}t\left(u^{(4)} + 10uu'' + 5(u')^2 + 10u^3\right) + u + x = 0\,.$$

Thanks !