"Integrable" gap probabilities for the Generalized Bessel process

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BESQ^α model

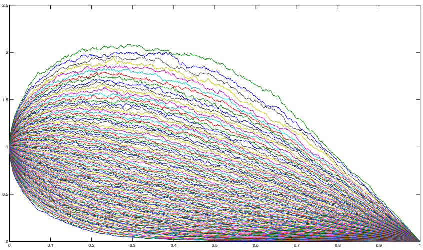
Consider a system of n independent squared Bessel paths BESQ^α

$$\{X_1(t), \ldots, X_n(t)\}$$

with parameter $\alpha > -1$, conditioned never to collide.

The process $\{\vec{X}(t)\}_{t\geq 0}$ is a diffusion process on $[0, +\infty)^n$. Additionally, we impose initial and final conditions

$$X_j(0) = a > 0 \text{ and } X_j(T) = 0 \quad \forall j = 1, \ldots, n.$$



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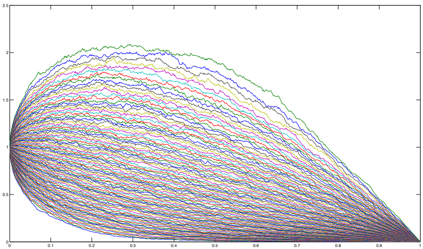


The joint probability density is given as

$$\frac{1}{Z_{n,t}} \det\left[x_k^{j-1} p_t^{\alpha+1-j(\mathrm{mod}\,2)}(a, x_k)\right]_{j,k=1}^n \det\left[x_j^{k-1} e^{-\frac{x_j}{2(T-t)}}\right]_{j,k=1}^n \mathrm{d}x_1 \dots \mathrm{d}x_n$$
$$= \frac{1}{n!} \det\left[K_n(x_i, x_j; t)\right]_{i,j=1}^n \mathrm{d}x_1 \dots \mathrm{d}x_n$$

where $p_t^\alpha(x, y)$ is the transition probability $p_t^\alpha(x, y) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\alpha/2} e^{-\frac{x+y}{2t}} I_\alpha\left(\frac{\sqrt{xy}}{t}\right)$ and the correlation kernel K_n given in terms of MOP with weights depending on the Bessel functions I_α.

Remark (Random Matrix interpretation)

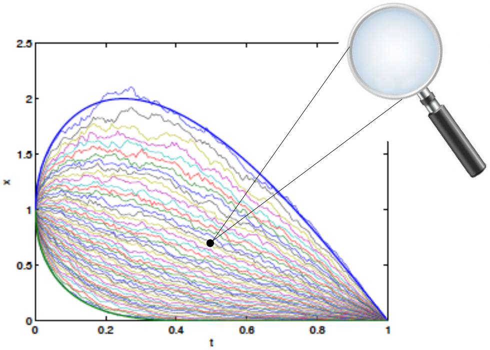
Let $M(t)$ be a $p \times n$ matrix with independent complex Brownian entries (with mean zero and variance $2t$).
The set of singular values

$$\{\lambda_1(t), \dots, \lambda_n(t)\}, \quad \lambda_i(t) \geq 0\ \forall i$$

i.e. the eigenvalues of the product $M(t)^* M(t)$, has the same distribution as the above noncolliding particle system $BESQ^\alpha$ with $\alpha = 2(n-p+1)$ (König, O'Connell, '01).

(Double) Scaling limit

Starting from the kernel K_n, one can perform a double scaling limit as $n \nearrow +\infty$ in different parts of the domain of the spectrum: the sine kernel appears in the bulk, the Airy kernel at the soft edges and the Bessel kernel appears at the hard edge $x = 0$ (Kuijlaars *et al.*, '09).



At a critical time t^*, there is a transition between the soft and the hard edges and the local dynamics is described by a new critical kernel.

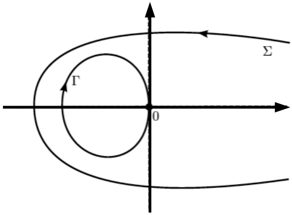
The Generalized Bessel kernel

Theorem (Kuijlaars, Martinez-Finkelshtein, Wielonsky, '11)

$$\lim_{n \nearrow +\infty} \frac{c^*}{n^{3/2}} K_n\left(\frac{c^* x}{n^{3/2}}, \frac{c^* y}{n^{3/2}}; t^* - \frac{c^* \tau}{\sqrt{n}}\right) = K_\alpha^{\mathrm{crit}}(x, y; \tau) \qquad x, y \in \mathbb{R}_+, \tau \in \mathbb{R},$$

with

$$K_\alpha^{\mathrm{crit}}(x, y; \tau) = \int_\Gamma \frac{\mathrm{d}u}{2\pi i} \int_\Sigma \frac{\mathrm{d}v}{2\pi i} \frac{e^{xu + \frac{\tau}{u} + \frac{1}{2u^2} - yv - \frac{\tau}{v} - \frac{1}{2v^2}}}{v - u} \left(\frac{u}{v}\right)^\alpha.$$



Gap probabilities of the Generalized Bessel process

Our object of study are the **gap probabilities**, meaning the probability of finding no points in a given domain.

For a determinantal process with kernel K_n, this boils down to calculating a Fredholm determinant:

$$\begin{aligned}\mathbb{P}\left(X_{\min} > s\right) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left[K_n(x_i, x_j)\right]_{i,j=1,\dots,k} \mathrm{d}x_1 \dots \mathrm{d}x_k \\ &= \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_n\Big|_{[0,s]}\right)\end{aligned}$$

and in the scaling limit regime

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_n\Big|_{\left[0, \frac{c^* s}{n^{3/2}}\right]}\right) \to \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) \qquad \text{as } n \nearrow +\infty.$$

Differential identity

Theorem (Girotti, '14)

Let $s > 0$ and K_α^{crit} be the integral operator acting on $L^2(\mathbb{R}_+)$ with kernel defined above. Then, the following differential formula for gap probabilites holds

$$\mathrm{d}_{s,\tau} \ln \det \left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\text{crit}} \Big|_{[0,s]} \right) = (Y_1)_{2,2}\, \mathrm{d}s - \left(\hat{Y}_0^{-1} \hat{Y}_1 \right)_{2,2} \mathrm{d}\tau$$

where Y is the solution to a suitable RH problem and Y_1 and $\hat{Y}_j$ are the coefficients appearing in the asymptotic expansion of Y at infinity and in a neighbourhood of zero, respectively.

The Riemann-Hilbert problem for Y

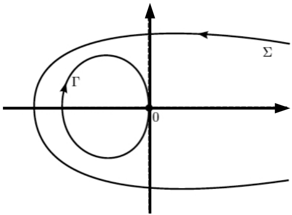
Find a 2×2 matrix-valued function $Y = Y(\lambda; s, \tau)$ such that

- Y is analytic on $\mathbb{C} \setminus (\Gamma \cup \Sigma)$
- Y admits a limit when approaching the contours from the left Y_+ or from the right Y_- (according to their orientation), and the following jump condition holds

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & -\lambda^{-\alpha} e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2}} \\ 0 & 1 \end{bmatrix} & \lambda \in \Sigma \\ \begin{bmatrix} 1 & 0 \\ -\lambda^{\alpha} e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2}} & 1 \end{bmatrix} & \lambda \in \Gamma \end{cases}$$

- Y has the following (normalized) behaviour at ∞:

$$Y(\lambda) = I + \frac{Y_1(s,\tau)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \qquad \lambda \to \infty.$$



Sketch of the proof

Proposition

The following identity holds

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) = \det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right)$$

where $\mathbb{H}$ *is an Its-Izergin-Korepin-Slavnov ('90) integral operator with kernel*

$$\mathbb{H} = \frac{\boldsymbol{f}(\lambda)^T \boldsymbol{g}(\mu)}{\lambda - \mu}$$

$$\boldsymbol{f}(\lambda) = \frac{1}{2\pi i}\begin{bmatrix} e^{-\frac{\lambda s}{2}}\chi_\Sigma(\lambda) \\ \chi_\Gamma(\lambda)\end{bmatrix} \qquad \boldsymbol{g}(\mu) = \begin{bmatrix} \mu^\alpha e^{\mu s + \frac{\tau}{\mu} + \frac{1}{2\mu^2}}\chi_\Gamma(\mu) \\ \mu^{-\alpha} e^{-\frac{\mu s}{2} - \frac{\tau}{\mu} - \frac{1}{2\mu^2}}\chi_\Sigma(\mu)\end{bmatrix}.$$

The result can be proved by noticing that $K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}$ is unitarily equivalent (via Fourier transform) to a certain integral operator that can be decomposed as the above operator $\mathbb{H}$.

IIKS operators naturally carry an associated RH problem, whose solution Y is tied to the invertibility of their resolvent operator.

Given such RH problem, we make use of a major (and more general) result due to Bertola ('10) and Bertola-Cafasso ('11) which, if applied to our case, reads as follows

Theorem (Bertola-Cafasso, '11)

Define the quantity for $\rho = s, \tau$

$$\omega(\partial_\rho) := \int_{\Sigma\cup\Gamma} \mathrm{Tr}\left[Y_-^{-1} Y_-' \left(\partial_\rho J\right) J^{-1}\right] \frac{\mathrm{d}\lambda}{2\pi i}.$$

Then, we have the equality

$$\omega(\partial_\rho) = \partial_\rho \ln\det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right).$$

By expanding the solution Y at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

$$\mathrm{d}_{s,\tau} \ln\det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right) = (Y_1)_{2,2}\,\mathrm{d}s - \left(\hat{Y}_0^{-1}\hat{Y}_1\right)_{2,2} \mathrm{d}\tau.$$

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A few more words on $\omega(\partial)$

The solution to the RH problem Y solves a rational ODE (up to a gauge transformation)

$$\frac{\mathrm{d}Y}{\mathrm{d}\lambda} = A(\lambda)Y(\lambda)$$

With this extra property, it turns out that (Bertola, '10) given

$$\omega(\partial) = \int_{\Sigma\cup\Gamma} \mathrm{Tr}\left[Y_-^{-1}Y_-'\left(\partial J\right)J^{-1}\right]\frac{\mathrm{d}\lambda}{2\pi i},$$

then ω is the logarithmic total differential of the isomonodromic τ function:

$$\mathrm{d}\omega = 0 \quad \text{and} \quad e^{\int\omega} = \tau_{\mathrm{JMU}}.$$

Conclusion

We give a specific geometrical meaning to a probabilistic quantity:

$$\tau_{\mathrm{JMU}} = \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) = \left\{\begin{array}{l}\text{infinitesimal fluctuation of}\\ \text{smallest path of BESQ}^\alpha\\ \text{at the critical time } t^*\end{array}\right\}$$

(up to a normalization constant).

What now?

Given

$$\mathrm{d}_{s,\tau} \ln \det \left(\mathrm{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, \mathrm{d}s - \left(\tilde{Y}_0^{-1} \tilde{Y}_1 \right)_{2,2} \mathrm{d}\tau$$

we can further study our RH problem to draw some interesting conclusions:

- asymptotic behaviour of gap probability (large/small gap, degeneration regimes) → Deift-Zhou steepest descent method
- integrability and differential equations (Tracy-Widom) → Lax pair, hamiltonian formalism

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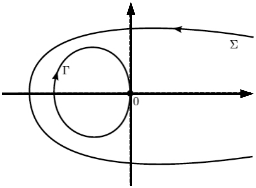
The Lax triplet

From the RH problem Y associated to our critical kernel K_α^{crit}

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & -\lambda^{-\alpha} e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2}} \\ 0 & 1 \end{bmatrix} & \lambda \in \Sigma \\ \begin{bmatrix} 1 & 0 \\ -\lambda^{\alpha} e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2}} & 1 \end{bmatrix} & \lambda \in \Gamma \end{cases}$$



we can derive the following Lax triplet:

$$\mathcal{A} = \mathcal{A}^{(\lambda)} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3},$$
$$\mathcal{B} = \mathcal{B}^{(s)} = \lambda B_1 + B_0,$$
$$\mathcal{C} = \mathcal{C}^{(\tau)} = \frac{C_{-1}}{\lambda}.$$

Up to a change of variables $\lambda \mapsto \frac{1}{\lambda}$, the Lax pair $\{\mathcal{A}, \mathcal{C}\}$ is

$$\mathcal{A} = \frac{\lambda}{2}\sigma_3 + A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \qquad \mathcal{C} = \frac{\lambda}{2}\sigma_3 + C_0$$

with coefficients

$$A_0 = \begin{bmatrix} \frac{\tau}{2} & uw \\ -\frac{1}{w}\left[v_\tau + u\left(v^2 - \Theta\right)\right] & -\frac{\tau}{2} \end{bmatrix}, \qquad A_{-2} = \begin{bmatrix} v & w \\ -\frac{1}{w}\left(v^2 - \Theta\right) & -v \end{bmatrix},$$

$$A_{-1} = \begin{bmatrix} u\left[v_\tau + u\left(v^2 - \Theta\right)\right] + \frac{\alpha}{2} & w\left[u_\tau - 2u^2 v + \tau u\right] \\ \frac{1}{w}\left[\left(u_\tau - 4u^2 v + \tau u\right)\left(v^2 - \Theta\right) - 2uvv_\tau - \alpha v + \tilde{\Theta}\right] & -u\left[v_\tau + u\left(v^2 - \Theta\right)\right] - \frac{\alpha}{2} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & uw \\ -\frac{1}{w}\left[v_\tau + u\left(v^2 - \Theta\right)\right] & 0 \end{bmatrix}.$$

We can recognize the Lax pair associated to the second member of the Painlevé III hierarchy defined by Sakka ('09):

$$\begin{cases} u_{\tau\tau} = (6uv - \tau)u_\tau - 6u^3 v^2 + 2\tau u^2 v + 2\Theta u^3 - (\alpha + 1)u + 1 \\ v_{\tau\tau} = -(6uv - \tau)v_\tau - 2u(3uv - \tau)(v^2 - \Theta) - \alpha v + \tilde{\Theta}. \end{cases}$$

The quest for a Garnier system...

As in the classical Painlevé theory (Jimbo, Miwa, Ueno, '81), we would like to find a completely integrable (Hamiltonian) system associated with the Lax triplet $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$.

In this case, we have two independent parameters that describe the flow,

the time τ and the space s,

therefore we need a 2-D version of Hamiltonian system (Garnier system, '26) for the canonical coordinates $(\mu_1, \mu_2; \lambda_1, \lambda_2)$:

$$\begin{cases} \frac{\partial \lambda_j}{\partial \tau} = \frac{\partial H_\tau}{\partial \mu_j} \\ \frac{\partial \mu_j}{\partial \tau} = -\frac{\partial H_\tau}{\partial \lambda_j} \end{cases} \qquad \begin{cases} \frac{\partial \lambda_j}{\partial s} = \frac{\partial H_s}{\partial \mu_j} \\ \frac{\partial \mu_j}{\partial s} = -\frac{\partial H_s}{\partial \lambda_j} \end{cases}$$

with rational Hamiltonians $H_\tau = H_t(\lambda_j, \mu_j; s, \tau)$ and $H_s = H_s(\lambda_j, \mu_j; s, \tau)$.

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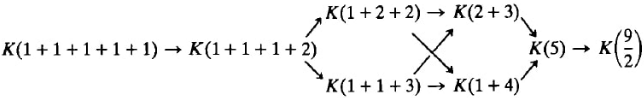

Action plan

Step 1: we identify the canonical coordinates in our system

$$\{\lambda_j\}_{j=1,2} \text{ as the solutions of the equation } (\mathcal{A}(\lambda; s, \tau))_{1,2} = 0$$
$$\{\mu_j\}_{j=1,2} \text{ as } \mu_j = (\mathcal{A}(\lambda_j; s, \tau))_{1,1}$$

Step 2: the compatibility equations of the Lax triplet yield a system of 8 differential equations (4 for the variable s, 4 for the variable τ) which can be represented as a Garnier system

$$\begin{cases} \dfrac{\partial \lambda_j}{\partial \tau} = \dfrac{\partial H_\tau}{\partial \mu_j} \\ \dfrac{\partial \mu_j}{\partial \tau} = -\dfrac{\partial H_\tau}{\partial \lambda_j} \end{cases} \qquad \begin{cases} \dfrac{\partial \lambda_j}{\partial s} = \dfrac{\partial H_s}{\partial \mu_j} \\ \dfrac{\partial \mu_j}{\partial s} = -\dfrac{\partial H_s}{\partial \lambda_j} \end{cases}$$

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$$H_\tau = -\frac{\lambda_1^2\mu_1^2}{\lambda_1-\lambda_2} + \frac{\lambda_2^2\mu_2^2}{\lambda_1-\lambda_2} - \frac{s^2\left(\lambda_1+\lambda_2\right)}{4\lambda_1^2\lambda_2^2} + \frac{\tau^2\left(\lambda_1+\lambda_2\right)}{4} - \frac{ks}{\lambda_1\lambda_2} - \frac{\tau\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2\right)}{2} + \frac{\lambda_1^3}{4} + \frac{\lambda_1^2\lambda_2}{4} + \frac{\lambda_1\lambda_2^2}{4} + \frac{\lambda_2^3}{4} - \frac{(\alpha+1)\lambda_1+2\alpha\lambda_2}{2}$$

$$H_s = -\frac{\lambda_1\lambda_2\left(\lambda_1\mu_1^2+\mu_1\right)}{s\left(\lambda_1-\lambda_2\right)} + \frac{\lambda_1\lambda_2\left(\lambda_2\mu_2^2+\mu_2\right)}{s\left(\lambda_1-\lambda_2\right)} + \frac{\tau^2\lambda_1\lambda_2}{4s} - \frac{k\left(\lambda_1+\lambda_2\right)}{\lambda_1\lambda_2} - \frac{\alpha\lambda_1\lambda_2}{2s} - \frac{s\left(\lambda_1+\lambda_2\right)}{4\lambda_1^2\lambda_2} - \frac{\tau\lambda_2\left(\lambda_1^2+\lambda_1\lambda_2\right)}{2s} + \frac{\lambda_1\lambda_2\left(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2-2\right)}{4s} - \frac{s}{4\lambda_2^2}$$

Remark

These Hamiltonians are different from the Hamiltonians of the $K(2+3)$ system defined in Okamoto-Kimura, '86. The identification process is on-going...

New horizons

Explicit connection between Hamiltonians and gap probabilities/RH problem for K_α^{crit}?

$$\mathrm{d}_{s,\tau} \ln\det\left(\mathrm{Id}_{L^2(R_+)} - K_\alpha^{\text{crit}}\Big|_{[0,s]}\right) = \mathcal{L}_1\left(H_\tau, H_s\right)\mathrm{d}s + \mathcal{L}_2\left(H_\tau, H_s\right)\mathrm{d}\tau$$

New horizons

Quantization?

Find a suitable canonical transformation of variables $(\mu_j, \lambda_j) \mapsto (\tilde{\mu}_i, \tilde{\lambda}_i)$ such that the Hamiltonians become polynomials or of the form $p^2 + V(q)$.

Via the classical substitution of the operators $\left\{x_j, \hbar \frac{\partial}{\partial x_j}\right\}$ into the canonical coordinates $(\tilde{\lambda}_j, \tilde{\mu}_j)$, study the Schrödinger system

$$\hbar \frac{\partial}{\partial \tau} \Phi(x; s, \tau) = \hat{H}_\tau \left(x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau \right) \Phi(x; s, \tau)$$

$$\hbar \frac{\partial}{\partial s} \Phi(x; s, \tau) = \hat{H}_s \left(x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau \right) \Phi(x; s, \tau)$$

Further work:

- what will the Lax pair $\{\mathcal{A}, \mathcal{B}\}$ yield?

$$\mathcal{A} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3}, \qquad \mathcal{B} = \lambda B_1 + B_0;$$

- asymptotic behaviour?
  Conjecture: degeneration of the gap probabilities of K_α^{crit} into gap probabilities of the Airy process (for $\tau \searrow -\infty$) or the Bessel process (for $\tau \nearrow +\infty$).

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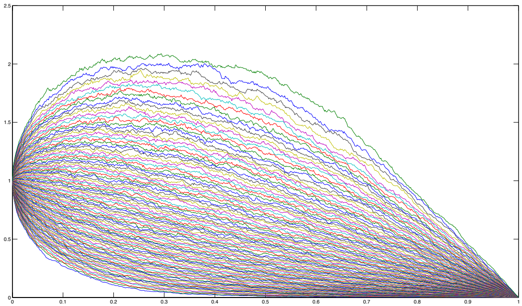

References

- M. Bertola, *Comm. Math. Phys.* **294**(2), 539–579, 2010.
- M. Bertola, M. Cafasso. *Int. Math. Res. Not.*, **2012**(7), 2012.
- M. Girotti, *Math. Phys. Anal. Geom.*, **17** (1), 183-211, 2014.
- A.R. Its, A.G. Izergin, V.E. Korepin, N.A. Slavnov. *Int. J. Mod. Phys.*, **B4**, 1990.
- W. König, N. O'Connell, *Electron. Commun. Probab.*, **6**(11), 107-114, 2001.
- A.B.J. Kuijlaars, A. Martinez-Finkelshtein, F. Wielonsky, *Comm. Math. Phys.*, **286**, 217–275, 2009 and *Comm. Math. Phys.*, **308**(1), 227–279, 2011.
- M. Jimbo, T. Miwa, K. Ueno, *Phys. D*, **2**(3), 407-448, 1981 and *Phys. D*, **2**(2), 306-352, 1981.
- K. Okamoto, H. Kimura, *Quart. J. Math. Oxford*, **2**(37), 61-80, 1986.
- A.H. Sakka. *J. Phys. A: Math. Theor.*, **42**, 1–19, 2009.

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