Polynomial Tau Functions and Bilinearization of the Drinfeld–Sokolov Hierarchies

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The Drinfeld–Sokolov Hierarchies

The KdV hierarchy

The KdV hierarchy is a sequence of PDE's on a function $u(t_1 = x, t_3, t_5, t_7, \ldots)$ of evolutionary type:

$$\frac{\partial u}{\partial t_i} = R_i'(u) \in \mathbb{R}\left[u, u', u'', \ldots\right], \qquad\qquad i \text{ odd},$$

where $\boxed{u' := \partial_x u}$

The $R_i'(u)$'s satisfy the Lenard recursion [1967]

$$\begin{cases} R_1(u) = u; \\ R_{i+2}'(u) = \left(\partial_x^3 + 4u\partial_x + 2u'\right) R_i(u). \end{cases}$$

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It satisfies the integrability condition

$$\partial_{t_i} R_j'(u) = \partial_{t_j} R_i'(u)$$

that is, $$\partial_{t_i}\partial_{t_j} u = \partial_{t_j}\partial_{t_i} u$$

The KdV hierarchy, explicitly

For a function $u(t_1 = x, t_3, t_5, t_7, \ldots)$ the KdV hierarchy reads

$$\begin{cases} \partial_{t_1} u = u' \\ \partial_{t_3} u = u''' + 6uu' & \leftarrow \text{KdV eq.} \\ \partial_{t_5} u = u^{(5)} + 10uu^{(3)} + 20u'u'' + 30u^2u' \\ \partial_{t_7} u = u^{(7)} + 14uu^{(5)} + 42u'u^{(4)} + 70u''u^{(3)} + 70u^2u^{(3)} + 280uu'u'' + \\ \quad \vdots \end{cases}$$

It is the simplest example of **integrable hierarchy**.

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It is the simplest example of **integrable hierarchy**.

To any solution u we associate a function τ called its **tau function**:

$$u = -2\frac{\partial^2 \log \tau}{\partial^2 x} = 2\frac{(\tau')^2 - \tau\tau''}{\tau^2}.$$

The Drinfeld–Sokolov hierarchies

Let a semi-simple Lie algebra $\mathfrak{g}$ and take its **Kac–Moody algebra** with the principal gradation

$$\mathcal{G} := \mathfrak{g} \otimes \mathbb{C}\left[\lambda, \lambda^{-1}\right] \oplus \mathbb{C} \cdot c.$$

Drinfeld and Sokolov showed [1985] how to associate to $\mathcal{G}$ an **integrable hierarchy** using mainly

- The set of exponents E of $\mathcal{G}$ and $E_+ = E \cap \mathbb{N}$;
- The Heisenberg generators $\Lambda_i \in \mathfrak{g} \otimes \mathbb{C}\left[\lambda, \lambda^{-1}\right]$, $i \in E$,
- and $\deg \Lambda_i = i$.

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- and $\deg \Lambda_i = i$.

Example: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ gives the KdV hierarchy, with

- The set of exponents $E = \mathbb{Z}^{\text{odd}}$;
- The Heisenberg generators $\Lambda_i = \Lambda^i$ where $\Lambda = \begin{pmatrix} 0 & \lambda \\ 1 & 0 \end{pmatrix}$.

Tau structure of the Drinfeld–Sokolov hierarchies

We fix a faithful representation

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(n, \mathbb{C}) = \mathrm{Mat}(n, \mathbb{C}).$$

Eventually, the Drinfeld–Sokolov hierarchy applies to functions

$$u_1(\mathbf{t}), \dots, u_n(\mathbf{t}), \qquad \text{where } \mathbf{t} = \{t_i\}_{i \in E_+} \text{ and } t_1 = x,$$

and takes the form of evolutionary PDE's

$$\frac{\partial u_\alpha}{\partial t_i} = K^{\alpha,i}(u_\beta) \in \mathbb{R}\left[u_1, \dots, u_n, u_1', \dots, u_n', \dots\right].$$

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Proposition (Tau structure)

There exist coordinates $u_\alpha(\mathbf{t})$ and a single function $\tau(\mathbf{t})$ s.t.

$$u_\alpha(\mathbf{t}) = -2 \frac{\partial^2 \log \tau}{\partial x \partial t_\alpha}.$$

Polynomial Tau Functions as Toeplitz Determinants

Laurent and Toeplitz block matrices

Let a loop

$$\gamma = \sum_{k\in\mathbb{Z}} \gamma_k \lambda^k, \qquad \gamma_k \in \mathsf{GL}(n, \mathbb{C}),$$

and define its Laurent block matrix:

$$\mathsf{L}(\gamma) := (\gamma_{i-j})_{i,j\in\mathbb{Z}} = \begin{pmatrix} \ddots & & & & \\ & \gamma_0 & \gamma_{-1} & \gamma_{-2} & \\ & \gamma_1 & \gamma_0 & \gamma_{-1} & \\ & \gamma_2 & \gamma_1 & \gamma_0 & \\ & & & & \ddots \end{pmatrix}$$

It satisfies $\mathsf{L}(\gamma\gamma') = \mathsf{L}(\gamma)\mathsf{L}(\gamma')$.

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Define its Toeplitz block matrix:

$$\mathsf{T}_N(\gamma) := (\gamma_{i-j})_{i,j=0}^{N}, \qquad \mathsf{T}(\gamma) := (\gamma_{i-j})_{i,j \in \mathbb{N}}.$$

Tau functions as Toeplitz determinants

Theorem (M. Cafasso, C.-Z. Wu, 2015)

For a Lie algebra $\mathfrak{g}$ *with exponents* E*, define*

$$\xi(\mathbf{t}) := \sum_{i \in E_+} t_i \Lambda_i.$$

Let $a \in \mathfrak{g}\left[\lambda, \lambda^{-1}\right]^{<0}$*, and associate a loop*

$$\gamma_a(\mathbf{t}) = e^{\xi(\mathbf{t})} e^a.$$

Then the function

$$\tau_a(\mathbf{t}) = \left(\lim_{N \to \infty} \det T_N(\gamma_a(\mathbf{t})) \right)^{\kappa}$$

is a tau function of the Drinfeld–Sokolov hierarchy of $\mathfrak{g}$ *type.*

Here κ *is the constant such that* $(X|Y)_0 = \kappa \cdot \mathrm{tr}(XY)$.

Schur expansion of tau functions

For $a \in \mathfrak{g}\left[\lambda, \lambda^{-1}\right]^{<0}$, and $\xi(\mathbf{t}) := \sum_{i \in E_+} t_i \Lambda_i$. Define

$$r = \mathsf{L}\left(\mathsf{e}^{a}\right), \qquad s(\mathbf{t}) = \mathsf{L}\left(\mathsf{e}^{\xi(\mathbf{t})}\right),$$

and for any partition μ with length $\ell(\mu)$

$$r_{\mu}^{(\mathsf{R})} = \det\left(r_{i-\mu_i-1, j-1}\right)_{i,j=1}^{\ell(\mu)}$$

$$s_{\mu}^{(\mathsf{L})}(\mathbf{t}) = \det\left(s_{i-1, j-\mu_j-1}\right)_{i,j=1}^{\ell(\mu)}$$

Schur polynomials of $\mathfrak{g}$ type

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Schur polynomials of $\mathfrak{g}$ type

Then we can expand $\tau_a(\mathbf{t})$ as

$$\tau_a(\mathbf{t})^{1/\kappa} = \lim_{N\to\infty} \det \mathsf{T}_N(\gamma_a(\mathbf{t})) = \sum_{\text{partitions } \mu} s_\mu^{(\mathsf{L})}(\mathbf{t}) \cdot r_\mu^{(\mathsf{R})}.$$

In particular, if $\pi(a)$ is a nilpotent matrix, then $\tau_a(\mathbf{t})$ is polynomial.

Example: $\mathsf{A}_1 = \mathfrak{sl}(2,\mathbb{C})$ = KdV – Polynomial tau functions

$$a_1 = \begin{pmatrix} 0 & 0 \\ \lambda^{-1} & 0 \end{pmatrix} \quad \longrightarrow \quad \tau_1(\mathbf{t}) = 1 + t_1$$

$$a_2 = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} \quad \longrightarrow \quad \tau_2(\mathbf{t}) = 1 + \frac{t_1^3}{3} + t_3$$

$$a_3 = \begin{pmatrix} 0 & 0 \\ \lambda^{-2} & 0 \end{pmatrix} \quad \longrightarrow \quad \tau_3(\mathbf{t}) = 1 - \frac{t_1^6}{45} + \frac{t_3\,t_1^3}{3} + \frac{t_1^3}{3} - t_5\,t_1 + t_3^2 + 2t_3$$

$$a_4 = \begin{pmatrix} 0 & \lambda^{-2} \\ 0 & 0 \end{pmatrix} \quad \longrightarrow \quad \tau_4(\mathbf{t}) = 1 - \frac{t_1^{10}}{4725} + \frac{t_3\,t_1^7}{105} - \frac{t_5\,t_1^5}{15} - \frac{t_1^5}{15} + \dots$$

Up to multiplying and shifting the variables, we recover the usual polynomial tau functions of KdV.

Example: $\mathsf{A}_1 = \mathfrak{sl}(2, \mathbb{C})$ = KdV – The Adler–Moser pol.

The Adler–Moser polynomials $\theta_k(q_1, q_3, \ldots, q_{2k-1})$ are defined recursively by $\theta_0 = 1$, $\theta_1 = q_1$ and

$$\frac{\partial \theta_{k+1}}{\partial q_1}\theta_{k-1} + \theta_{k+1}\frac{\partial \theta_{k-1}}{\partial q_1} = (2k-1)\theta_k^2,$$

where the integration constant is q_{2h-1}.

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Theorem (C. D.)

The following change of variables transforms the Adler–Moser polynomials into the polynomial tau functions of KdV:

$$\sum_{j\geq 1} q_j z^{2j-1} = \tanh\left(\sum_{j\geq 1} \alpha_j t_j z^{2j-1}\right),$$

where $\alpha_j = (-1)^{j-1}(2j-1)(2j-3)!!^2$.

Bilinearization of the Drinfeld–Sokolov hierarchies

Hirota derivatives

Let $f(\mathbf{t}), g(\mathbf{t})$, we denote by D_i the Hirota derivatives defined by

$$
\begin{aligned}
f(\mathbf{t}+\mathbf{h}) \cdot g(\mathbf{t}-\mathbf{h}) &= \exp\left(\sum h_i D_i\right)(f,g) \\
&= f \cdot g + h_1 D_1(f,g) + \ldots + \frac{h_1^2}{2} D_1^2(f,g) + \ldots
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For example,

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D_1(f, g) &= \frac{\partial f}{\partial t_1} g - f \frac{\partial g}{\partial t_1}, \\
D_1 D_2(f, g) &= \frac{\partial^2 f}{\partial t_1 \partial t_2} g + f \frac{\partial^2 g}{\partial t_1 \partial t_2} - \frac{\partial f}{\partial t_1} \frac{\partial g}{\partial t_2} - \frac{\partial f}{\partial t_2} \frac{\partial g}{\partial t_1}.
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\end{aligned}
$$

We have a grading $\deg D_i = i$.

A **Hirota equation** is an equation of the following form, for any *even* polynomial $P(\mathbf{t})$,

$$P(D)(\tau,\tau) = 0,$$

Hirota equations for the A_n case

The A_n hierarchy applied to the tau function defined by

$$u_\alpha(\mathbf{t}) = -2\frac{\partial^2 \log \tau}{\partial x \partial t_\alpha}$$

is equivalent to two strings of Hirota equations:

$$\mathbb{Y}_k^{(n)}: \quad \left(s_{k+1}(\tilde{D}) - \tfrac{1}{2} D_1 D_k\right)(\tau, \tau) = 0,$$

$$\mathbb{Y}_{1,k}^{(n)}: \quad \left(D_1 s_{k+1}(\tilde{D}) - D_1 D_{k+1} + \tfrac{1}{2} D_2 D_k\right)(\tau, \tau) = 0,$$

where $\tilde{D} = \left(D_1, \frac{1}{2} D_2, \frac{1}{3} D_3, \ldots\right)$,
and s_k is the usual Schur polynomial for A_n (so $t_{k(n+1)} = 0$).

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Example A_1 = KdV: The KdV equation on $u = -2\partial_{t_1}^2 \log\tau$ reads

$$\left(D_1^4 - 4D_1 D_3\right)(\tau,\tau) = 0 \qquad \left(\mathbb{Y}_3^{(1)} = \mathbb{Y}_{1,2}^{(1)}\right)$$

$$\partial_{t_3} u = \partial_{t_1}^3 u + 6u\partial_{t_1} u \qquad \text{(KdV equation)}$$

Bilinearization of the B_2 Hierarchy

We compute tau functions of the B_2 hierarchy. For example:

$$\tau = 1 + \frac{1}{2}t_1 + \frac{1}{4}at_1^2 - \frac{1}{192}a^2 t_1^4 - \frac{1}{8}a^2 t_1 t_3$$

Proposition (M. Cafasso, D. Yang, C. D.)

The computed tau functions of the B_2 hierarchy satisfy:

- *No Hirota equation of degree 2 or 4;*
- *Only 1 Hirota equation of degree 6 and 8 resp.:*

$$(D_1^6 - 5D_1^3 D_3 - 5D_3^2 + 9D_1 D_5)(\tau,\tau) = 0,$$
$$(D_1^8 + 7D_1^5 D_3 - 35D_1^2 D_3^2 - 21D_1^3 D_5 - 42D_3 D_5 + 90D_1 D_7)(\tau,\tau) = 0.$$

Moreover, it satisfies no Hirota equation of mixed degree less than 4, 6 nor 8 that is linearly independent from the above equations.

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Moreover, it satisfies no Hirota equation of mixed degree less than 4, 6 nor 8 that is linearly independent from the above equations.

In comparison: The KdV hierarchy has 1 equation of degree 4 and 2 linearly independent equations for all even degree ≥ 6.

Thank you for your attention and enjoy the conference!

Examples of generalized Schur polynomials

Classification of simple Lie algebras: A_n, B_n, C_n, D_n, $\mathrm{E}_{6,7,8}$, F_4, G_2,

- $\mathrm{A}_n = \mathfrak{sl}(n+1,\mathbb{C})$, positive exponents $E_+ = \mathbb{N} \setminus (n+1)\mathbb{N}$:

Traditional Schur polynomials (with $t_{k(n+1)} = 0$):

$$s_1 = t_1\,, \qquad s_2 = \tfrac{1}{2}t_1^2 + t_2\,,$$
$$s_3 = \tfrac{1}{3}t_1^3 + t_1 t_2 + t_3\,, \qquad s_{1^2} = \tfrac{1}{2}t_1^2 - t_2\,.$$

- $\mathrm{B}_n = \mathfrak{so}(2n+1,\mathbb{C})$, positive exponents $E_+ = \mathbb{N}^{\mathrm{odd}}$:

$$s_1 = 0, \qquad s_2 = \tfrac{1}{2}t_1\,,$$
$$s_3 = \tfrac{1}{4}t_1^2\,, \qquad s_{1^2} = -\tfrac{1}{2}t_1\,.$$

- $\mathrm{C}_n = \mathfrak{sp}(2n,\mathbb{C})$, positive exponents$E_+ = \mathbb{N}^{\mathrm{odd}}$:

$$s_1 = t_1\,, \qquad s_2 = \tfrac{1}{2}t_1^2\,,$$
$$s_3 = \tfrac{1}{3}t_1^3 + 2t_3\,, \qquad s_{1^2} = \tfrac{1}{2}t_1^2\,.$$

Schur expansion of tau functions

For $a \in \mathfrak{g}\left[\lambda, \lambda^{-1}\right]^{<0}$ and $\gamma_a = \mathrm{e}^{\xi(\mathbf{t})}\mathrm{e}^a$, define

$$s(\mathbf{t}) := \mathsf{L}(\mathrm{e}^{\xi(\mathbf{t})}), \qquad s_N(\mathbf{t}) = \left(\mathsf{L}(\mathrm{e}^{\xi(\mathbf{t})})_{i,j}\right)_{i=0,\dots,N}^{j=-N,\dots,N},$$

$$r := \mathsf{L}\left(\mathrm{e}^a\right), \qquad r_N = \left(\mathsf{L}\left(\mathrm{e}^a\right)_{i,j}\right)_{i=-N,\dots,N}^{j=0,\dots,N}.$$

So that

$$\mathsf{L}(\gamma_a) = \mathsf{L}(\mathrm{e}^{\xi(\mathbf{t})})\mathsf{L}(\mathrm{e}^a) = s(\mathbf{t})r,$$

and because $\pi(a)$ is nilpotent, for large N,

$$\mathsf{T}_N(\gamma_a) = (\mathsf{L}(\gamma_a)_{i,j})_{i=0,\dots,N}^{j=0,\dots,N} = s_N(\mathbf{t})r_N.$$

Therefore, we can rewrite the tau function as

$$\begin{aligned}\tau_a(\mathbf{t})^{1/\kappa} &= \lim_{N\to\infty} \det \mathsf{T}_N(\gamma_a) \\ &= \lim_{N\to\infty} \det s_N(\mathbf{t}) \det r_N.\end{aligned}$$