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Bi-flat F-manifolds, logarithmic connections and complex reflection groups.

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Plan of the talk

1. Flat and bi-flat F-manifolds.
2. Complex reflection groups and bi-flat F-manifolds.

Flat F-manifolds (Manin)

Definition

An F-manifold with compatible flat structure (or flat F-manifold) $(M, \circ, \nabla, e)$ is a manifold equipped with a product $\circ : TM \times TM \to TM$ on the tangent spaces, a connection ∇ and a distinguished vector field e such that

- *the one parameter family of connections*

$$\nabla - \lambda\circ$$

  is flat and torsionless for any λ.
- *e is the unit of the product and it is flat: $\nabla e = 0$.*

Let Γ_{ij}^k be the Christoffel symbols of ∇ and c_{ij}^k the structure constants of the product.
The fact that $\Gamma_{ij}^k - \lambda c_{ij}^k$ is flat and torsionless for any λ tell us that:

1. the connection ∇ is torsionless and the product $\circ$ is commutative
2. the connection ∇ is flat and the product $\circ$ is associative.
3. the tensor field $\nabla_l c_{ij}^k$ is symmetric in the lower indices.

Oriented associativity equations

From condition 1,2,3 it follows that, in **flat coordinates** for ∇ we have

$$c^i_{jk} = \partial_j \partial_k A^i.$$

The vector potential A^i satisfies the associativity equations:

$$\partial_j \partial_l A^i \partial_k \partial_m A^l = \partial_k \partial_l A^i \partial_k \partial_m A^l$$

Principal hierarchy for F-manifolds with compatible flat connection

Integrable hierarchy:

$$u_{t_{(p,l)}} = X_{(p,l)} \circ u_x, \qquad p = 1, ..., n \qquad l = 0, 1, 2, 3, ...$$

where

$$(\nabla - \lambda\circ)\left(X_{(p,0)} + X_{(p,1)}\lambda + X_{(p,2)}\lambda^2 + ...\right) = 0.$$

This means

$$\nabla X_{(p,0)} = 0$$

and

$$\nabla X_{(p,l+1)} = X_{(p,l)} \circ .$$

In flat coordinates $(v^1, ..., v^n)$ the flows of the hierarchy are systems of conservation laws:

$$v_{t_{(p,l)}} = X_{(p,l)} \circ v_x = \partial_x X_{(p,l+1)}.$$

Bi-flat F-manifolds

Definition

A bi-flat F-manifold is a manifold equipped with two different flat structures $(\nabla, \circ, e)$ and $(\nabla^*, *, E)$ related by the following conditions

1. $X * Y = E^{-1} \circ X \circ Y, \quad \forall X, Y.$
2. $[e, E] = e,$
3. $\mathrm{Lie}_E \circ = \circ,$
4. $(d_{\nabla} - d_{\nabla^*})(X \circ) = 0, \quad \forall X.$

The invariant metric

If ∇ is the Levi-Civita connection of η and η is invariant w.r.t the product:

$$< X \circ Y, Z >=< X, Y \circ Z >, \qquad \forall X, Y, Z,$$

where $< \cdot, \cdot >$ is the bilinear form defined by η, the bi-flat F-manifolds becomes Frobenius manifolds.

Consequence

- $\eta_{il} A^l = \partial_i F$: oriented associativity equations become WDVV associativity equations:

$$\begin{aligned} \partial_j \partial_h \partial_i F \eta^{il} \partial_l \partial_k \partial_m F &= \partial_j \partial_k \partial_i F \eta^{il} \partial_l \partial_h \partial_m F \\ \partial_n \partial_i \partial_j F &= \eta_{ij} \end{aligned}$$

- the principal hierarchy becomes Hamiltonian w.r.t. the Dubrovin-Novikov bracket associated with η. In flat coordinates

$$\eta_{ij} X^j_{(p,l)} = \partial_i h_{(p,l)}$$

and the flows of the principal hierarchies can be written as

$$v_{t_{(p,l)}} = X_{(p,l)} \circ v_x = \partial_x X_{(p,l+1)} = P \delta H_{(p,l+1)}$$

where $H[v] = \int h_{(p,l+1)}(v)\, dx$ and $P^{ij} = \eta^{ij} \partial_x$.

Three dimensional regular case and Painlevé trascendents

Theorem (A.Arsie, P.L. 2015): Three dimensional regular bi-flat F-manifolds are locally parameterized by solutions of the full Painlevé IV, V, and VI equations according to the Jordan canonical form J of $L = E\circ$. More precisely,

- *PVI* in the case

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

- *PV* in the case

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

- *PIV* in the case

$$J = \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}.$$

Kawakami and Mano (2017) proved that PII-PVI appear as special cases in dimension 4.

Complex reflection groups

A complex (pseudo)-reflection is a unitary transformation of $\mathbb{C}^n$ of finite period that leaves invariant a hyperplane. A finite complex reflection group is a finite group generated by complex reflections.

Irreducible finite complex reflection groups were classified by Shephard and Todd, and consist in an infinite family depending on 3 positive integers and 34 exceptional cases $G_i, i = 4..37$. They include finite Coxeter groups and Shephard groups (symmetry groups of regular complex polytopes).

The ring of invariant polynomials of a complex reflection group is generated by n algebraically independent invariant polynomials $(u_1, ..., u_n)$, where n is the dimension of the complex vector space on which the group acts.

Well generated complex reflection groups and Frobenius manifolds without metric

Well-generated complex reflection groups are complex reflection groups of rank n, whose minimal generating set consists of n reflections.

Kato, Mano and Sekiguchy proved that the orbit space of well generated complex reflection group has a flat structure (more precisely it is a Frobenius manifold without metric) and provided the explicit formula for the vector potential in the case $G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, G_{34}$.

Coxeter groups

Let $\{p_1, ..., p_n\}$ be euclidean coordinates and $\{u_1, ..., u_n\}$ be a set of basic polynomial invariants and $d_i = \deg(u^i)$.In the case of Coxeter groups:

1. the flat coordinates for ∇ are basic invariants.
2. In the coordinates $\{u_1, ..., u_n\}$ we have $e = \frac{\partial}{\partial u_n}$.
3. In the coordinates $\{p_1, ..., p_n\}$ we have $E = \sum_{i=1}^{n} p^i \frac{\partial}{\partial p_i}$.
4. the dual product has the form

$$* = \frac{1}{N} \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \pi_H,$$

where α_H is a linear form defining the mirror H, $\mathcal{H}$ is the collection of the reflecting hyperplanes, π_H denotes the orthogonal projection onto the orthogonal complement of the hyperplane H and N is a normalizing factor chosen in such a way that $\sum_{H \in \mathcal{H}} \pi_H = Id$. This is an example of $\vee$-system (Veselov).

Assumptions

We assume 1,2,3 and we substitute 4 with

$$* = \frac{1}{N} \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \sigma_H \pi_H,$$

where

- α_H is a linear form defining a reflecting hyperplane H,
- $\mathcal{H}$ is the collection of the reflecting hyperplanes H,
- π_H is unitary projection onto the unitary complement of H w.r.t. a suitable Hermitian metric,
- the weights σ_H coincide with the order of the reflection that leaves invariant H.
- N is a normalizing factor chosen in such a way that

$$\sum_{H \in \mathcal{H}} \sigma_H \pi_H = Id_{\mathbb{C}^n},$$

Complex reflection groups and bi-flat F-manifolds

We have considered all rank 2 and rank 3 exceptional well generated complex reflection groups, the families $G(m,1,2)$ and $G(m,1,3)$.

- ∇^* has the form

$$\nabla^* = \frac{1}{N}\sum_{H\in\mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \tau_H \pi_H,$$

  and the collection $\{\tau_H\}_{H\in\mathcal{H}}$ of weights is G-invariant (this implies flatness, Looijenga).

- ∇^* might depend on a parameter. In this case also ∇ depend on a parameter. There is a standard choice s.t. $\Gamma^{*s}_{jk}(p) = -c^{*s}_{jk}(p)$. In this case the flat coordinates are the basic invariants satisfying

$$\frac{\partial^2 u^i}{\partial p^j \partial p^k} = (d_i - 1) c^{*s}_{jk} \frac{\partial u^i}{\partial p^s}.$$

A conjecture on Coxeter groups

Conjecture: The number of parameters coincides with the number of orbits for the action of G on the collection of reflecting hyperplanes minus one.

The above conjecture has been verified for Weyl groups of rank 2, 3, and 4 and for the groups $I_2(m)$.

The case of B_3

The dual product has the form

$$c^{*i}_{lp}(p) = \frac{1}{3}\left(\sum_{s=1}^{9} \frac{1}{||\alpha_s||^2} \frac{(\alpha_s)_l\,(\alpha_s)_p\,(\check{\alpha}_s)^i}{\alpha_s(p)}\right)$$

with

$$\alpha_1 = [1,0,0]\,,\ \alpha_2 = [0,1,0]\,,\ \alpha_3 = [0,0,1]\,,\ \alpha_4 = [1,0,-1]\,,\ \alpha_5 = [0,1,-$$
$$\alpha_6 = [1,-1,0]\,,\ \alpha_7 = [1,1,0]\,,\ \alpha_8 = [1,0,1]\,,\ \alpha_9 = [0,1,1]\,.$$

Basic invariants

$$
\begin{aligned}
u_1 &= p_1^2 + p_2^2 + p_3^2, \\
u_2 &= p_1^4 + p_2^4 + p_3^4 + c_1(p_1^2 + p_2^2 + p_3^2)^2, \\
u_3 &= p_1^6 + p_2^6 + p_3^6 + c_2(p_1^2 + p_2^2 + p_3^2)(p_1^4 + p_2^4 + p_3^4) + c_3(p_1^2 + p_2^2 + p_3^2)^3,
\end{aligned}
$$

with $c_1 = \frac{1}{3} + \frac{2}{3}c_2$ and $c_3 = \frac{2}{9}c_2^2$.
The dual connection has the form $\Gamma_{jk}^{(2)i} = -c_{jk}^{*i} + \lambda C_{jk}^i$ where

$$
C_{lp}^i(p) = \sum_{s=1}^{9} \frac{\tau_s}{||\alpha_s||^2} \frac{(\alpha_s)_l\,(\alpha_s)_p\,(\check{\alpha}_s)^i}{\alpha_s(p)}.
$$

with $\tau_s = 2$ if $s = 1, 2, 3$, $\tau_s = -1$ if $s = 4, 5, 6, 7, 8, 9$ and $\lambda = \frac{5}{3} + \frac{4}{3}c_2$.

$$
\begin{aligned}
A^1_{B_3} &= \frac{1}{72}(24c_2^2 + 62c_2 + 40)u_1^4 + \frac{1}{72}(-72c_2 - 90)u_2u_1^2 + u_1u_3 + \frac{3}{8}u_2^2, \\
A^2_{B_3} &= \frac{1}{90}(8c_2^3 + 34c_2^2 + 46c_2 + 20)u_1^5 - \frac{1}{9}(c_2 + 1)u_2u_1^3 \\
&\quad + \frac{1}{90}(-45c_2 - 45)u_2^2u_1 + u_2u_3, \\
A^3_{B_3} &= \frac{1}{2}u_3^2 + \frac{1}{2160}(-128c_2^4 - 640c_2^3 - 1200c_2^2 - 1000c_2 - 312)u_1^6 + \\
&\quad \frac{1}{3}\left(c_2 + \frac{5}{4}\right)(c_2 + 1)\left(c_2 + \frac{3}{2}\right)u_2u_1^4 - \frac{1}{2}(c_2 + 1)\left(c_2 + \frac{3}{2}\right)u_2^2u_1^2 \\
&\quad + \frac{1}{2160}(270c_2 + 405)u_2^3.
\end{aligned}
$$

For $c_2 = -\frac{5}{4}$ we get a Frobenius manifold.

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