Painlevé functions, Fredholm determinants and combinatorics

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PAUL PAINLEVÉ

(1863–1933)

Richard Fuchs

(1873-1944)

BERTRAND GAMBIER

(1879–1954)

Painlevé equations are nonlinear 2nd order ODEs of the form

$$w'' = F(w, w', t)$$

where $F(w, w', t)$ is a rational function of w, w', t.

Their solutions $w\,(t; C_1, C_2)$ satisfy Painlevé property

- $w\,(t; C_1, C_2)$ do not have critical points depending on $C_1,\ C_2$

Examples:

- $w' = w \quad \Longrightarrow \quad w = e^{t-C}$ (essential singularity $t = \infty$)
- $w' = w^2 \quad \Longrightarrow \quad w = \frac{1}{C-t}$ (movable pole)
- $w' = w^3 \quad \Longrightarrow \quad w \sim \frac{1}{\sqrt{t-C}}$ (movable branchpoint)

Classification at order 1 [L. Fuchs, 1884]

The only ODE $w' = F(w, t)$ without movable critical points is the generalized Riccati equation

$$w' = p_2(t)\, w^2 + p_1(t)\, w + p_0(t)\,.$$

Painlevé equations [P. Painlevé & B. Gambier, 1900–1910]:

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)(w')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)w' + \\ + \frac{2w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{w^2} + \frac{\gamma(t-1)}{(w-1)^2} + \frac{\delta t(t-1)}{(w-t)^2}\right) \qquad (\mathrm{P_{VI}})$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \\ + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1}, \qquad (\mathrm{P_{V}})$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4tw^2 + 2\left(t^2 - \alpha\right)w + \frac{\beta}{w}, \qquad (\mathrm{P_{IV}})$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{t} + \frac{\alpha w^2 + \beta}{t} + \gamma w^3 + \frac{\delta}{w}, \qquad (\mathrm{P_{III}})$$

$$w'' = 2w^3 + tw + \alpha, \qquad (\mathrm{P_{II}})$$

$$w'' = 6w^2 + t. \qquad (\mathrm{P_{I}})$$

Isomonodromic deformations (PVI)

Fuchsian system on $\mathbb{P}^1$:

$$\partial_z \Phi = \Phi A(z),$$
$$A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}, \qquad A_\nu \in \mathfrak{sl}_2$$

- 4 regular singular points $0, t, 1, \infty$

Monodromy representation:

$$\rho : \pi_1(\mathbb{P}^1 \backslash \{a\}) \to SL_2(\mathbb{C})$$

Riemann-Hilbert correspondence:

$$\mathcal{RH} : \begin{array}{c}\text{parameter set } \mathcal{P} \\ \text{of the linear system}\end{array} \longrightarrow \begin{array}{c}\text{space } \mathcal{M} \\ \text{of monodromy data}\end{array}$$

Monodromy provides a convenient labeling of Painlevé functions.

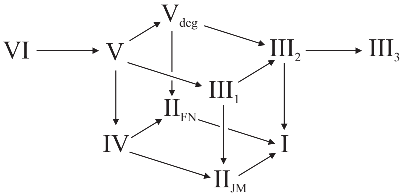
$$\begin{array}{c}\text{solution of} \\ \text{Painlevé equations}\end{array} = \begin{array}{c}\text{construction of} \\ \text{inverse map } \mathcal{RH}^{-1}\end{array}$$

Painlevé VI:

$$\left(t(t-1)\zeta''\right)^2 = -2\det\begin{pmatrix} 2\theta_0^2 & t\zeta'-\zeta & \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 \\ t\zeta'-\zeta & 2\theta_t^2 & (t-1)\zeta'-\zeta \\ \zeta'+\theta_0^2+\theta_t^2+\theta_1^2-\theta_\infty^2 & (t-1)\zeta'-\zeta & 2\theta_1^2 \end{pmatrix}$$

- $\zeta(t) = t(t-1)\frac{d}{dt}\ln\tau$, where $\tau(t)$ is Painlevé VI tau function

Coalescence diagram:



	PVI	PV	$PIII_1$	$PIII_2$	$PIII_3$	PIV	PII	PI
♯(parameters)	4	3	2	1	0	2	1	0

(Special) solutions of Painlevé VI:

1. Hypergeometric Riccati family [Forrester, Witte, '02]

$$\tau_{\mathrm{Riccati}}(t) = (1-t)^{-\frac{N\left(N+\nu+\nu'\right)}{2}} \det\Big[A_{j-k}(t)\Big]_{j,k=0}^{N-1},$$

$$A_m(t) = \frac{\Gamma(1+\nu')\, t^{\frac{\eta-m}{2}}\,(1-t)^{\nu}}{\Gamma(1+\eta-m)\,\Gamma(1-\eta+m+\nu')}\,{}_2F_1\left[\begin{matrix}-\nu, 1+\nu' \\ 1+\eta-m\end{matrix}\,\middle|\,\frac{t}{t-1}\right]$$
$$+\frac{\xi\,\Gamma(1+\nu)\, t^{\frac{m-\eta}{2}}\,(1-t)^{\nu'}}{\Gamma(1-\eta+m)\,\Gamma(1+\eta-m+\nu)}\,{}_2F_1\left[\begin{matrix}1+\nu, -\nu' \\ 1-\eta+m\end{matrix}\,\middle|\,\frac{t}{t-1}\right]$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \frac{1}{2}(\eta, N, -N-\nu-\nu', \nu-\nu'+\eta)$ depend on $\nu, \nu', \eta \in \mathbb{C}$ and $N \in \mathbb{Z}_{\geq 0}$
- 1-parameter family of initial conditions depending on $\xi \in \mathbb{C}$

2. Elliptic Picard family [Kitaev, Korotkin, '98]

$$\tau_{\mathrm{Picard}}(t) = \frac{e^{i\pi\sigma^2\bar{\tau}}}{t^{\frac{1}{8}}(1-t)^{\frac{1}{8}}}\,\frac{\vartheta_3(\sigma\pi\bar{\tau}+\sigma'\pi|\bar{\tau})}{\vartheta_3(0|\bar{\tau})}, \qquad \bar{\tau} = \frac{iK'(t)}{K(t)}$$

- PVI parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$
- 2-parameter family of initial conditions depending on $\sigma, \sigma' \in \mathbb{C}$

3. Algebraic solutions

$$\tau_{H_3'}(t) = \frac{(1-s)^{\frac{1}{20}}\, s^{\frac{1}{20}}\, (1+3s)^{\frac{1}{12}}}{(1+s)^{\frac{3}{20}}\,(1-3s)^{\frac{11}{300}}\,(1+4s-s^2)^{\frac{1}{25}}},$$

$$t = \frac{(s-1)^5(3s+1)^3(s^2+4s-1)}{(s+1)^5(3s-1)^3(s^2-4s-1)}.$$

- $(\theta_0, \theta_t, \theta_1, \theta_\infty) = \left(0, 0, 0, -\frac{1}{5}\right)$, no parameters in the initial conditions
- great icosahedron solution from [Dubrovin, Mazzocco, '98]

4. Fredholm determinant solutions [Borodin, Deift, '01]

$$\tau_{\mathrm{BD}}(t) = \det\left(\mathbf{1} - \lambda K\big|_{(0,t)}\right),$$

where continuous ${}_2F_1$ kernel $K(x,y) = \dfrac{\psi(x)\varphi(y) - \varphi(x)\psi(y)}{x-y}$ is defined by

$$\varphi(x) = x^{\theta_0}(1-x)^{\theta_1}\, {}_2F_1\left[\begin{array}{c}\theta_0+\theta_1+\theta_\infty, \theta_0+\theta_1-\theta_\infty \\ 2\theta_0\end{array}; x\right],$$

$$\psi(x) = x^{1+\theta_0}(1-x)^{\theta_1}\, {}_2F_1\left[\begin{array}{c}1+\theta_0+\theta_1+\theta_\infty, 1+\theta_0+\theta_1-\theta_\infty \\ 2+2\theta_0\end{array}; x\right].$$

- PVI parameters $(\theta_0, \theta_t = 0, \theta_1, \theta_\infty)$
- 1-parameter family of initial conditions depending on $\lambda \in \mathbb{C}$

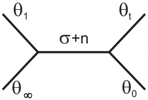
Question 1:

Can the general solution of Painlevé VI be expressed in terms of a Fredholm determinant?

General solution of PVI:

[Gamayun, Iorgov, OL, 1207.0787]

PVI tau function is a Fourier transform of $c = 1$ Virasoro conformal block:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\, \mathcal{B}(\vec{\theta}, \sigma + n, t) = \sum_{n\in\mathbb{Z}} e^{in\eta}$$  (t)

- $\mathcal{B}(\vec{\theta}, \sigma, t) = t^{\alpha} \sum_{k=0}^{\infty} B_k(\vec{\theta}, \sigma)\, t^k$, with B_k rational in $\vec{\theta}, \sigma$ and determined by commutation relations of Vir
- as $c \to \infty$ (Vir $\to \mathfrak{sl}_2$), conformal block $\mathcal{B}(t) \sim {}_2F_1(t)$
- all 4 parameters $(\theta_0, \theta_t, \theta_1, \theta_\infty) \Longleftrightarrow$ external momenta
- 2 integration constants $(\sigma, \eta) \Longleftrightarrow$ internal momentum + Fourier conjugate variable

CFT derivations:

[Iorgov, OL, Teschner, 1401.6104]

- understood in the framework of Liouville CFT and generalized to an arbitrary number of punctures (Garnier system)
- uses quantum monodromy of conformal blocks with additional level 2 degenerate insertions

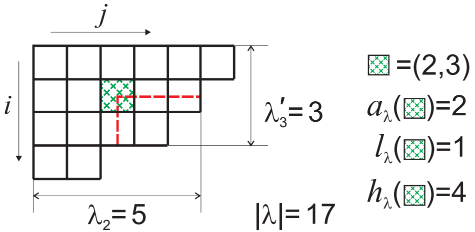
[Bershtein, Shchechkin, 1406.3008]

- bilinear differential-difference equations for conformal blocks coming from an embedding $\mathrm{Vir} \oplus \mathrm{Vir} \subset \mathrm{NSR} \oplus \mathrm{F}$
- extends to arbitrary values of central charge c

AGT correspondence [Alday, Gaiotto, Tachikawa, '09]

$$\mathcal{B}(t) = \mathcal{Z}_{\rm inst}(t) = \begin{array}{c}\text{combinatorial sum}\\ \text{over tuples of partitions}\end{array}$$ [Nekrasov, '04]

- coefficients of $\mathcal{B}(t)$ are explicit rational functions of parameters determined by geometry of appropriate Young diagram
- proved in [Alba, Fateev, Litvinov, Tarnopolsky, '10]
- provides series representation for general Painlevé VI function!



Young diagram associated to partition
$\lambda = \{6, 5, 4, 2\}$.

Conjecture [Gamayun, Iorgov, OL, 1207.0787]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta}\mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2}(1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta},\sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta},\sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j-i+\mu_i-j+1+2\sigma\right)^2} \times$$

$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j-i+\lambda_i-j+1-2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Question 2:

How to understand this combinatorial structure without reference to CFT/gauge theory ?

$$\tau(t) \sim \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n)\, t^{(\sigma+n)^2+|\lambda|+|\mu|}$$

Question 2:

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$$\tau(t) \sim \sum_{n\in\mathbb{Z}} e^{in\eta} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}(\sigma+n)\, t^{(\sigma+n)^2+|\lambda|+|\mu|}$$

Remark: Similar Fourier transform structure also appears for

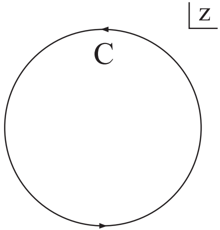
- regular type expansions for PV, $\mathrm{PIII}_{1,2,3}$ at $t=0$ (irregular CBs)
- irregular expansions for PI–PV "nonlinear" Stokes rays at $t=\infty$

A digression on (block) Toeplitz determinants

- symbol $J(z) \in \mathrm{Hom}(\mathcal{C}, \mathrm{GL}_N(\mathbb{C}))$
- continues into an annulus $\mathcal{A} \supset \mathcal{C}$

$$J(z) = \sum_{k \in \mathbb{Z}} J_k z^k,$$

- $\det J(z) = 1$



(Block) Toeplitz matrix associated to J :

$$T_K[J] = \begin{pmatrix} J_0 & J_{-1} & J_{-2} & \dots & J_{-K} \\ J_1 & J_0 & J_{-1} & \dots & J_{-K+1} \\ J_2 & J_1 & J_0 & \dots & J_{-K+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_K & J_{K-1} & J_{K-2} & \dots & J_0 \end{pmatrix}.$$

1st Widom's theorem ['76]

The limit $W[J] = \lim_{K\to\infty} \det T_K[J]$ exists and is equal to

$$W[J] = \det_{\mathcal{V}_+} \Pi_+ J^{-1} \Pi_+ J \Pi_+$$

- $N = 1 \implies$ strong Szegö limit theorem

Assume that J admits factorizations

$$J(z) = \Psi^{int}(z)^{-1}\Psi^{ext}(z) = \tilde{\Psi}_+(z)\,\tilde{\Psi}_-(z)^{-1}$$

where $\Psi^{int}(z)$, $\tilde{\Psi}_-(z)$ and $\Psi^{ext}(z)$, $\tilde{\Psi}_+(z)$ are analytic, respectively, outside and inside $\mathcal{C}$.

Remark. For symbols admitting 1st factorization, the Widom's constant $W[J]$ may be rewritten as

$$W[J] = \det_{\mathcal{V}}(\mathbf{1}+U), \qquad U = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \mathrm{End}(\mathcal{V}_+ \oplus \mathcal{V}_-),$$

where the operators $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$, $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ are defined by

$$\mathsf{a} = \Psi^{ext}\Pi_+\Psi^{ext\,-1} - \Pi_+, \qquad d = \Psi^{int}\Pi_-\Psi^{int\,-1} - \Pi_-.$$

They thus have integrable kernels

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{a}(z,z')\,g(z')\,dz', \quad \mathsf{a}(z,z') = \frac{\Psi^{ext}(z)\,\Psi^{ext}(z')^{-1} - \mathbf{1}}{z-z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{d}(z,z')\,g(z')\,dz', \quad \mathsf{d}(z,z') = \frac{\mathbf{1} - \Psi^{int}(z)\,\Psi^{int}(z')^{-1}}{z-z'}.$$

2nd Widom's theorem ['74]

Suppose that $J(z)$ smoothly depends on an additional parameter t.

For symbols admitting left and right factorizations, the log-derivatives of the Widom's constant wrt parameters are given by

$$\partial_t \ln W[J] = \frac{1}{2\pi i} \oint_C \mathrm{Tr}\left(J^{-1} \partial_t J \left[\partial_z \left(\tilde{\Psi}_- \right) \tilde{\Psi}_-^{-1} + \Psi^{ext\,-1} \partial_z \left(\Psi^{ext} \right) \right] \right) dz.$$

- Widom's constant $=$ tau function
- [Malgrange, Bertola, Cafasso]

von Koch formula

Write U in the Fourier basis and expand Fredholm determinant using

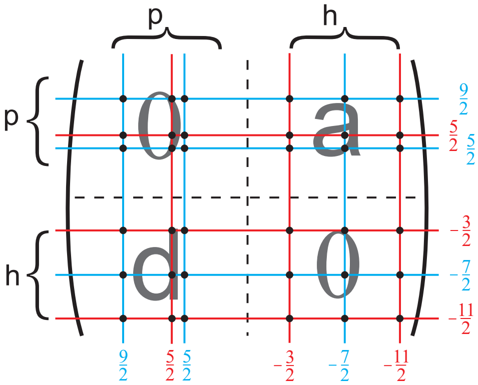
$$\det(\mathbf{1}+U) = \sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}} \det U_{\mathfrak{Y}}, \qquad U \in \mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$$

$$= 1 + \sum_{m\in\mathfrak{X}} U_{mm} + \frac{1}{2!}\sum_{m,n\in\mathfrak{X}} \begin{vmatrix} U_{mm} & U_{mn} \\ U_{nm} & U_{nn} \end{vmatrix} + \ldots$$

- multi-indices of principal minors

$$\det U_{\mathfrak{Y}} = \det \begin{pmatrix} 0 & \mathsf{a}_{\mathsf{h}}^{\mathsf{p}} \\ \mathsf{d}_{\mathsf{p}}^{\mathsf{h}} & 0 \end{pmatrix}$$

incorporate color indices $\alpha = 1,\ldots N$ and (half-)integer Fourier indices



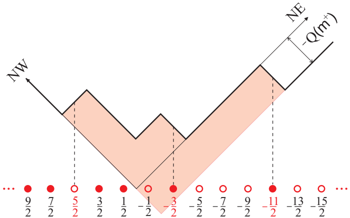
- combinatorial expansion

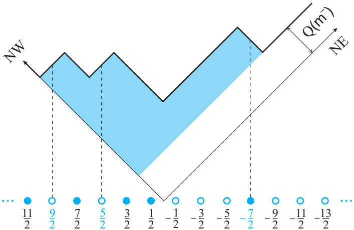
$$\det(\mathbf{1}+U)=\sum_{(\mathsf{p},\mathsf{h})}(-1)^{|\mathsf{p}|}\det \mathsf{a}_{\mathsf{h}}^{\mathsf{p}}\det \mathsf{d}_{\mathsf{p}}^{\mathsf{h}},$$

  with balance condition $|\mathsf{p}|=|\mathsf{h}|$

- Fourier indices in p and h are resp. positive and negative

- A Maya diagram is a map $\mathrm{m} : \mathbb{Z}' \to \{-1, 1\}$ subject to the condition $\mathrm{m}(p) = \pm 1$ for all but finitely many $p \in \mathbb{Z}'_{\pm}$ (positions of particles and holes)
- $\mathrm{charge}(\mathrm{m}) = \sharp(\mathrm{particles}) - \sharp(\mathrm{holes})$
- Maya diagram = charged partition/Young diagram





- balanced configurations (p, h) are in bijection with N-tuples of Young diagrams of zero total charge
- $\mathbb{M}_0^N \cong \mathbb{Y}^N \times \mathfrak{Q}_N$, where $\mathfrak{Q}_N$ denotes the A_{N-1} root lattice:

$$\mathfrak{Q}_N := \left\{ \vec{Q} \in \mathbb{Z}^N \,\middle|\, \sum_{\alpha=1}^{N} Q_\alpha = 0 \right\}.$$

- in the case $N = 2$, Widom's constant

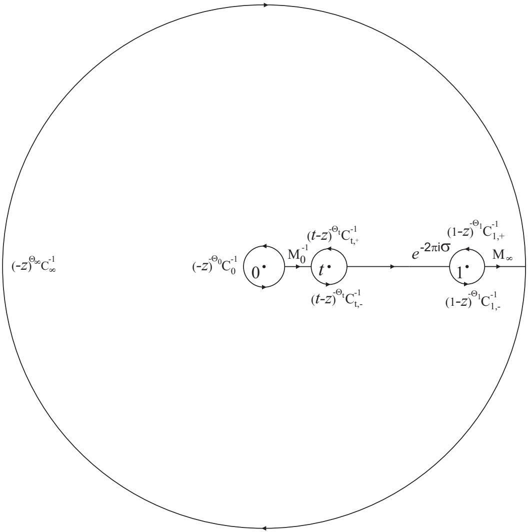
$$\det(\mathbf{1} + U) = \sum_{(\lambda,\mu;n)\in\mathbb{Y}^2\times\mathbb{Z}} (-1)^{|\mathsf{p}|} \det \mathsf{a}_{\lambda,\mu,n} \det \mathsf{d}_{\lambda,\mu,n}$$

Scheme of the proof

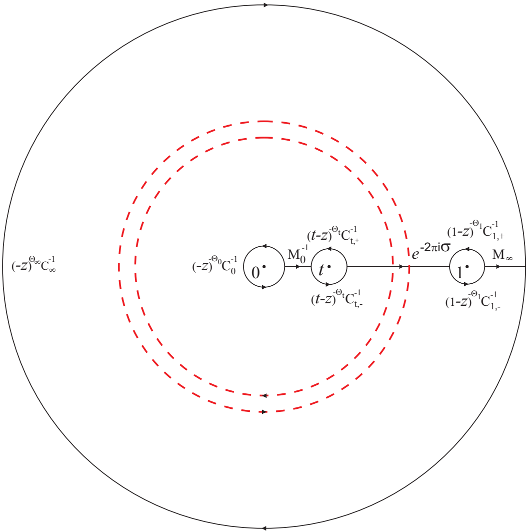
Step 1: Represent the isomonodromic tau function in the form of Fredholm determinant

- arbitrary rank N, $n = 4$ regular singularities $0, t, 1, \infty$

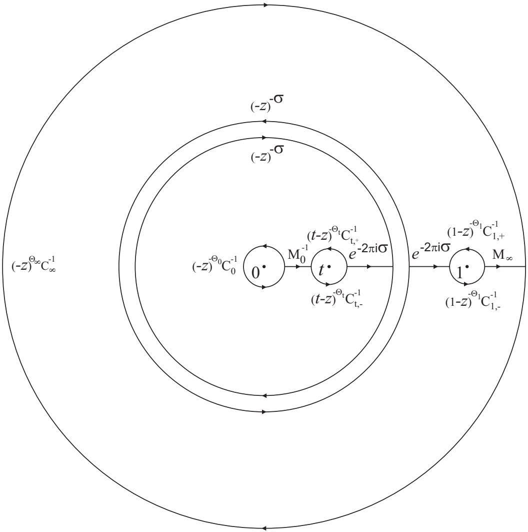
Isomonodromic RHP I



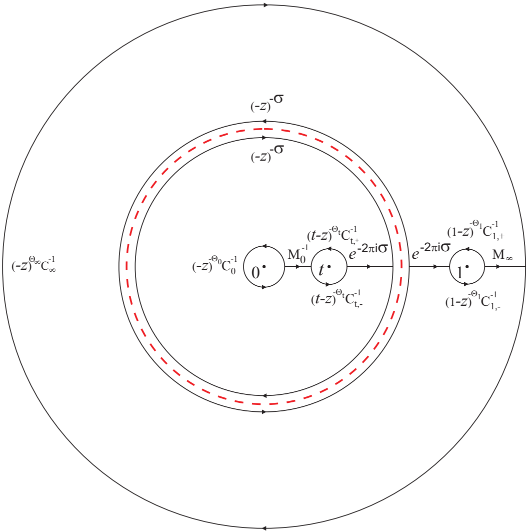
Isomonodromic RHP I



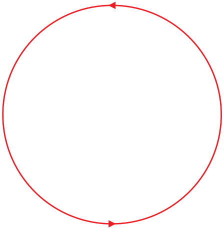
Isomonodromic RHP II



Isomonodromic RHP II



Isomonodromic RHP III



- For a circle $\mathcal{C} \subset \mathcal{A}$ define

$$\tilde{\Psi}(z) = \begin{cases} \Psi^{ext}(z)^{-1}\hat{\Psi}(z), & \text{outside } \mathcal{C}, \\ \Psi^{int}(z)^{-1}\hat{\Psi}(z), & \text{inside } \mathcal{C}. \end{cases}$$

- contour $\tilde{\Gamma} = \mathcal{C}$ (single circle !!!), jump $J : \mathcal{C} \to \mathrm{GL}(N, \mathbb{C})$ is

$$J(z) = \Psi^{int}(z)^{-1}\Psi^{ext}(z) = \tilde{\Psi}_+(z)\tilde{\Psi}_-(z)^{-1}$$

2nd Widom's theorem then implies that

$$\tau_{\mathrm{JMU}}(t) = t^{\frac{1}{2}\operatorname{Tr}\left(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2\right)} \det\left(\mathbf{1} + K\right),$$

where

$$K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix} \in \operatorname{End}(\mathcal{V})$$

with $\mathsf{a} : \mathcal{V}_- \to \mathcal{V}_+$ and $\mathsf{d} : \mathcal{V}_+ \to \mathcal{V}_-$ given by

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{a}(z,z')\, g(z')\, dz', \quad \mathsf{a}(z,z') = \frac{\Psi^{ext}(z)\,\Psi^{ext}(z')^{-1} - \mathbf{1}}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \mathsf{d}(z,z')\, g(z')\, dz', \quad \mathsf{d}(z,z') = \frac{\mathbf{1} - \Psi^{int}(z)\,\Psi^{int}(z')^{-1}}{z - z'}.$$

- tau function for 4-point Fuchsian system written via 3-point solutions
- hypergeometric representations for $N = 2$

For $N=2$:

$$\mathrm{a}\left(z,z'\right)=\frac{(1-z')^{2\theta_1}\begin{pmatrix} K_{++}(z) & K_{+-}(z) \\ K_{-+}(z) & K_{--}(z)\end{pmatrix}\begin{pmatrix} K_{--}(z') & -K_{+-}(z') \\ -K_{-+}(z') & K_{++}(z')\end{pmatrix}-\mathbf{1}}{z-z'},$$

$$\mathrm{d}\left(z,z'\right)=\frac{\mathbf{1}-(1-\frac{t}{z'})^{2\theta_t}\begin{pmatrix} \bar{K}_{++}(z) & \bar{K}_{+-}(z) \\ \bar{K}_{-+}(z) & \bar{K}_{--}(z)\end{pmatrix}\begin{pmatrix} \bar{K}_{--}(z') & -\bar{K}_{+-}(z') \\ -\bar{K}_{-+}(z') & \bar{K}_{++}(z')\end{pmatrix}}{z-z'},$$

with

$$K_{\pm\pm}(z)={}_2F_1\left[\begin{matrix}\theta_1+\theta_\infty\pm\sigma,\theta_1-\theta_\infty\pm\sigma \\ \pm 2\sigma\end{matrix};z\right],$$

$$K_{\pm\mp}(z)=\pm\frac{\theta_\infty^2-(\theta_1\pm\sigma)^2}{2\sigma(1\pm 2\sigma)}\,z\,{}_2F_1\left[\begin{matrix}1+\theta_1+\theta_\infty\pm\sigma,1+\theta_1-\theta_\infty\pm\sigma \\ 2\pm 2\sigma\end{matrix};z\right],$$

$$\bar{K}_{\pm\pm}(z)={}_2F_1\left[\begin{matrix}\theta_t+\theta_0\mp\sigma,\theta_t-\theta_0\mp\sigma \\ \mp 2\sigma\end{matrix};\frac{t}{z}\right],$$

$$\bar{K}_{\pm\mp}(z)=\mp\,t^{\mp 2\sigma}e^{\mp i\eta}\frac{\theta_0^2-(\theta_t\mp\sigma)^2}{2\sigma(1\mp 2\sigma)}\,\frac{t}{z}\,{}_2F_1\left[\begin{matrix}1+\theta_t+\theta_0\mp\sigma,1+\theta_t-\theta_0\mp\sigma \\ 2\mp 2\sigma\end{matrix};\frac{t}{z}\right].$$

Step 2: Write K in the Fourier basis and expand Fredholm determinant using von Koch formula:

$$\det\left(\mathbf{1}+K\right)=\sum_{\mathfrak{Y}\in 2^{\mathfrak{X}}}\det K_{\mathfrak{Y}}\,,\qquad K\in\mathbb{C}^{\mathfrak{X}\times\mathfrak{X}}$$

We have seen that e.g. for $N=2$

$$\det\left(\mathbf{1}+K\right)=\sum_{(\lambda,\mu;n)\in\mathbb{Y}^{2}\times\mathbb{Z}}(-1)^{|\mathbf{p}|}\det \mathsf{a}_{\lambda,\mu,n}\det \mathsf{d}_{\lambda,\mu,n}$$

Step 3: Explicit computation of elementary determinants $\det \mathrm{a}_{\mathrm{h}}^{\mathrm{p}}$, $\det \mathrm{d}_{\mathrm{p}}^{\mathrm{h}}$ of Plemelj operators

- in the case $N = 2 \Longrightarrow$ Cauchy determinants $\det \dfrac{1}{x_i - y_j}$
- rewrite resulting factorized expressions using lengths of rows/columns instead of positions of particles/holes of different colors

Theorem [Gavrylenko, OL, 1608.00958]

Complete expansion of Painlevé VI tau function at $t = 0$ is given by

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{in\eta} \mathcal{B}(\vec{\theta}, \sigma + n; t),$$

where the function $\mathcal{B}(\vec{\theta}, \sigma; t)$ is explicitly given by

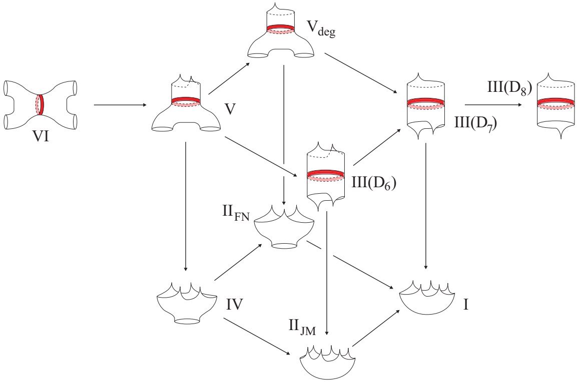
$$\mathcal{B}\left(\vec{\theta}, \sigma; t\right) = N^{\theta_1}_{\theta_\infty,\sigma} N^{\theta_t}_{\sigma,\theta_0} t^{\sigma^2-\theta_0^2-\theta_t^2} (1-t)^{2\theta_t\theta_1} \sum_{\lambda,\mu\in\mathbb{Y}} \mathcal{B}_{\lambda,\mu}\left(\vec{\theta}, \sigma\right) t^{|\lambda|+|\mu|},$$

$$\mathcal{B}_{\lambda,\mu}(\boldsymbol{\theta}, \sigma) = \prod_{(i,j)\in\lambda} \frac{\left((\theta_t+\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1+\sigma+i-j)^2-\theta_\infty^2\right)}{h_\lambda^2(i,j)\left(\lambda'_j - i + \mu_i - j + 1 + 2\sigma\right)^2} \times$$

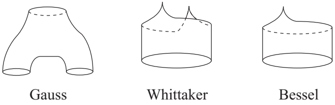
$$\times \prod_{(i,j)\in\mu} \frac{\left((\theta_t-\sigma+i-j)^2-\theta_0^2\right)\left((\theta_1-\sigma+i-j)^2-\theta_\infty^2\right)}{h_\mu^2(i,j)\left(\mu'_j - i + \lambda_i - j + 1 - 2\sigma\right)^2},$$

$$N^{\theta_2}_{\theta_3,\theta_1} = \frac{\prod_{\epsilon=\pm} G\left(1+\theta_3+\epsilon(\theta_1+\theta_2)\right) G\left(1-\theta_3+\epsilon(\theta_1-\theta_2)\right)}{G(1-2\theta_1)G(1-2\theta_2)G(1+2\theta_3)}.$$

Other Painlevé equations



Chekhov-Mazzocco-Rubtsov confluence diagram



Some solvable RHPs in rank $N = 2$

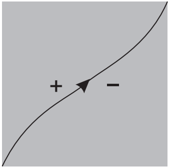
General case

Step 1: Represent the tau function of the Schlesinger system in the form of Fredholm determinant

- arbitrary rank N, arbitrary number n of regular singularities

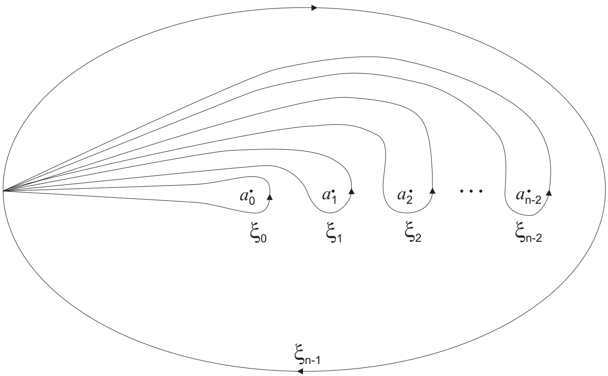
Riemann-Hilbert setup

- contour Γ on a Riemann surface Σ
- jump matrix $J : \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$



RHP defined by (Γ, J) is to find analytic invertible matrix function $\Psi : \Sigma \backslash \Gamma \to \mathrm{GL}\,(N, \mathbb{C})$ whose boundary values satisfy

$$\Psi_+ = J\Psi_-$$

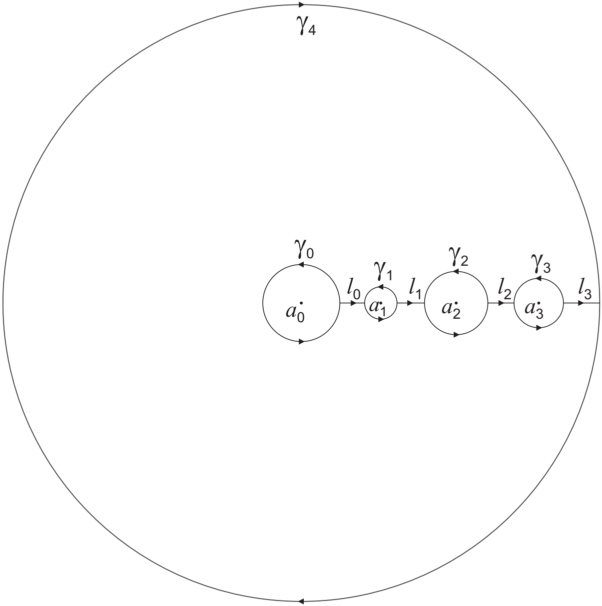


Monodromy representation $\rho : \pi_1\left(\mathbb{P}^1\backslash a\right) \to \mathrm{GL}\left(N, \mathbb{C}\right)$ generated by

$$M_k = \rho\left(\xi_k\right) = M_{1\to k-1}{}^{-1} M_{1\to k}$$

Assume that all $M_{1\to k} = M_1 \ldots M_k$ are diagonalizable,

$$M_{1\to k} = S_k e^{2\pi i \mathfrak{S}_k} S_k^{-1}, \qquad \mathfrak{S}_k = \mathrm{diag}\left\{\sigma_{k,1}, \ldots, \sigma_{k,N}\right\}.$$



Contour Γ for $n = 5$

Fundamental matrix solution

$$\Phi(z) = \begin{cases} \Psi(z), & z \text{ outside } \gamma_{1\ldots n}, \\ C_k (a_k - z)^{\Theta_k} \Psi(z), & z \text{ inside } \gamma_k, \quad k = 1, \ldots, n-1, \\ C_n (-z)^{-\Theta_n} \Psi(z), & z \text{ inside } \gamma_n. \end{cases}$$

- only piecewise constant jumps on $\mathbb{R}_{>0}$
- matrix $\Phi^{-1}\partial_z\Phi$ meromorphic on $\mathbb{P}^1$ with poles only possible at $a_1, \ldots, a_n$
- local analysis shows that

$$\partial_z\Phi = \Phi A(z), \qquad A(z) = \sum_{k=1}^{n} \frac{A_k}{z - a_k}$$

with $A_k = \Psi(a_k)^{-1}\Theta_k\Psi(a_k)$

Jump data

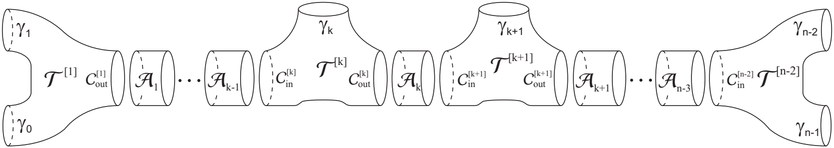
- local exponents: n diagonal non-resonant $N \times N$ matrices $\Theta_k = \operatorname{diag}\{\theta_{k,1}, \ldots, \theta_{k,N}\}$ $(k = 1, \ldots, n)$ satisfying a consistency relation $\sum_{k=1}^{n} \operatorname{Tr} \Theta_k = 0$
- $2n$ connection matrices $C_{k,\pm} \in \mathrm{GL}(N, \mathbb{C})$ satisfying the constraints

$$
\begin{aligned}
M_{1\to k} &:= C_{k,-} e^{2\pi i \Theta_k} C_{k,+}^{-1} = C_{k+1,-} C_{k+1,+}^{-1}, \qquad k = 1, \ldots n-2, \\
M_{1\to n-1} &:= C_{n-1,-} e^{2\pi i \Theta_{n-1}} C_{n-1,+}^{-1} = C_{n,-} e^{-2\pi i \Theta_n} C_{n,+}^{-1}, \\
M_{1\to n} &:= \mathbf{1} = C_{n,-} C_{n,+}^{-1} = C_{1,-} C_{1,+}^{-1},
\end{aligned}
$$

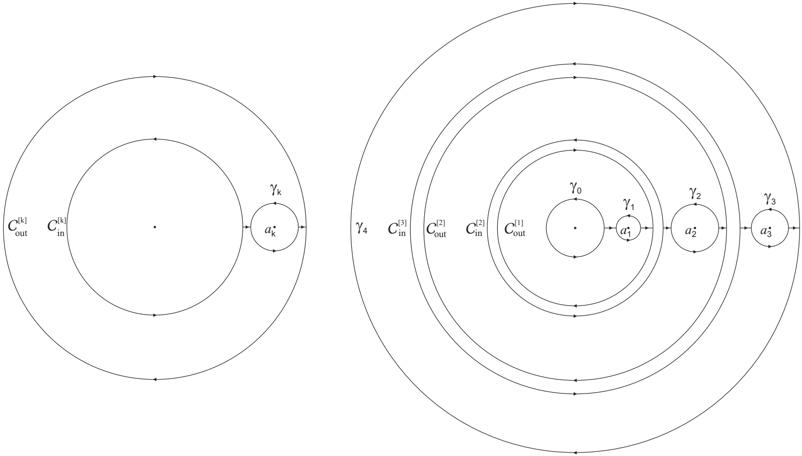
Jump matrix J

$$
\begin{aligned}
J(z)\Big|_{\ell_k} &= M_{1\to k}^{-1}, \qquad k = 1, \ldots, n-1, \\
J(z)\Big|_{\gamma_k} &= (a_k - z)^{-\Theta_k} C_{k,\pm}^{-1}, \qquad \Im z \gtrless 0, \quad k = 1, \ldots, n-1, \\
J(z)\Big|_{\gamma_n} &= (-z)^{\Theta_n} C_{n,\pm}^{-1}, \qquad \Im z \gtrless 0.
\end{aligned}
$$

Auxiliary 3-point RHPs



- we are going to associate to the n-point RHP $n-2$ 3-point RHPs assigned to different trinions



Contour $\Gamma^{[k]}$ (left) and $\hat{\Gamma}$ for $n = 5$ (right)

- $\hat{\Psi}(z) = \begin{cases} \Psi(z) & \text{outside the annuli,} \\ (-z)^{-\mathfrak{S}_k} S_k^{-1} \Psi(z) & \text{inside.} \end{cases}$
- jumps on the boundary circles $\mathcal{C}^{\text{out}}_{k-1}$, $\mathcal{C}^{\text{in}}_{k}$ mimic regular singularities characterized by counterclockwise monodromies $M_{1 \to k}$

Cauchy-Plemelj operators

- associate to every trinion $\mathcal{T}_k$ with $k = 2, \ldots, n-3$ the spaces of vector-valued functions

$$\mathcal{H}^{[k]} = \bigoplus_{\epsilon=\mathrm{in},\mathrm{out}} \left(\mathcal{H}^{[k]}_{\epsilon,+} \oplus \mathcal{H}^{[k]}_{\epsilon,-} \right), \qquad \mathcal{H}^{[k]}_{\epsilon,\pm} = \mathbb{C}^N \otimes \mathcal{V}_\pm \left(\mathcal{C}^{\epsilon}_k \right).$$

- elements $f^{[k]} \in \mathcal{H}^{[k]}$ will be written as

$$f^{[k]} = \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix}.$$

- define an operator $\mathcal{P}^{[k]} : \mathcal{H}^{[k]} \to \mathcal{H}^{[k]}$ by

$$\mathcal{P}^{[k]} f^{[k]} (z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k \cup \mathcal{C}^{\mathrm{out}}_k} \frac{\Psi^{[k]}_+ (z) \, \Psi^{[k]}_+ (z')^{-1} f^{[k]} (z') \, dz'}{z - z'}$$

Lemma. We have $\left(\mathcal{P}^{[k]}\right)^2 = \mathcal{P}^{[k]}$ and $\ker \mathcal{P}^{[k]} = \mathcal{H}^{[k]}_{\mathrm{in},+} \oplus \mathcal{H}^{[k]}_{\mathrm{out},-}$. Moreover, $\mathcal{P}^{[k]}$ can be explicitly written as

$$\mathcal{P}^{[k]} : \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} f^{[k]}_{\mathrm{in},+} \\ f^{[k]}_{\mathrm{out},-} \end{pmatrix} \mapsto \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a}^{[k]} & \mathsf{b}^{[k]} \\ \mathsf{c}^{[k]} & \mathsf{d}^{[k]} \end{pmatrix} \begin{pmatrix} f^{[k]}_{\mathrm{in},-} \\ f^{[k]}_{\mathrm{out},+} \end{pmatrix},$$

where the operators $\mathsf{a}^{[k]}$, $\mathsf{b}^{[k]}$, $\mathsf{c}^{[k]}$, $\mathsf{d}^{[k]}$ are defined by

$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{in}}_k,$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{in}}_k} \Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k,$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}^{\mathrm{out}}_k} \left[\Psi^{[k]}_+(z)\, \Psi^{[k]}_+(z')^{-1} - \mathbf{1}\right] \frac{g(z')\, dz'}{z - z'}, \qquad z \in \mathcal{C}^{\mathrm{out}}_k.$$

- introduce the total space

$$\mathcal{H} := \bigoplus_{k=1}^{n-2} \mathcal{H}^{[k]}.$$

- there is a splitting

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

$$\mathcal{H}_\pm := \mathcal{H}^{[1]}_{\mathrm{out},\pm} \oplus \left(\mathcal{H}^{[2]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[2]}_{\mathrm{out},\pm}\right) \oplus \ldots \oplus \left(\mathcal{H}^{[n-3]}_{\mathrm{in},\mp} \oplus \mathcal{H}^{[n-3]}_{\mathrm{out},\pm}\right) \oplus \mathcal{H}^{[n-2]}_{\mathrm{in},\mp}.$$

- combine the 3-point projections $\mathcal{P}^{[k]}$ into an operator $\mathcal{P}_\oplus : \mathcal{H} \to \mathcal{H}$ given by the direct sum

$$\mathcal{P}_\oplus = \mathcal{P}^{[1]} \oplus \ldots \oplus \mathcal{P}^{[n-2]}.$$

- similarly, define another projection $\mathcal{P}_\Sigma : \mathcal{H} \to \mathcal{H}$ by

$$\mathcal{P}_\Sigma f(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_\Sigma} \frac{\hat{\Psi}_+(z)\, \hat{\Psi}_+(z')^{-1} f(z')\, dz'}{z - z'}, \quad \mathcal{C}_\Sigma := \bigcup_{k=1}^{n-3} \mathcal{C}_k^{\mathrm{out}} \cup \mathcal{C}_{k+1}^{\mathrm{in}}.$$

- it is easy to show that $\mathcal{P}_\Sigma \mathcal{P}_\oplus = \mathcal{P}_\oplus$ and $\mathcal{P}_\oplus \mathcal{P}_\Sigma = \mathcal{P}_\Sigma$
- the space
$$\mathcal{H}_\mathcal{T} := \operatorname{im} \mathcal{P}_\oplus = \operatorname{im} \mathcal{P}_\Sigma.$$
can be thought of as the subspace of functions on the union of boundary circles $\mathcal{C}_k^{\mathrm{in}}$, $\mathcal{C}_k^{\mathrm{out}}$ that can be continued inside $\bigcup_{k=1}^{n-2} \mathcal{T}_k$ with monodromy and singular behavior of the n-point fundamental matrix solution $\Phi(z)$
- varying the positions of singular points, one obtains a trajectory of $\mathcal{H}_\mathcal{T}$ in the infinite-dimensional Grassmannian $\mathrm{Gr}(\mathcal{H})$ defined with respect to the splitting $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$
- each of the subspaces $\mathcal{H}_\pm$ may be identified with $N(n-3)$ copies of the space $L^2(S^1)$ of functions on a circle; the factor $n-3$ corresponds to the number of annuli and N is the rank of the appropriate RHP

- introduce operators $\mathcal{P}_{\oplus,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ and $\mathcal{P}_{\Sigma,+} : \mathcal{H}_+ \to \mathcal{H}_\mathcal{T}$ given by restrictions of $\mathcal{P}_\oplus$ and $\mathcal{P}_\Sigma$ to $\mathcal{H}_+$
- define $L \in \mathrm{End}\,(\mathcal{H}_+)$ defined by

$$L := \mathcal{P}_{\oplus,+}{}^{-1}\mathcal{P}_{\Sigma,+}$$

- there exists a basis in which $L^{-1} = \mathbf{1} - K$, with

$$K = \begin{pmatrix} U_1 & V_1 & 0 & . & 0 \\ W_1 & U_2 & V_2 & . & 0 \\ 0 & W_2 & U_3 & . & . \\ . & . & . & . & V_{n-4} \\ 0 & 0 & . & W_{n-4} & U_{n-3} \end{pmatrix}, \quad \vec{g} = \begin{pmatrix} \tilde{g}_1 \\ \tilde{g}_2 \\ \vdots \\ \tilde{g}_{n-3} \end{pmatrix}, \quad \tilde{g}_k = \begin{pmatrix} g^{[k]}_{\mathrm{out},+} \\ g^{[k+1]}_{\mathrm{in},-} \end{pmatrix},$$

$$U_k = \begin{pmatrix} 0 & \mathrm{a}^{[k+1]} \\ \mathrm{d}^{[k]} & 0 \end{pmatrix}, \quad V_k = \begin{pmatrix} \mathrm{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{c}^{[k+1]} \end{pmatrix}$$

Definition

The tau function associated to the Riemann-Hilbert problem for Ψ is defined as

$$\tau(a) := \det\left(L^{-1}\right)$$

Theorem

We have

$$\tau(a) = \Upsilon(a)^{-1}\tau_{\mathrm{JMU}}(a),$$

where $\tau_{\mathrm{JMU}}(a)$ is defined up to a prefactor independent of a by

$$d_a \ln \tau_{\mathrm{JMU}} = \sum_{1\leq k<l\leq n-1} \operatorname{Tr} A_k A_l \, d\ln(a_k - a_l),$$

and $\Upsilon(a) = \prod_{k=2}^{n-2} a_k^{\bar{\Delta}_k - \bar{\Delta}_{k-1} - \Delta_k}$, with $\Delta_k = \frac{1}{2}\operatorname{Tr}\Theta_k^2$, $\bar{\Delta}_k = \frac{1}{2}\operatorname{Tr}\mathfrak{S}_k^2$

Fourier basis

Let us represent the elements of $\mathcal{H}_{\mathcal{C}}$ by their Laurent series inside $\mathcal{A}$,

$$f(z) = \sum_{p\in\mathbb{Z}'} f^p z^{-\frac{1}{2}+p}, \qquad f^p \in \mathbb{C}^N,$$

and write integral kernels of 3-point projection operators $\mathrm{a}^{[k]}$, $\mathrm{b}^{[k]}$, $\mathrm{c}^{[k]}$, $\mathrm{d}^{[k]}$ as

$$\mathrm{a}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1} - \mathbf{1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{a}^{[k]}{}_{-q}^{\;p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}, \qquad z,z' \in \mathcal{C}_k^{\mathrm{in}},$$

$$\mathrm{b}^{[k]}(z,z') := -\frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{b}^{[k]}{}_{q}^{p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}-q}, \quad z\in\mathcal{C}_k^{\mathrm{in}}, z'\in\mathcal{C}_k^{\mathrm{out}}$$

$$\mathrm{c}^{[k]}(z,z') := \frac{\Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{c}^{[k]}{}_{-q}^{-p}\, z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}+q}, \; z\in\mathcal{C}_k^{\mathrm{out}}, z'\in\mathcal{C}_k^{\mathrm{in}}$$

$$\mathrm{d}^{[k]}(z,z') := \frac{\mathbf{1} - \Psi_+^{[k]}(z)\,\Psi_+^{[k]}(z')^{-1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_+} \mathrm{d}^{[k]}{}^{-p}{}_{q} z^{-\frac{1}{2}-p} z'^{-\frac{1}{2}-q}, \quad z,z' \in \mathcal{C}_k^{\mathrm{out}}.$$

Principal minor

$$
\begin{pmatrix}
0 & \left(\mathsf{a}^{[2]}\right)^{I_1}_{J_1} & \left(\mathsf{b}^{[2]}\right)^{I_1}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
\left(\mathsf{d}^{[1]}\right)^{J_1}_{I_1} & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & \left(\mathsf{a}^{[3]}\right)^{I_2}_{J_2} & \left(\mathsf{b}^{[3]}\right)^{I_2}_{I_3} & \cdot & \cdot & 0 & 0 \\
0 & \left(\mathsf{c}^{[2]}\right)^{J_2}_{J_1} & \left(\mathsf{d}^{[2]}\right)^{J_2}_{I_2} & 0 & 0 & \cdot & \cdot & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \left(\mathsf{c}^{[3]}\right)^{J_3}_{J_2} & \left(\mathsf{d}^{[3]}\right)^{J_3}_{I_3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \left(\mathsf{b}^{[n-3]}\right)^{I_{n-2}}_{I_{n-3}} & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot & 0 & 0 & \\
0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & \left(\mathsf{a}^{[n-2]}\right)^{I_{n-3}}_{J_{n-3}} \\
0 & 0 & 0 & 0 & \cdot & \cdot & \left(\mathsf{c}^{[n-3]}\right)^{J_{n-3}}_{J_{n-4}} & \left(\mathsf{d}^{[n-3]}\right)^{J_{n-3}}_{I_{n-3}} & 0
\end{pmatrix}
$$

- vanishes unless balance condition $|I_k| = |J_k|$ is satisfied
- factorization into a product of elementary determinants

$$
Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right) := (-1)^{|I_k|} \det \begin{pmatrix} \left(\mathsf{a}^{[k]}\right)^{I_{k-1}}_{J_{k-1}} & \left(\mathsf{b}^{[k]}\right)^{I_{k-1}}_{I_k} \\ \left(\mathsf{c}^{[k]}\right)^{J_k}_{J_{k-1}} & \left(\mathsf{d}^{[k]}\right)^{J_k}_{I_k} \end{pmatrix}
$$

Corollary: Fredholm determinant $\tau(a)$ is given by

$$\tau(a) = \sum_{(\vec{I},\vec{J})\in \mathrm{Conf}_+} \prod_{k=1}^{n-2} Z^{I_{k-1},J_{k-1}}_{I_k,J_k}\left(\mathcal{T}^{[k]}\right)$$

- The set Conf_+ of proper balanced configurations $\left(\vec{I},\vec{J}\right)$ may be described in terms of Maya diagrams and charged partitions
- balanced configurations (I_k, J_k) are in one-to-one correspondence with N-tuples of Maya diagrams of zero total charge

Theorem

Fredholm determinant $\tau(a)$ can be written as a combinatorial series

$$\tau(a) = \sum_{\vec{Q}_1,\ldots\vec{Q}_{n-3}\in\mathfrak{Q}_N} \sum_{\vec{Y}_1,\ldots\vec{Y}_{n-3}\in\mathbb{Y}^N} \prod_{k=1}^{n-2} Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}\left(\mathcal{T}^{[k]}\right)$$

- elementary determinants $Z^{\vec{Y}_{k-1},\vec{Q}_{k-1}}_{\vec{Y}_k,\vec{Q}_k}$ are constructed from matrix elements of 3-point Plemelj operators in Fourier basis
- in rank $N=2$, they are given by Cauchy matrices conjugated by diagonal factors $\Rightarrow$ explicitly computable !!!
- the result coincides with dual Nekrasov partition function for $U(2)$ linear quiver gauge theory with $\epsilon_1+\epsilon_2=0$
- series representation for general solution of PVI/Garnier system
- rank $N \Rightarrow$ a sum of $N-1$ Cauchy matrices
  (unless additional spectral conditions are imposed)

Conclusions

1. Isomonodromic tau functions of Fuchsian systems can be written as block Fredholm determinants whose kernels are built of fundamental solutions of 3-point Fuchsian systems
2. Expanding these determinants in Fourier basis leads to combinatorial series over tuples of partitions
3. The coefficients of the series can be computed explicitly when 3-point solutions have hypergeometric representations (in particular for $N = 2$)