On the DR/DZ equivalence conjecture

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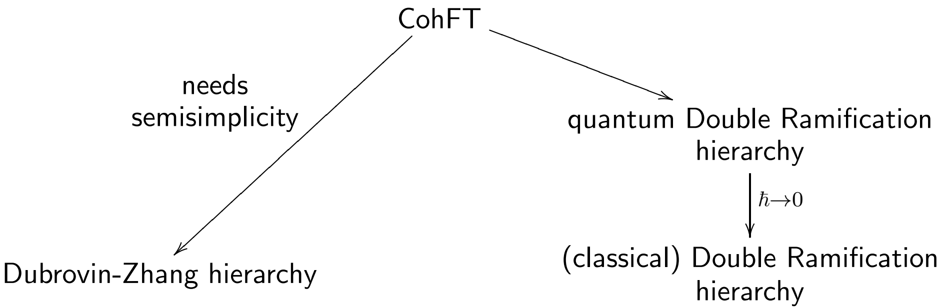
Integrable hierarchies from CohFTs



<u>CohFT</u>: V $\mathbb{C}$-vector space, η symm. nondeg. bilinear form, $e_1 \in V$

$c_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ linear maps

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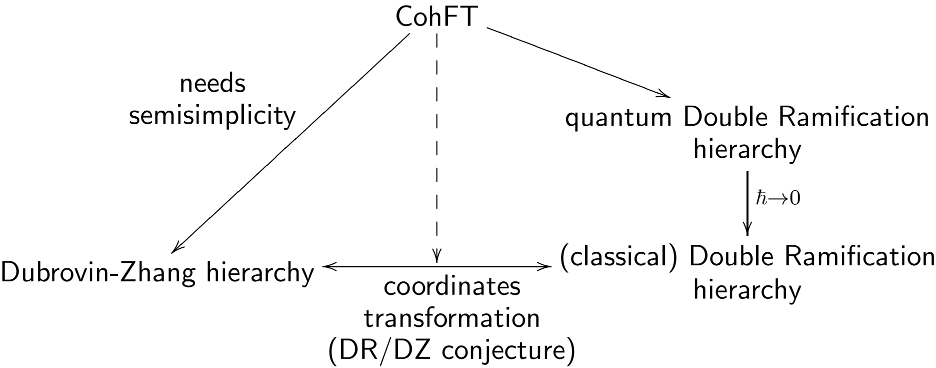

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Integrable hierarchies from CohFTs

Dependent variables:

$$w^\alpha = w^\alpha(x), \qquad w^\alpha_k := \partial_x^k w^\alpha, \qquad 1 \le \alpha \le N = \dim V$$

Hamiltonian densities:

$$\{h_{\beta,d}(w^*_*;\varepsilon)\}_{\substack{1\le\beta\le N\\ d\ge -1}} \qquad h_{\beta,d} \in \mathcal{A}^{[0]}_w := \mathbb{C}[[w^*_*,\varepsilon]]^{[0]}$$

$$(\deg\varepsilon = -1,\ \deg w^*_k = k)$$

Hamiltonians:

$$\overline{h}_{\alpha,d} := \int h_{\alpha,d}\, dx \qquad \in \Lambda_w := \mathcal{A}_w/\mathrm{Im}\partial_x \oplus \mathbb{C}$$

Integrable hierarchies from CohFTs

Poisson structure:

$$\{\cdot,\cdot\}_K : \Lambda_w \times \Lambda_w \to \Lambda_w$$
$$(\overline{f},\overline{h}) \mapsto \{\overline{f},\overline{h}\}_K = \int \left(\frac{\delta \overline{f}}{\delta w^\mu} K^{\mu\nu} \frac{\delta \overline{h}}{\delta w^\nu} \right) dx$$

$$K^{\mu\nu} = \sum_{j\geq 0} K_j^{\mu\nu} \partial_x^j = \eta^{\mu\nu}\partial_x + O(\varepsilon^2), \qquad K_j^{\mu\nu} \in \mathcal{A}_w^{[-j+1]}$$

Equations of the hierarchy:

$$\boxed{\frac{\partial w^\alpha}{\partial t_d^\beta} = \left\{ w^\alpha, \overline{h}_{\beta,d} \right\}_K , \qquad \{\overline{h}_{\beta_1,d_1}, \overline{h}_{\beta_2,d_2}\}_K = 0}$$

Integrable hierarchies from CohFTs

Tau structure: $\{f, \overline{h}_{1,0}\}_K = \partial_x f$

$\overline{h}_{\alpha,-1}$ are N independent Casimirs of $\{\cdot,\cdot\}_K$,

$\{h_{\alpha,p-1}, \overline{h}_{\beta,q}\}_K = \{h_{\beta,q-1}, \overline{h}_{\alpha,p}\}_K$.

Tau functions: For any solution $u^*(x,t,\varepsilon)$ there exists $F(t^*_*,\varepsilon)$

$$\left.\frac{\partial h_{\alpha,p-1}}{\partial t^\beta_q}\right|_{w^*_*=w^*_*(x,t^*_*;\varepsilon)|_{x=0}} = \frac{\partial^3 F}{\partial t^1_0 \partial t^\alpha_p \partial t^\beta_q}$$

Normal coordinates: $w^\alpha = \eta^{\alpha\mu} h_{\mu,-1}$

Integrable systems

Normal Miura transf.: (a Miura that preserves the tau structure)

$$\widetilde{u}^\alpha = w^\alpha + \eta^{\alpha\mu}\ \partial_x \partial_{t^\mu_0}\ P(w^*_*;\varepsilon)$$

where w^α are normal coordinates

and $P(w^*_*;\varepsilon) \in \mathcal{A}_w^{[-2]}$

Effect on tau functions: $\widetilde{F}(t^*_*;\varepsilon) = F(t^*_*;\varepsilon) + P(w^*_*;\varepsilon)|_{w^*_* = w^*_*(x,t^*_*;\varepsilon)|_{x=0}}$

Dubrovin-Zhang hierarchy of a semisimple CohFT

$$F(t^*_*;\varepsilon) = \sum_{g\geq 0} F_g(t^*_*)\varepsilon^{2g}, \qquad F_g(t^*_*) := \sum_{n,d_i} \langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_n}(e_{\alpha_n})\rangle_g \frac{t^{\alpha_1}_{d_1}\dots t^{\alpha_n}_{d_n}}{n!}.$$

<u>Genus 0</u>: $\begin{cases} \overline{h}^{[0]}_{\alpha,p}(v^*) = \frac{\partial^2}{\partial t^1_0 \partial t^\alpha_{p+1}} F_0(t^*_0 = v^*, t^*_{>0} = 0) \\ (K^{\mathrm{DZ}}_v)^{\alpha\beta} = \eta^{\alpha\beta}\partial_x \end{cases}$

$v^\alpha(x,t^*_*)$ solution with initial datum $v^\alpha(x,t^*_*=0) = x\delta^\alpha_1$

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Genus g:

Proposition (Eguchi-Getzler-Xiong, DZ, Buryak-Posthuma-Shadrin)

$$F_g(t^*_*) = F_g\left(t^*_{\leq 3g-2} = P^*_{\leq 3g-2}(v^*_{\leq 3g-2}), t^*_{>3g-2} = 0\right)|_{x=0}, \quad g>0$$

$$w^\alpha(v^*_*;\varepsilon) = v^\alpha + \eta^{\alpha\mu}\ \partial_x\partial_{t^\mu_0}\left(\textstyle\sum_{g\geq 1}\varepsilon^{2g}F_g(v^*_{\leq 3g-2})\right)$$ not a Miura transf.!

$\begin{cases} \overline{h}^{[0]}_{\alpha,p}(v^*) \mapsto \overline{h}^{\mathrm{DZ}}_{\alpha,p}(w^*_*;\varepsilon) \\ K^{\mathrm{DZ}}_v \mapsto K^{\mathrm{DZ}}_w \end{cases}$

polynomial anyway!
[Buryak-Posthuma-Shadrin]

Double ramification hierarchy of any CohFT

From Sasha's talk:

$$\left\{ \begin{array}{c} \text{Intersection numbers} \\ \text{of the CohFT and the DR cycle} \end{array} \right\} \quad \mapsto \quad g_{\alpha,d}(u^*_*;\varepsilon) \in \mathcal{A}_u^{[0]}$$

Poisson structure : $K_u^{\mathrm{DR}} = \eta\partial_x$ (Standard Poisson $\{\cdot,\cdot\} := \{\cdot,\cdot\}_{\eta\partial_x}$)

Nice properties :

Recursion : $\partial_x(D-1)g_{\alpha,p+1} = \left\{g_{\alpha,p}, \overline{g}_{1,1}\right\}$

Quantization : $\{\cdot,\cdot\} \mapsto \frac{1}{\hbar}[\cdot,\cdot]$

$g_{\alpha,p}(u^*_*;\varepsilon) \mapsto G_{\alpha,p}(u^*_*;\varepsilon,\hbar)$

Tau-structure :

$$h^{\mathrm{DR}}_{\alpha,p}(u^*_*;\varepsilon) = \frac{\delta g_{\alpha,p+1}}{\delta u^1}$$

$$\widetilde{u}^\alpha = \eta^{\alpha\mu} h^{\mathrm{DR}}_{\mu,-1}(u^*_*,\varepsilon) \qquad \eta\partial_x \mapsto K^{\mathrm{DR}}_{\widetilde{u}}$$

$$\widetilde{u}^\alpha(x, t^*_* = 0) = x\delta^\alpha_1 \implies F^{\mathrm{DR}}(t^*_*;\varepsilon)$$

Comparing the DZ and DR tau-functions

$$F^{\mathrm{DZ}}(t^*_*;\varepsilon) = \sum_{\substack{2g+n-2>0\\ d_1,\dots,d_n\geq 0}} \langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_n}(e_{\alpha_n})\rangle_g^{\mathrm{DZ}} \frac{t^{\alpha_1}_{d_1}\dots t^{\alpha_n}_{d_n}}{n!}\varepsilon^{2g}$$

$$F^{\mathrm{DR}}(t^*_*;\varepsilon) = \sum_{\substack{2g+n-2>0\\ d_1,\dots,d_n\geq 0}} \langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_n}(e_{\alpha_n})\rangle_g^{\mathrm{DR}} \frac{t^{\alpha_1}_{d_1}\dots t^{\alpha_n}_{d_n}}{n!}\varepsilon^{2g}$$

Proposition (Buryak-Dubrovin-Guéré-R. '16)

$$\langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_m}(e_{\alpha_m})\rangle_g^{\mathrm{DR}} = 0 \qquad \textit{unless} \qquad 2g-1 \leq \sum_{i=1}^m d_i \leq 3g-3+n.$$

Compare:

$$\langle \tau_{d_1}(e_{\alpha_1})\dots\tau_{d_m}(e_{\alpha_m})\rangle_g^{\mathrm{DZ}} = 0 \qquad \text{unless} \qquad \sum_{i=1}^m d_i \leq 3g-3+n.$$

Strong DR/DZ equivalence conjecture

$$\langle\tau_{d_1}(e_{\alpha_1})\dots\tau_{d_m}(e_{\alpha_m})\rangle_g^{\mathrm{DR}}=0 \qquad \text{for} \qquad \sum_{i=1}^m d_i<2g-1.$$

<u>Remark</u>: This does not hold for $F(t^*_*;\varepsilon)$.

<u>Idea</u>: We know that normal Miura transformations of the form

$\widetilde{u}^\alpha=w^\alpha+\eta^{\alpha\mu}\,\partial_x\partial_{t^\mu_0}\,P(w^*_*;\varepsilon), \qquad P\in\mathcal{A}_w^{[-2]}$

changes tau functions by

$\widetilde{F}(t^*_*;\varepsilon)=F(t^*_*;\varepsilon)+P(w^*_*(x,t^*_*;\varepsilon);\varepsilon)|_{x=0}$

Can we find $P(w^*_*;\varepsilon)$ so that $\widetilde{F}(t^*_*;\varepsilon)$ has no small correlators?

Strong DR/DZ equivalence conjecture

$$\langle \tau_{d_1}(e_{\alpha_1}) \ldots \tau_{d_m}(e_{\alpha_m}) \rangle_g^{\mathrm{DR}} = 0 \qquad \text{for} \qquad \sum_{i=1}^{m} d_i < 2g - 1.$$

Remark: This does not hold for $F(t^*_*;\varepsilon)$.

Theorem (BDGR '16)

$\exists!\ P(w^*_*;\varepsilon) \in \mathcal{A}_w^{[-2]}$ such that $F^{\mathrm{red}} := F + P(w^*_*(x,t^*_*;\varepsilon);\varepsilon)|_{x=0}$ satisfies the above selction rules.

Conjecture (Strong DR/DZ equivalence)

The DR and DZ hierarchies are equivalent via the normal Miura transformation generated by the unique $P(w^*_*;\varepsilon)$ found above ($\iff$ $F^{\mathrm{red}} = F^{\mathrm{DR}}$).

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Conjecture (Strong DR/DZ equivalence, generalization)

For any (even non semisimple) CohFT, $F^{\mathrm{red}} = F^{\mathrm{DR}}$.

Another characterization of the reduced DZ hierarchy

$$\sum_{g\geq 1} \varepsilon^{2g} F_g^{\mathrm{DZ}}(t^*_*) = \underbrace{\sum_{\substack{g\geq 1\\ k\leq 2g-2}} \varepsilon^{2g} F_{g,k}(v^*_*) (v^1_1)^k}_{F^{\mathrm{DZ}}_{\geq 1}(v^*_*;\varepsilon)} \Big|_{v^*_*=v^*_*(x,t^*_*)|_{x=0}}$$

$$F_{g,k}(v^*_*) \in \mathbb{C}[[v^*_*,\varepsilon]]^{[2g-k-2]}$$

where:

$$v^\alpha(x,t^*_*=0) = x\delta^\alpha_1 \qquad \Longrightarrow \qquad v^\alpha_k(x=0,t^*_*) = t^\alpha_k + \delta^\alpha_1\delta_{k,1} + O((t^*_*)^2)$$

Another characterization of the reduced DZ hierarchy

$$\sum_{g\geq 1} \varepsilon^{2g} \; F_g^{\mathrm{DZ}}(t^*_*) \;\; = \;\;\;\; F_{\geq 1}^{\mathrm{DZ}}(v^*_*;\varepsilon) \;\Big|_{v^*_*=v^*_*(x,t^*_*)|_{x=0}}$$

Another characterization of the reduced DZ hierarchy

$$\textstyle\sum_{g\geq 1} \varepsilon^{2g} \; F_g^{\mathrm{DZ}}(t^*_*) \;=\; F^{\mathrm{DZ}}_{\geq 1}(v^*_*;\varepsilon) \;\Big|_{v^*_* = v^*_*(x,t^*_*)|_{x=0}}$$

$$\begin{cases} F^{\mathrm{DZ}}(t^*_*;\varepsilon) \;=\; F_0(t^*_*) + \; F^{\mathrm{DZ}}_{\geq 1} \;\Big|_{v^*_* = v^*_*(x,t^*_*)|_{x=0}} \\ F^{\mathrm{red}}(t^*_*;\varepsilon) \;=\; F_0(t^*_*) + \left(F^{\mathrm{DZ}}_{\geq 1}\right)^{\mathrm{sing}} \;\Big|_{v^*_* = v^*_*(x,t^*_*)|_{x=0}} \end{cases}$$

$$\begin{cases} w^\alpha \;=\; v^\alpha + \eta^{\alpha\mu}\; \partial_x \partial_{t^\mu_0} \; F^{\mathrm{DZ}}_{\geq 1} \\ \qquad\;\; v^\alpha + \sum_{\substack{g\geq 1 \\ k\leq 2g}} \varepsilon^{2g} \; w^k_{g,k}(v^*_*) \; (v^1_1)^k, \qquad w^k_{g,k}(v^*_*) \in \mathbb{C}[[v^*_*,\varepsilon]]^{[2g-k]} \\ \widetilde{u}^\alpha \;=\; v^\alpha + \eta^{\alpha\mu}\; \partial_x \partial_{t^\mu_0} \; (F^{\mathrm{DZ}}_{\geq 1})^{\mathrm{sing}} \\ \qquad\;\; v^\alpha + \sum_{\substack{g\geq 1 \\ k\leq 0}} \varepsilon^{2g} \; \widetilde{u}^k_{g,k}(v^*_*) \; (v^1_1)^k, \qquad \widetilde{u}^k_{g,k}(v^*_*) \in \mathbb{C}[[v^*_*,\varepsilon]]^{[2g-k]} \\ u^\alpha \;=\; v^\alpha + \sum_{\substack{g\geq 1 \\ k< 0}} \varepsilon^{2g} \; u^k_{g,k}(v^*_*) \; (v^1_1)^k, \qquad u^k_{g,k}(v^*_*) \in \mathbb{C}[[v^*_*,\varepsilon]]^{[2g-k]} \end{cases}$$

What we already proved

Theorem (BDGR'16)

For any semisimple CohFT, the Miura transformation $u^\alpha = u^\alpha(w^*_*;\varepsilon)$, *transforms the DZ Poisson structure to the DR Poisson structure.*

Theorem (B '14, BR '14, BG'15, BDGR '16, BGDR'17, BGR'17)

The strong DR/DZ equivalence conjecture holds for:

- *the trivial CohFT,*
- *the full Hodge class,*
- *Witten's* 3-, 4- *and* 5-*spin classes,*
- *the Fan-Jarvis-Ruan-Witten* D_4 *CohFT,*
- *the GW theory of* $\mathbb{P}^1$.

It holds up to genus 5 *for any rank* 1 *CohFT.*
It holds up to genus 2 *for any semisimple CohFT.*

Why is this interesting?

- The DZ hiearchies contain basically all known examples of (1+1) integrable Hamiltonian systems of evolutionary PDEs possessing a tau-symmetry (KdV, Toda, ILW, Gelfand-Dickey, Drinfeld-Sokolov, Ablowitz-Ladik, etc.).
- $\mathrm{DZ} \mapsto \mathrm{DR}$ is the quintessential normal form for such systems:
  - the Poisson structure is reduced to $\eta\partial_x$,
  - the Hamiltonian densities satisfy a universal recursion,
  - the topological tau-function is reduced to the simplest possible form.
- The route $\mathrm{DZ} \mapsto \mathrm{DR}$ leads to systematic quantization of all systems of topological type.
- The fact that the DR hiearchy exists for non semisimple CohFTs suggests that it might be possible to construct the DZ hierarchy in that case too.
- Makes the relation with geometry of $\overline{\mathcal{M}}_{g,n}$ much more transparent.
- Might be the road to finally prove polynomiality of the second Poisson structure for the DZ hiearchy.

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DR/DZ equivalence and tautological relations

Theorem (BGR'17)

The DR/DZ conjecture holds iff, for $2g-1 \leq \sum d_i \leq 3g-3+n$*:*

$$\int_{A^g_{d_1,\ldots,d_n}} c_{g,n}(e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n}) = \int_{B^g_{d_1,\ldots,d_n}} c_{g,n}(e_{\alpha_1}\otimes\ldots\otimes e_{\alpha_n})$$

where

$$A^g_{d_1,\ldots,d_n},\ B^g_{d_1,\ldots,d_n} \in R^*(\overline{\mathcal{M}}_{g,n}) \subset H^*(\overline{\mathcal{M}}_{g,n})$$

represent certain cycles in $\overline{\mathcal{M}}_{g,n}$*.*

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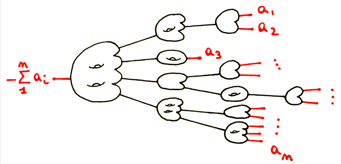
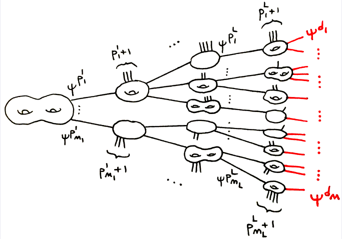
$$A^g_{d_1,\dots,d_n} = \sum \mathrm{Coeff}_{a_1^{d_1}\dots a_n^{d_n}} \frac{1}{\sum a_i}$$



$$B^g_{d_1,\dots,d_n} = \sum (-1)^{L-1}\pi_*$$



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Conjecture (BDGR'17, sufficient for the DR/DZ equivalence)

$$A^g_{d_1,\ldots,d_n} = B^g_{d_1,\ldots,d_n} \quad \in R^d(\overline{\mathcal{M}}_{g,n})$$

$$2g-1 \leq d = \sum d_i \leq 3g-3+n$$

Results towards proving the conjecture

Theorem (BGR'17)

- $A = B$ *holds in* $\overline{\mathcal{M}}_{0,n}$ *and* $\overline{\mathcal{M}}_{1,n}$.
- A *and* B *classes behave the same way upon pullback and pushforward along* $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$.
- $i^*\pi_*A = i^*\pi_*B$ *with* $\overline{\mathcal{M}}_{g,n+m} \xrightarrow{\pi} \overline{\mathcal{M}}_{g,n} \xleftarrow{i} \mathcal{M}_{g,n}$.
- *For semisimple CohFTs, checking* $A = B$ *in degree* $\sum d_i = 2g$ *and with* $d_i > 0$ *is enough to prove the DR/DZ equivalence (this is how we proved it for* $g \leq 2$*).*

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