Methods of tangent and cotangent coverings for Dubrovin-Novikov integrability operators

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Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \ldots) = 0$$

admits a Hamiltonian formulation if

$$u^i_t = A^{ij}\left(\frac{\delta H}{\delta u^j}\right)$$

where $A = (A^{ij})$ is a Hamiltonian operator, i.e. a differential operator

$$A^{ij} = a^{ij\sigma} D_\sigma \quad \text{such that} \quad A^* = -A \quad \text{and} \quad [A, A] = 0$$

$D_\sigma = D_x \circ \cdots \circ D_x$ (total x-derivatives σ times).

Finding Hamiltonian operators/PDEs is hard.

Symmetries

A Hamiltonian equation shows a *correspondence* between conservation laws and symmetries.

Generalized *symmetries* are vector functions $\varphi^i = \varphi^i(u^j, u^j_x, u^j_{xx}, \ldots)$ such that

$$\begin{cases} \ell_F(\varphi^i) = D_t\varphi^i - \frac{\partial f^i}{\partial u^j_\sigma} D_\sigma \varphi^j = 0, \\ F^k = 0 \end{cases}$$

where ℓ_F is the Fréchet derivative of F.

Conservation laws

A *conservation law* is a one-form $\omega = Adx + Bdt$ which is closed modulo the equation:

$$\bar{d}\omega = \nabla F$$

where $\nabla = a_k^{\tau\sigma} D_{\tau\sigma} F^k$. The vector function

$$\psi_k = \psi_k(u^j, u^j_x, u^j_{xx}, \dots) = (-1)^{|\tau\sigma|} D_{\tau\sigma} a_k^{\tau\sigma}|_{F=0}$$

represents uniquely the conservation law and fulfills the equation

$$\begin{cases} \ell_F^*(\varphi^i) = -D_t\psi_i + (-1)^{|\sigma|} D_\sigma \left(\frac{\partial f^j}{\partial u^i_\sigma}\psi_j\right) = 0 \\ F^k = 0 \end{cases}$$

where ℓ_F^* is the formal adjoint of ℓ_F;

A necessary condition

If an equation admits a Hamiltonian formulation, this implies that A maps conservation laws into symmetries:

$$\ell_F \circ A = (A')^* \circ \ell_F^* \qquad A\text{: almost-Hamiltonian op.}$$

The condition can be extended to all *integrability operators*:

$$\ell_F^* \circ S = S' \circ \ell_F \qquad S\text{: almost-symplectic op.}$$

$$\ell_F \circ R = R' \circ \ell_F \qquad R\text{: recursion operator}$$

$$\ell_F^* \circ C = (C')^* \circ \ell_F^* \qquad C\text{: co-recursion operator}$$

Note that A', S', R', C' are arbitrary.

Almost: it is a necessary condition ...

Cotangent covering

Kersten, Krasil'shchik, Verbovetsky, JGP 2003.
Introducing new variables p_k, p_{kx}, p_{kxx}, ... we can represent operators by linear functions:

$$A(\psi) = a^{ij\sigma} D_\sigma \psi_j \quad \Leftrightarrow \quad A = a^{ij\sigma} p_{j\sigma}$$

Then a Hamiltonian operator fulfills the equations

$$\mathcal{T}^*\colon \begin{cases} \ell_F^*(\mathbf{p}) = -p_{i,t} + (-1)^{|\sigma|} D_\sigma \left(\frac{\partial f^j}{\partial u^i_\sigma} p_j\right) = 0 \\ F = u^i_t - f^i = 0 \end{cases} \quad \text{and} \quad \ell_F(A) = 0.$$

The system $\mathcal{T}^*$ is the **cotangent covering**. It is *invariant*.

Tangent covering

Introducing new variables q^k, q^k_x, q^k_{xx}, … we can represent operators by linear functions:

$$R(\varphi) = a_j^{i\sigma} D_\sigma \varphi^j \quad \Leftrightarrow \quad R = a_j^{i\sigma} q^j_\sigma$$

Then a recursion operator fulfills the equations

$$\mathcal{T}\colon \begin{cases} \ell_F(\mathbf{q}) = q^i_t - \frac{\partial f^i}{\partial u^j_\sigma} q^j_\sigma = 0 \\ F = u^i_t - f^i = 0 \end{cases} \quad \text{and} \quad \ell_F(R) = 0.$$

The system $\mathcal{T}$ is the *tangent covering*. It is *invariant*.

Example: Hamiltonian operators for KdV

The KdV equation: $u_t = uu_x + u_{xxx}$
The linearization: $\ell_F = D_t - uD_x - u_x - D_{xxx}$
The adjoint linearization: $\ell_F^* = -D_t + uD_x + D_{xxx}$
The cotangent covering for the KdV equation:

$$\begin{cases} p_t = p_{xxx} + up_x \\ u_t = u_{xxx} + uu_x \end{cases}$$

The equation $\ell_F(A) = 0$ has the two solutions:

$$A_1 = p_x \quad \text{or} \quad A_1 = D_x$$

$$A_2 = \frac{1}{3}(3p_{3x} + 2up_x + u_x p) \quad \text{or} \quad A_2 = \frac{1}{3}(3D_{xxx} + 2uD_x + u_x)$$

For example, $\ell_F(A_1) = D_t p_x - uD_x p_x - u_x p_x - D_{xxx} p_x$.

Example: recursion operator for KdV

The tangent covering of KdV:

$$\mathcal{T}\colon \begin{cases} q_t = u_x q + u q_x + q_{xxx} \\ u_t = u_{xxx} + u u_x \end{cases}$$

Unfortunately, the equation for recursion operators $\ell_F(R) = 0$ has the only trivial solution $R = q$.
However, there is a conservation law on $\mathcal{T}$:
$\omega = q dx + (uq + q_{xx}) dt$. We can introduce a new non-local variable w such that

$$w_x = q, \qquad w_t = uq + q_{xx}.$$

Then we have the non-local recursion operator

$$R = q_{xx} + \frac{2}{3} uq + \frac{1}{3} u_x w \quad \text{or} \quad R = D_{xx} + \frac{2}{3} u + \frac{1}{3} u_x D_x^{-1}$$

Applications to Dubrovin–Novikov operators

The cotangent covering of a hydrodynamic-type system is:

$$\begin{cases} p_{i,t} = (V^k_{i,j} - V^k_{j,i}) u^j_x p_k + V^k_i p_{k,x} \\ u^i_t = V^i_j(\mathbf{u}) u^j_x \end{cases}$$

A first-order Dubrovin–Novikov Hamiltonian operator:

$$A^i = g^{ij} p_{jx} + \Gamma^{ij}_k u^k_x p_j.$$

Tsarev's compatibility conditions are the coefficients of the linear equation in $p_{k,\sigma}$, $\ell_F(A) = 0$:

$$D_t A^i - V^i_{j,k} u^j_x A^k - V^i_j D_x A^j = 0 \quad \Leftrightarrow \quad \begin{cases} g^{ik} V^j_k = g^{jk} V^i_k \\ \nabla_i V^k_j = \nabla_j V^k_i \end{cases}$$

Further applications

- Higher order Dubrovin–Novikov Hamiltonian operators.
- Symplectic Dubrovin–Novikov operators.
- Recursion operators for cosymmetries.
- Nonlocal Dubrovin–Novikov first-order operators, also known as **Ferapontov–Mokhov operators**. Higher order analogue.

Application to third-order DN operators

Dubrovin–Novikov operators can be defined for arbitrary orders. Here we consider the third order ones:

$$
\begin{aligned}
A_3^{ij} = & g^{ij}(\mathbf{u})D_x^3 + b_k^{ij}(\mathbf{u})u_x^k D_x^2 \\
& + [c_k^{ij}(\mathbf{u})u_{xx}^k + c_{km}^{ij}(\mathbf{u})u_x^k u_x^m]D_x \\
& + d_k^{ij}(\mathbf{u})u_{xxx}^k + d_{km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n,
\end{aligned}
$$

Potemin's canonical form in Casimirs:

$$A_3^{ij} = D_x(g^{ij}D_x + c_k^{ij}u_x^k)D_x$$

Remark: g_{ij} is the Monge form of a **quadratic line complex** (Ferapontov, Pavlov, V. JGP 2014, IMRN 2016).
We restrict our consideration to hydrodynamic-type systems in these Casimirs. Then they can be written in conservative form: $V^i_j = (V^i)_{,j}$

Compatibility conditions (Ferapontov, Pavlov, V. 2017)

Theorem Let A_3 be a Hamiltonian operator. Then $u^i_t = V^i_j u^j_x = (V^i)_{,j} u^j_x$ admits a Hamiltonian formulation with A_3 if and only if

$$\begin{cases} g_{im} V^m_j = g_{jm} V^m_i \\ c_{mkj} V^m_i + c_{mik} V^m_j + c_{mji} V^m_k = 0, \\ V^k_{i,j} = g^{ks} c_{smj} V^m_i + g^{ks} c_{smi} V^m_j \end{cases} \tag{1}$$

Theorem. The above system is in involution. Its solution depends on at most $(1/2)n(n+3)$ parameters. **The solution is reduced to a linear algebra problem either if the unknown is g_{ij} or if the unknown is V^i.**

Remark. No Hamiltonian needed at this stage!

Properties of the systems of conservation laws

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system $u^i_t = (V^i)_{,j} u^j_x$ a *congruence of lines* in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

Theorem.

- The congruence is *linear*: there are n linear relations between u^i, V^i, $u^i V^j - u^j V^i$.
- The system is *linearly degenerate*, and *non diagonalizable*.
- $V^i = \psi^i_\alpha w^\alpha$, where ψ^i_α is determined by $g_{ij} = \varphi_{\alpha\beta} \psi^\alpha_i \psi^\beta_j$ and w^α are linear functions. This means that $V^i = p(u^j)/q(u^j)$ where $p(u^j)$, $q(u^j)$ are polynomials of degree n, $n-1$.

Hamiltonian, momentum and more

The above systems of conservation laws all admit non-local Hamiltonian, momentum and Casimirs. They all are new non-local conserved quantities.
Let us set $\psi^\gamma_k = \psi^\gamma_{km} u^m + \omega^\gamma_k$, and $w^\gamma = \eta^\gamma_m u^m + \xi^\gamma$.
Let us set $u^i = b^i_x$; the system becomes $b^i_t = V^i(\mathbf{b}_x)$.
Theorem.

- Hamiltonian op. $A_3 = -g^{ij}(\mathbf{b}_x) D_x - c^{ij}_k(\mathbf{b}_x) b^k_{xx}$
- Hamiltonian $H = -\int \varphi_{\beta\gamma}[\left(\frac{1}{3}\eta^\gamma_p \psi^\beta_{qm} b^m_x + \frac{1}{2}\omega^\beta_p \eta^\gamma_q\right) b^p b^q + x\left(\frac{1}{2}\psi^\beta_{pq}\xi^\gamma b^p b^q_x + \xi^\gamma \omega^\beta_q b^q\right)] dx$
- n Casimirs $C^\alpha = \int \left(\frac{1}{2}\psi^\alpha_{mk} b^k_x + \omega^\alpha_m\right) b^m dx$
- momentum $P = -\int \left(\frac{1}{3}\varphi_{\beta\gamma}\omega^\beta_q \psi^\gamma_{pm} b^m_x + \frac{1}{2}\varphi_{\beta\gamma}\omega^\beta_p \omega^\gamma_q\right) b^p b^q dx$

Invariance of the hydrodynamic-type system

Theorem. The class of conservative systems of hydrodynamic type possessing third-order Hamiltonian formulation is invariant under reciprocal transformations of the form

$$d\tilde{x} = (a_i u^i + a)dx + (a_i V^i + b)dt$$
$$d\tilde{t} = (b_i u^i + c)dx + (b_i V^i + d)dt$$

Classification results

Theorem. Let $u^i_t = (V^i)_x$ be a hydrodynamic-type system, and suppose that it admits a Hamiltonian formulation via a third-order Dubrovin-Novikov operator whose Casimirs are u^i. Then:

- $n = 2$ The system is linearisable.
- $n = 3$ The system is either linearisable, or equivalent to the system of WDVV equations (to be discussed); from Castelnuovo's classification of linear line congruences.
- $n = 4$ Far more complicated: there exists no classification of linear congruences in $\mathbb{P}^5$. There exist one generic nontrivial integrable example.

Example: WDVV equations in 3-comp.

From $f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xtt}$ setting $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ we have

$$\begin{aligned} u^1_t &= u^2_x, \\ u^2_t &= u^3_x, \\ u^3_t &= ((u^2)^2 - u^1 u^3)_x, \end{aligned}$$

endowed with a third-order Hamiltonian operators with nonlocal Hamiltonian

$$H = -\int \left(\frac{1}{2} u^1 \left(\partial_x{}^{-1} u^2 \right)^2 + \partial_x{}^{-1} u^2 \partial_x{}^{-1} u^3 \right) dx.$$

(Ferapontov, Galvao, Mokhov, Nutku, 1995). It is **bi-Hamiltonian** and up to a non-trivial transformation is the **3-wave equation** (Zakharov, Manakov, ∼1970).

Example: WDVV system in 6-comp.

Dubrovin 1996; Ferapontov-Mokhov 1998; Pavlov-V. 2015. We have a pair of hydrodynamic type systems in conservative form:

$$a^i_y = (v^i(\mathbf{a}))_x, \quad a^i_z = (w^i(\mathbf{a}))_x,$$

where

$$v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6,$$
$$v^4 = f_{yyy} = \frac{2a^5 + a^2 a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3 a^4 + a^6}{a^1},$$
$$v^6 = w^5 = f_{yzz} = \frac{2a^3 a^5 - a^2 a^6}{a^1},$$
$$w^6 = f_{zzz} = (a^5)^2 - a^4 a^6 + \frac{(a^3)^2 a^4 + a^3 a^6 - 2a^2 a^3 a^5 + (a^2)^2 a^6}{a^1}.$$

Monge metric for 6-components WDVV

$$g_{ik}(\mathbf{a}) = \begin{pmatrix} (a^4)^2 & -2a^5 & 2a^4 & -(a^1a^4+a^3) & a^2 & 1 \\ -2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\ 2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\ -(a^1a^4+a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\ a^2 & a^1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Remark: the metric can be found in few seconds by computer.

Example: generic value of n

The system of conservation laws:

$$u^1_t = u^2_x, \quad u^2_t = u^3_x, \ldots, \quad u^{n-1}_t = u^n_x, \quad u^n_t = [u^1 u^3 - (u^2)^2]_x.$$

The third-order Hamiltonian operator's Monge metric:

$$g_{ij} = \begin{pmatrix} 2a^2 & -a^1 & 0 & & 1 \\ -a^1 & 0 & & 1 & \\ 0 & & 1 & & \\ & 1 & & & 0 \\ 1 & & & 0 & 0 \end{pmatrix}$$

and the Hamiltonian is

$$H = -\frac{1}{2} a^1 (D^{-1} a^2)^2 + \frac{1}{2} \sum_{m=2}^{N} (D^{-1} a^m)(D^{-1} a^{N+2-m}).$$

Problem: integrability for $n \geqslant 4$?

Open problems

- Integrability for $n \geqslant 4$ of the systems of conservation laws.
- Geometry of WDVV equations. All of them have a third-order H.o. and all of them are linear line congruences.
- What happens when the fluxes are functions of first or second order derivatives?
- Non-local Hamiltonian operators of second and third order, and their compatibility with hydrodynamic-type systems.
- Extension to symplectic operators, local and non-local.

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `http://gdeq.org`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

Forthcoming book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, to appear in the series Texts and Monographs in Symbolic Computation, Springer, 2017.

Thank you!

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