Geometry of Integrable Systems

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The many faces of Ricci curvature

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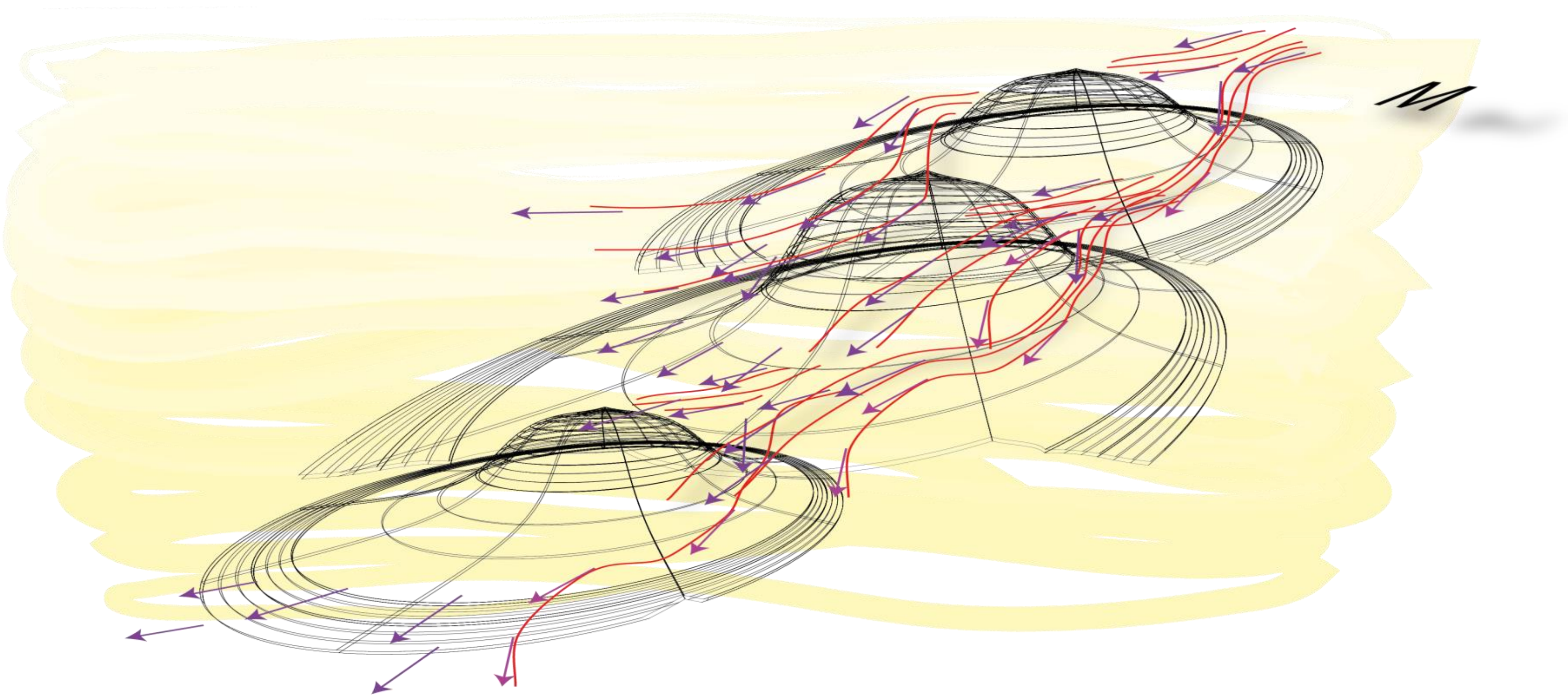
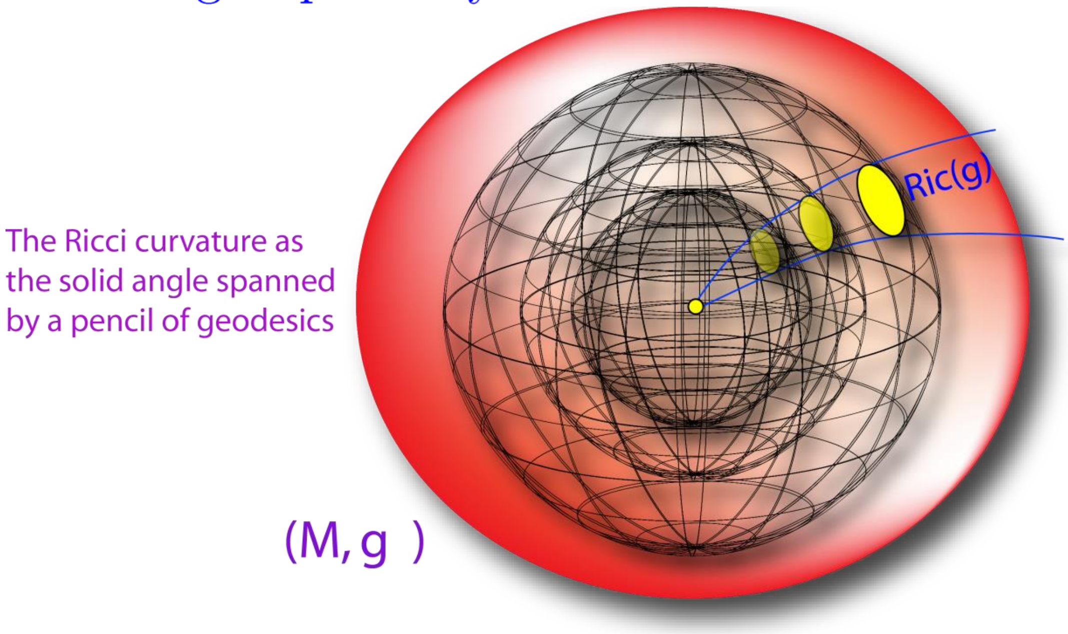


- Powered by Ricci flow theory, our understanding of the role of Ricci curvature in geometry & physics has undergone remarkable developments in recent years. We have seen the conjunction of a variety of ideas from topology, geometric analysis, optimal transport, synthetic geometry, and renormalization group theory.

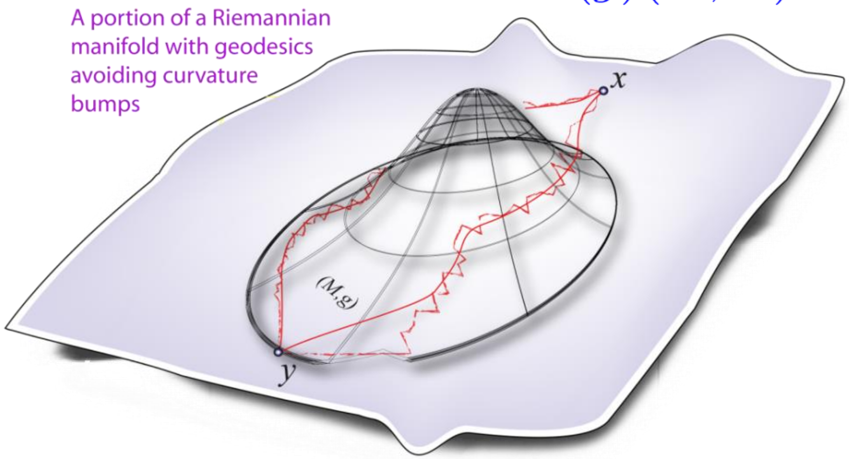


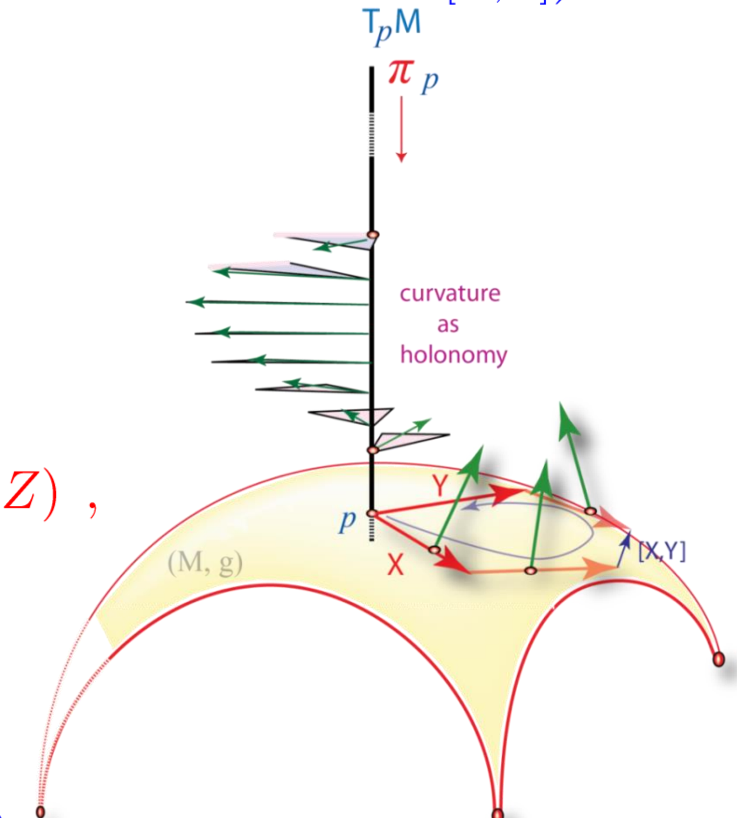
- In this talk (partly based on joint work with Christine Guenther) we touch upon some of these themes as well as on some unconventional aspects that Ricci curvature still holds in store.

- The *CLASSICAL* FRAMEWORK:

- (M, g) a C^∞ compact or complete n–dimensional manifold, $(n \geq 3)$, endowed with a Riemannian metric g, the associated Levi–Civita connection ∇, and its Riemann curvature

$$\mathcal{R}m(g)(X,Y)Z := \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z \,,$$





- $\mathcal{R}ic(g)(Y,Z) := trace_g \left(X \longmapsto \mathcal{R}m(g)(X,Y)Z\right)$,
- $\mathcal{R}(g) := trace_g \left(\mathcal{R}ic(g)\right)$: Scalar curvature
- Equivariance under the action of $\mathcal{D}iff(M)$:

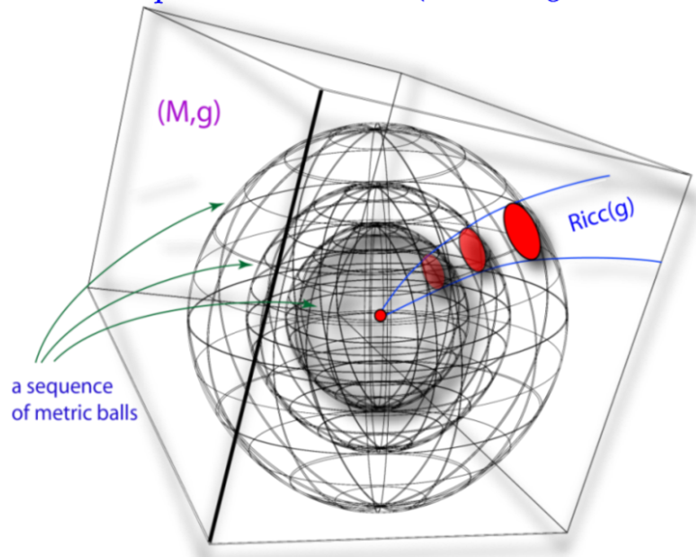
- $Ric(\phi^* g) = \phi^* Ric(g) \Longrightarrow \nabla^i \mathcal{R}_{ik} \doteq \frac{1}{2} \nabla_k \mathcal{R}$: contracted Bianchi identity (D.Hilbert, J. Kazdan, role in prescribing Ricci curvature D. DeTurck)

If we think of the Christoffel symbols $\Gamma^k_{ij}(g) = \frac{1}{2}\, g^{kl}\, \left(\frac{\partial}{\partial x^i}\, g_{jl} + \frac{\partial}{\partial x^j}\, g_{li} - \frac{\partial}{\partial x^l}\, g_{ij}\right)$ as the components of a $\mathfrak{so}(n)$–valued 1–form, we have the familiar expression for $Ric(g)$ requiring a...meditative approach ...

$$\mathcal{R}_{ik} = \nabla_h \Gamma^h_{ik} - \nabla_i \Gamma^h_{hk}$$

Elementary insight in the Ricci curvature is provided by its expression in *normal geodesic coordinates* based at $p \in M$, $(\Gamma^k_{ij}(g)\,|_p = 0)$ (Puiseaux)

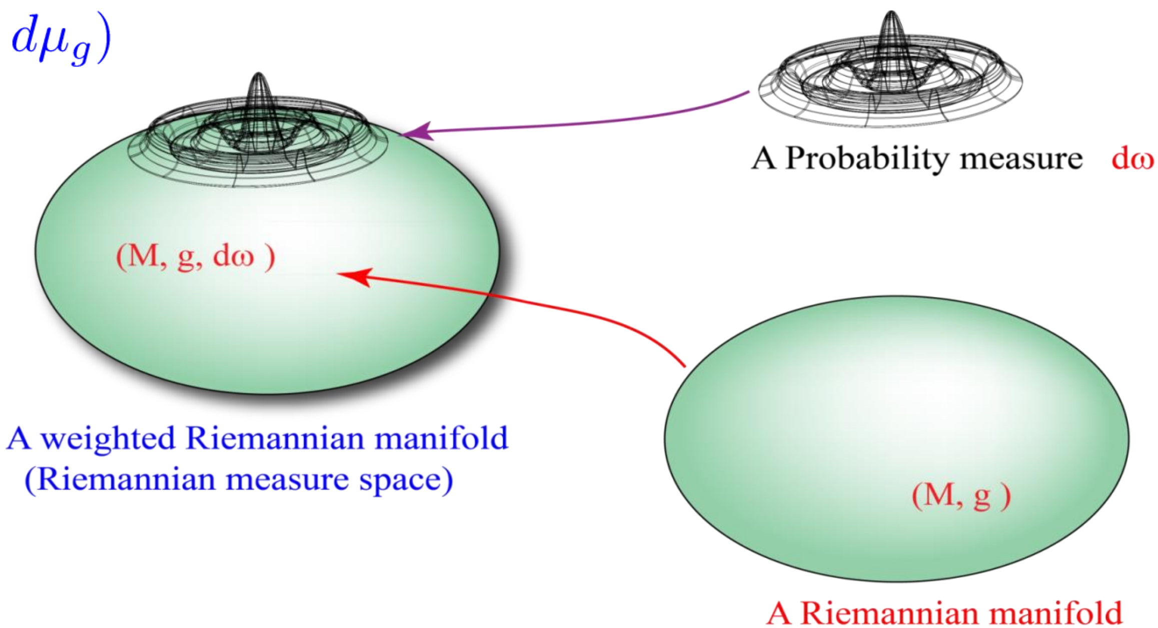
- $\exp^*_p (d\mu_g) = \left(1 - \frac{1}{6}\, \mathcal{R}_{ik}(p)\, x^i(q) x^k(q) + \ldots\right)\, d\mu_E$



$d\mu_E$: Euclidean measure on T_pM

- *Ricci curvature*: distortion (w.r.t. $d\mu_E$) of the solid angle subtended by a small pencil of geodesics issued from p in the direction $X = \exp_p^{-1}(q)$.

Given the interplay between $Ric(g)$ and $d\mu_g$, it is not surprising that in geometric analysis and physics (scaling & diffeomorphisms group, diffusive evolution, QFT) the analysis of Ricci curvature often calls for weighted Riemannian manifolds (or Riemannian measure spaces) $(M, g, d\omega)$ *i.e.* smooth n–dim Riemannian manifolds endowed with a probability measure $d\omega$, (not necessarily absolutely continuous with respect to the Riemannian measure; but here we assume $d\omega << d\mu_g$)



- We shall see the rationale for $(M, g, d\omega)$. For now, notice that in the extension $(M, g) \Longrightarrow (M, , d\omega = e^{-f} d\mu_g)$ the role of $Ric(g)$ is taken over by the Bakry–Emery Ricci curvature

$$Ric^{BE}(g, d\omega) := \mathcal{R}ic(g) + Hess_g\, f$$

In such a setting the familiar (second contracted) Bianchi identity

$$\nabla^i \mathcal{R}_{ik} \doteq \frac{1}{2}\nabla_k \mathcal{R} \quad \Longrightarrow \quad \nabla^i_{(\omega)} \mathcal{R}^{BE}_{ik} \doteq \frac{1}{2}\nabla_k \mathcal{R}^{Per},$$ where

$$\mathcal{R}^{Per}(g,f) := \mathcal{R}(g) + 2\Delta_g f - |\nabla f|^2_g = \mathcal{R}(g) + 2\Delta^{(\omega)}_g f + |\nabla f|^2_g :$$

is Perelman's modified scalar curvature.

The Einstein–Hilbert functional $\int_M \mathcal{R}(g)\, d\mu_g$ becomes

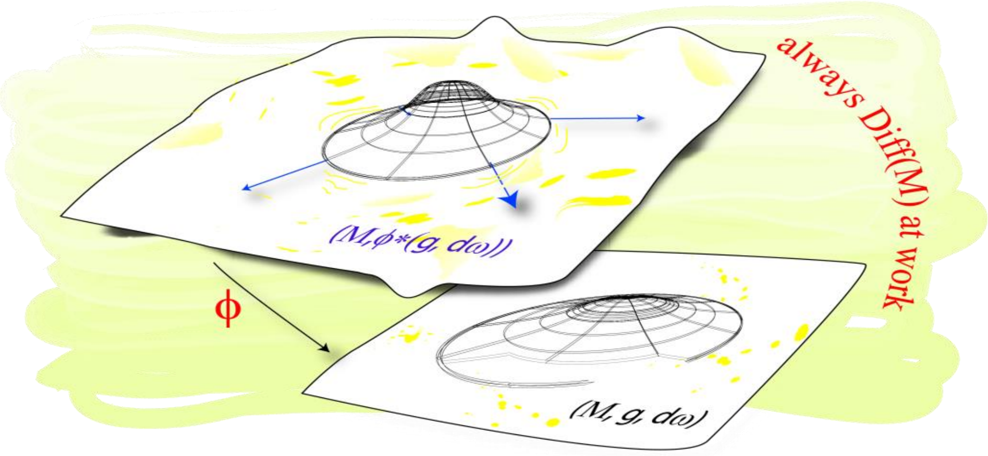
$$\int_M \mathcal{R}^{Per}(g,f)\, d\omega = \int_M \left(\mathcal{R}(g) + |\nabla f|^2_g\right) e^{-f} d\mu_g =: \mathcal{F}[g,f]$$

is Perelman's $\mathcal{F}$–energy associated with $(M, g, d\omega)$. If we take its inf

$$\inf_{\left\{f:\, \int d\omega = 1\right\}} \mathcal{F}[g,f] := \mathcal{F}[g]$$

we get Perelman's $\mathcal{F}$–energy (entropy) associated with (M, g).

Best constant in a Poincaré type inequality.



Notation

$$\nabla_{(\omega)} \circ := e^f\, \nabla\, \left(e^{-f} \circ\right):$$ the $d\omega$–weighted divergence

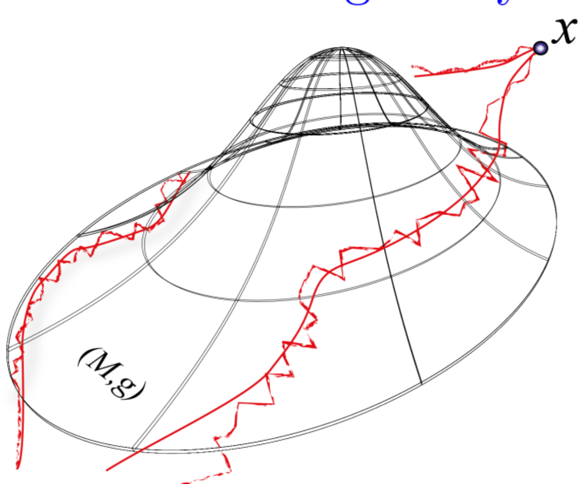
$$\Delta^{(\omega)}_g \psi := (\Delta_g - \nabla f \cdot \nabla)\, \psi:$$ the $d\omega$–weighted Laplacian on $(M, g, d\omega)$

$\mathcal{R}ic(g)$ and $\mathcal{R}ic^{BE}(g)$ carry a few Laplacians with them...

From the point of view of geometric analysis insight on the nature of Ricci curvature is provided by the expression of its components in local harmonic coordinates $(U, \{x^i\}; \triangle_g x^i = 0)$, (C. Lanczos, D. DeTurck, J. Kazdan, ...):

$$\mathcal{R}_{ik} =_{har} -\frac{1}{2}\triangle_{(g)}\left(f_{(ik)}\right) + Q_{ik}\left(g^{-1}, \partial g\right)\ , \quad f_{(ik)} := g_{ik}\ ,$$

Hence in harmonic coordinates the Ricci curvature acts as a semi–linear elliptic operator on each scalar function $f_{(ik)} := g_{ik}$: the metric tensor components g_{ik} have maximal regularity in harmonic coordinates (Jost–Karcher).

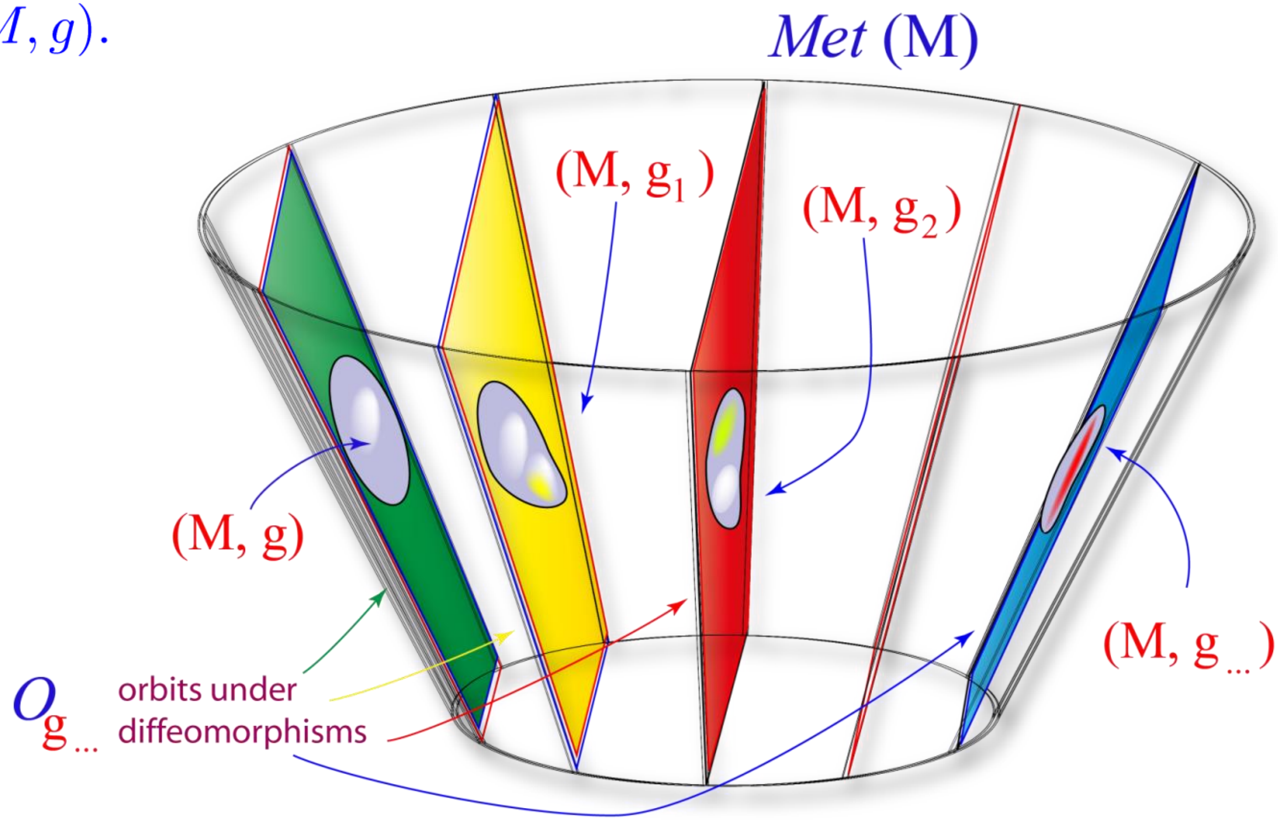


$\frac{1}{2}\triangle_{(g)}$:generator Brownian motion...Ricci and diffusion. $(M, g, d\omega)'s$ lurk in the background.

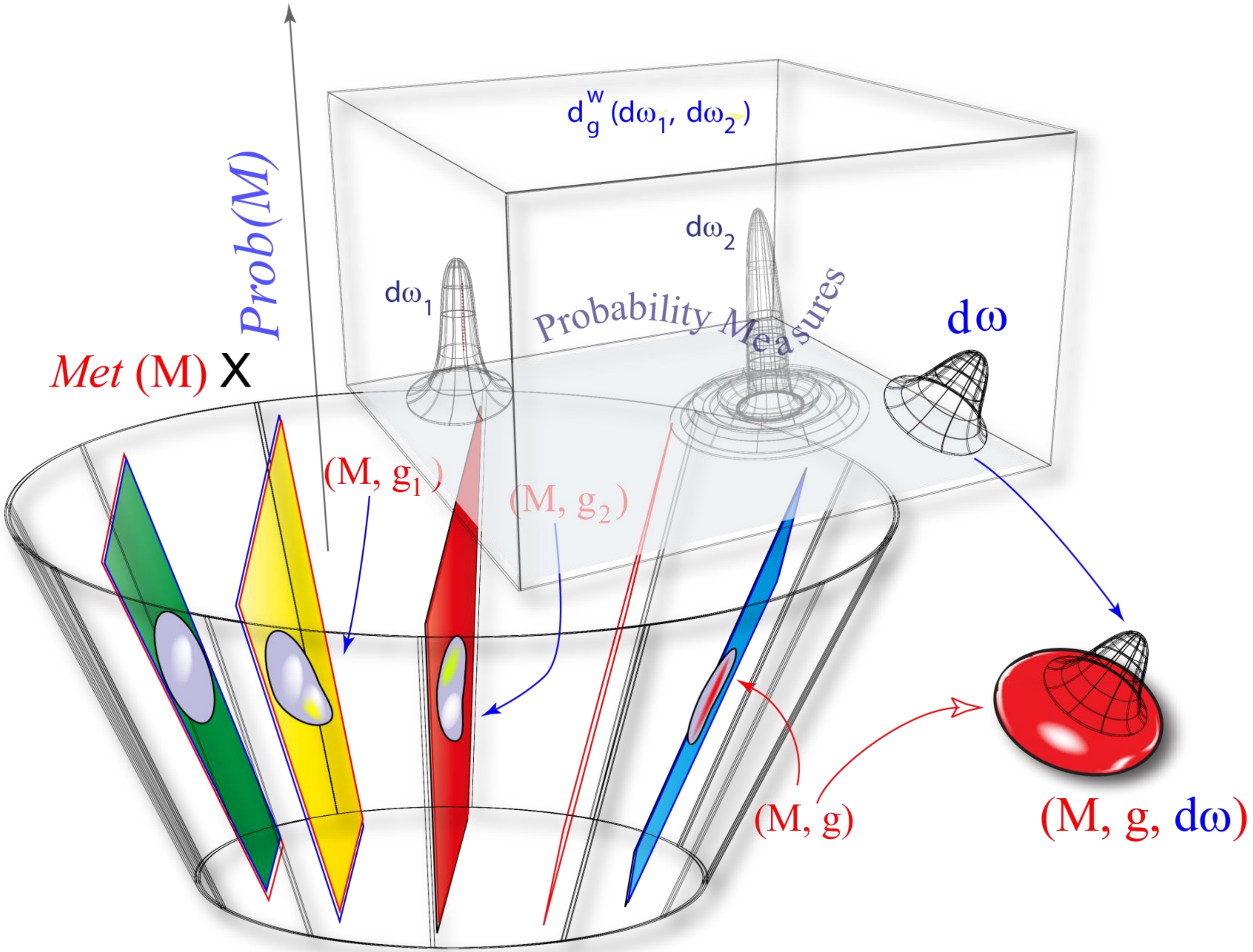
When passing from normal geodesic coordinates to harmonic coordinates we gain control on the components of the metric tensor in terms of the Ricci curvature rather than of the full Riemann tensor.

Since $Ric(g)$ and $Ric^{BE}(g,f)$ can be seen as pde operators acting on metrics and measures it is useful to take the appropriate ∞–dimensional perspective. To begin with...

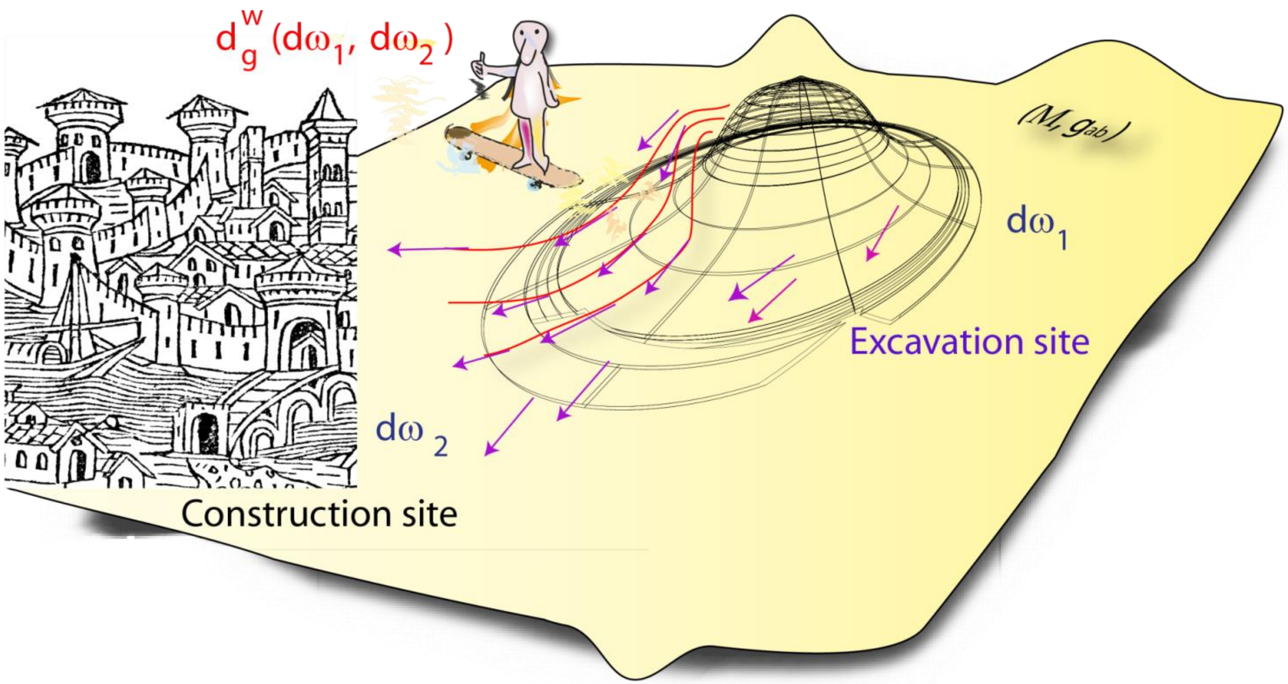
- $\mathcal{M}et(M)$: the space of all smooth Riemannian metrics over M.
- $\mathcal{O}_g := \{g' \in \mathcal{M}et(M) |\ g' = \phi^*(g) |\ \phi \in \mathcal{D}iff(M)\}$,
- $\mathcal{O}_g \in \frac{\mathcal{M}et(M)}{\mathcal{D}iff(M)}$ defines the Riemannian Structure associated with the Riemannian manifold (M,g).



More generally we can use the Space of weighted Riemannian manifolds $(M, g, d\omega)$, i.e. $\mathcal{M}et(M) \times [Prob(M), d_g^W]$, where $Prob(M)$ denotes the space of probability measures $d\omega$ over M endowed with the quadratic Wasserstein distance $d_g^W(d\omega_1, d\omega_2)$.



Intuitively, $d_g^w(d\omega_1, d\omega_2)$ represents, as we consider all possible couplings between the measures $d\omega_1$ and $d\omega_2$, the minimal cost needed to transport $d\omega_1$ into $d\omega_2$ provided that the cost to transport the point x into the point y is given by $d_g^2(x, y)$. The distance $d_g^w(d\omega_1, d\omega_2)$ metrizes $Prob(M)$ turning it into a geodesic space.

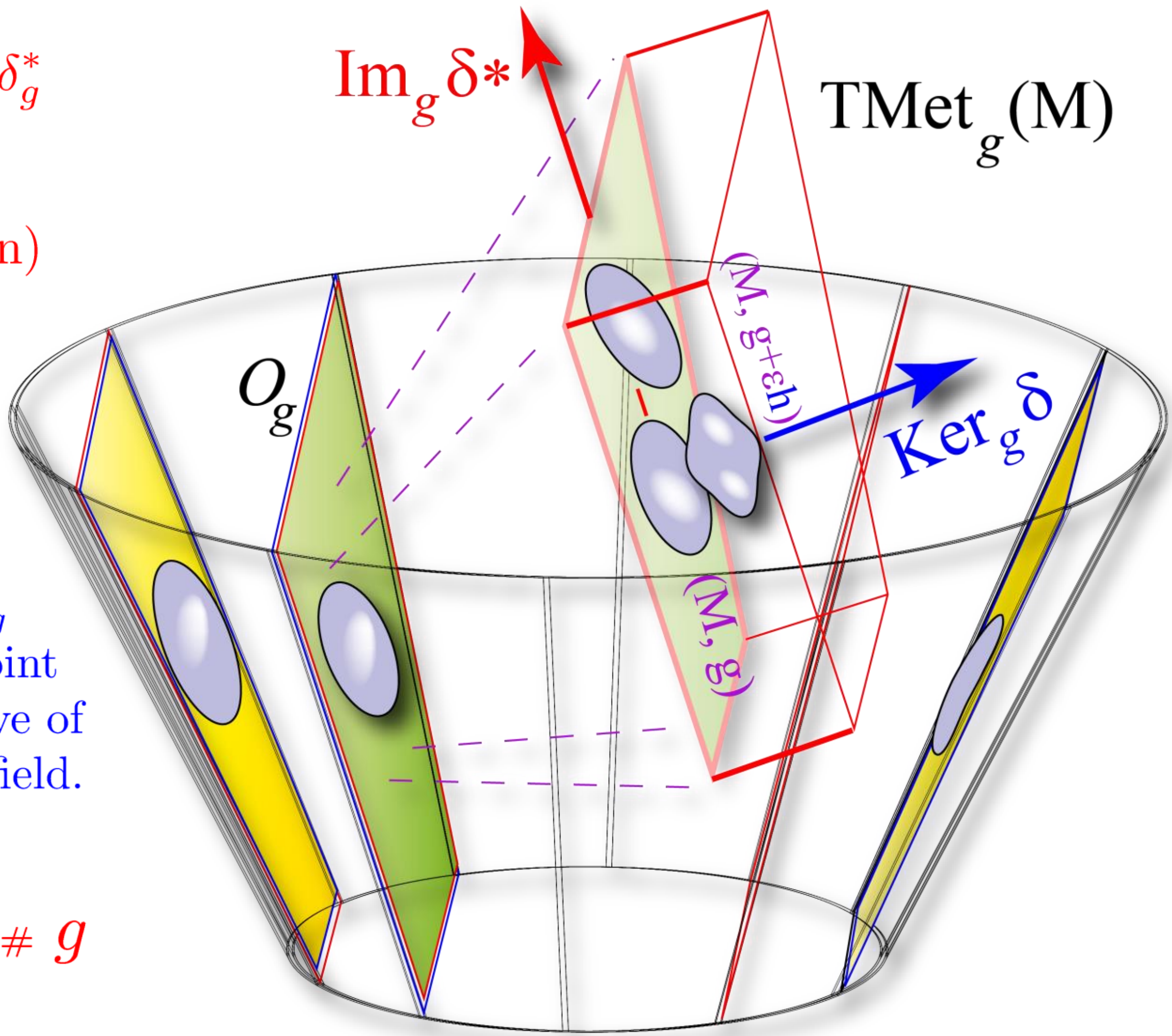


The quadratic Wasserstein distance $d_g^w(d\omega_1, d\omega_2)$ plays a basic role in the Monge-Kantorovich problem of optimally transporting one distribution of mass $d\omega_1$, (say from an excavation site on the manifold (M, g)) onto another distribution $d\omega_2$, (realized at the construction site on (M, g)), where optimality is measured against the cost function $d_g^2(x, y)$.

$\mathrm{Met}(M) \times \mathrm{Prob}(M)$ is modelled on its tangent space $T_g\mathrm{Met}(M) \oplus T_\omega \mathrm{Prob}(M, g)$. Tangent vectors $h \in T_g\mathrm{Met}(M)$ are symmetric 2–tensor fields on M. There is a $Diff(M)$ induced factorization

$$T_g\mathcal{M}et(M) \cong ker\ \delta_g \oplus Im\ \delta_g^*$$

(The $L^2(M, g)$ orthogonal Berger–Ebin decomposition)



Notation

The divergence operator $-\delta_g$ and its $L^2(M, g)$ formal adjoint δ_g^*: (1/2 of) the Lie derivative of the metric g along a vector field.

$$\delta_g^* \left(w_a\, dx^a \right) \doteq \frac{1}{2}\, \mathcal{L}_{w\#}\, g$$

$$\delta_g \left(h_{ab}\, dx^a \otimes dx^b \right) \doteq -\, g^{ij}\, \nabla_i h_{jk}\, dx^k\ ,$$

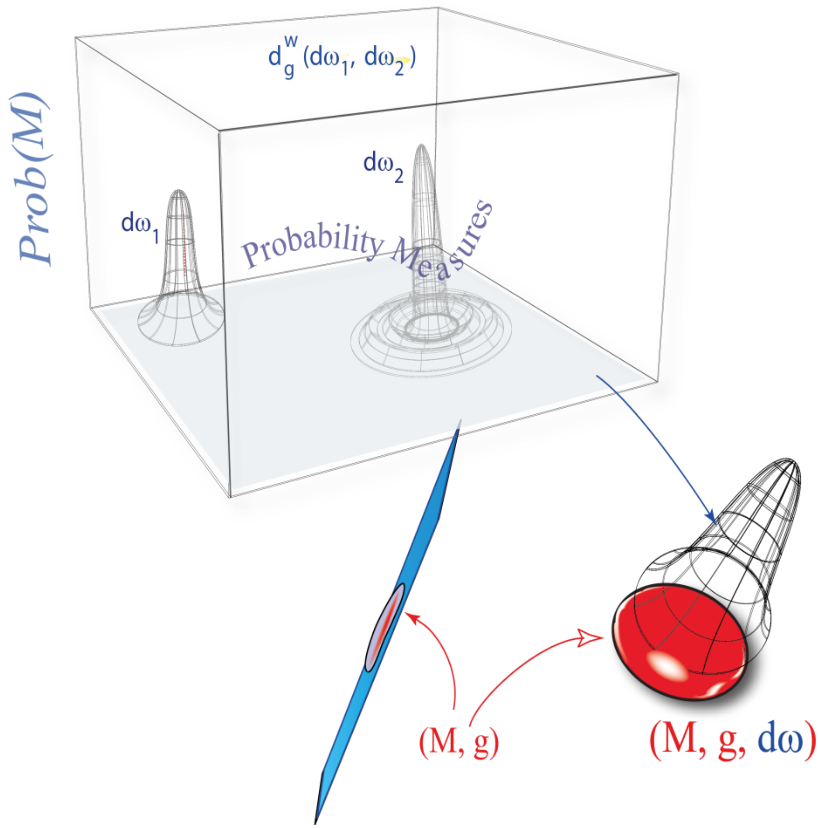
...for the $T_\omega Prob_{ac}(M)$ factor we have...

$$T_\omega Prob_{ac}(M,g) \simeq \{h \in C^\infty(M,\mathbb{R}),\ \int_M h\, d\omega = 0\}\ ,$$

This can be parametrized *a la Otto* in terms of the solutions ψ of the elliptic pde

$$\triangle_\omega \psi = -h \implies \nabla^i \left(e^{-f}\,\nabla_i \psi\right) = -h\, e^{-f}\ ,$$

under the equivalence relation identifying any two such solutions differing by an additive constant.



One considers the Hilbert space completion of $T_\omega \mathrm{Prob}_{ac}(M,g)$ with respect to the $L^2(M, d\omega)$ Otto metric

$$\langle \nabla\psi, \nabla\phi \rangle_{(g,d\omega)} = \int_M g^{ik} \nabla_i \psi \nabla_k \phi\, d\omega$$

$$\overline{T}_\omega \mathrm{Prob}_{ac}(M,g) := \overline{\{\psi \in C^\infty(M,\mathbb{R})/\mathbb{R}\,|\, \triangle_\omega \psi = -h\,,\quad h \in T_\omega \mathrm{Prob}_{ac}(M,g)\}}^{L^2(M,d\omega)}$$

Also to $T_\omega \mathrm{Prob}(M)$ we can associate a $\mathcal{D}iff(M)$–induced factorization

$$\left(T_\phi \mathcal{D}iff(M),\ \langle, \rangle_{(\phi,d\omega)}\right) \simeq \left(T_\omega \mathrm{Prob}(M),\ \langle, \rangle_{(Otto)}\right) \oplus \left(T_\phi \mathcal{D}iff_\omega(M),\ \langle, \rangle_{(\phi,d\omega)}\right)$$

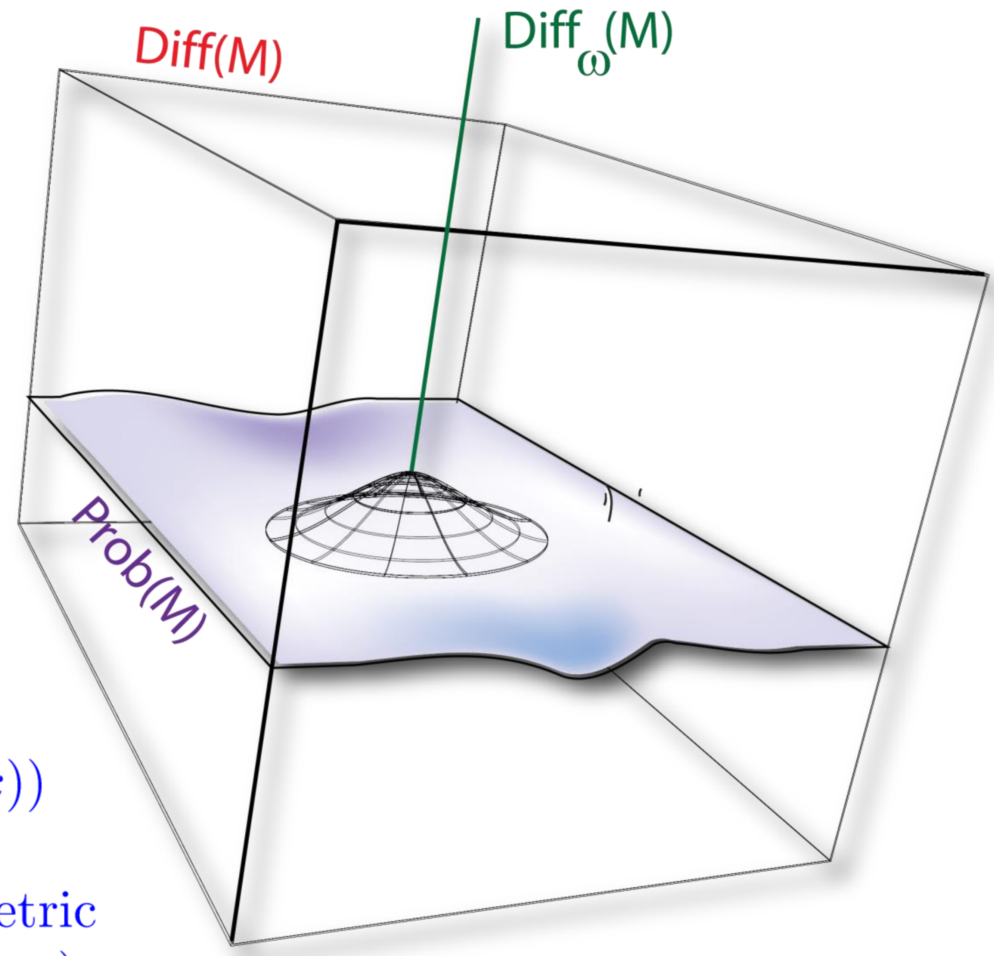
induced by the fibration

$$\mathrm{Prob}(M) \simeq \mathcal{D}iff(M)/\mathcal{D}iff_\omega(M)$$

and with respect to which

$$\pi : \mathcal{D}iff(M) \longrightarrow \mathrm{Prob}(M)$$

is a Riemannian submersion (see *e.g.* Boris Khesin)



Notation

$\mathcal{D}iff_\omega(M)$: $d\omega$–preserving Diffeomorphisms

$$\langle u, v\rangle_{(\phi,d\omega)} := \int_M g(u,v)_{\phi(x)}\, d\omega(\phi(x))$$

The $L^2(M, d\omega)$ weak Riemannian metric on $\mathcal{D}iff(M)$. (Arnol'd, Ebin, Marsden)

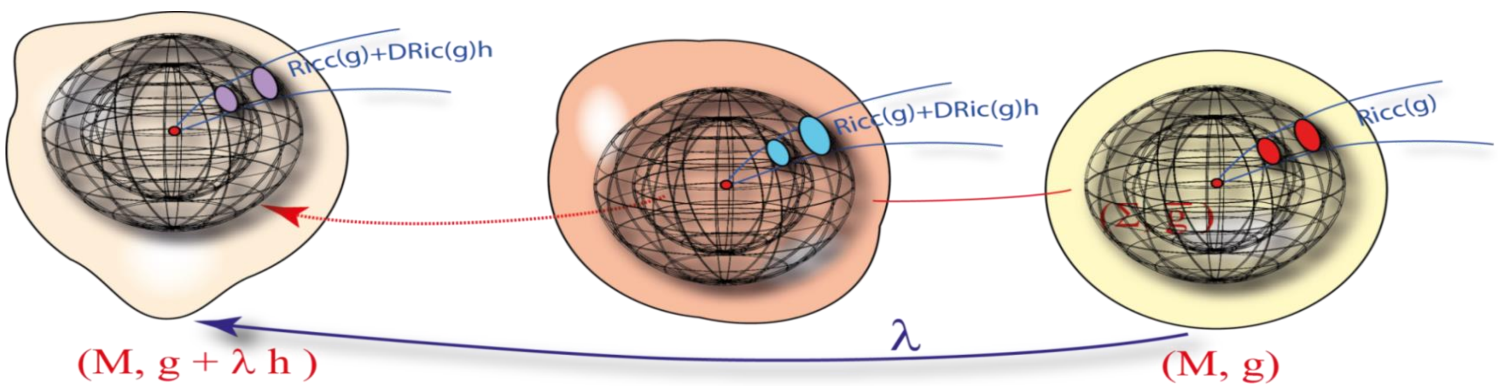
$\mathcal{R}ic(g)$ (as well as $\mathcal{R}ic^{BE}(g)$) $\in T_g\mathrm{Met}(M)$.
Is it a vector field $\mathrm{Met}(M) \ni g \mapsto \mathcal{R}ic(g) \in T\mathrm{Met}(M)$?

If we linearize the Ricci tensor in the direction of the metric variation $h \in C_0^\infty(M, \otimes_s^2 T^*M) \simeq T_g\mathcal{M}et(M)$

$$(-\epsilon, \epsilon) \ni \lambda \longmapsto g(\lambda) := g + \lambda h \in \mathcal{M}et(M)\,,$$

$$D\mathcal{R}ic \cdot h_{ij} := \left.\frac{d}{d\lambda}\mathcal{R}_{ij}(g(\lambda))\right|_{\lambda=0} = -\frac{1}{2}\left(\Delta_L h_{ij} - \mathcal{L}_{Bianchi_g(h)}\, g_{ij}\right)$$

- where $\Delta_L h_{ij} := \Delta h_{ij} + 2\mathcal{R}_{kijl}h^{kl} - \mathcal{R}_i^k h_{jk} - \mathcal{R}_j^k h_{ik}$ is the Lichnerowicz Laplacian operator
- $\xi := -Bianchi_g(h) := \frac{1}{2}\nabla\, tr_g(h) - div\, h$ is the Bianchi (operator) mapping symmetric 2–tensors to vector fields



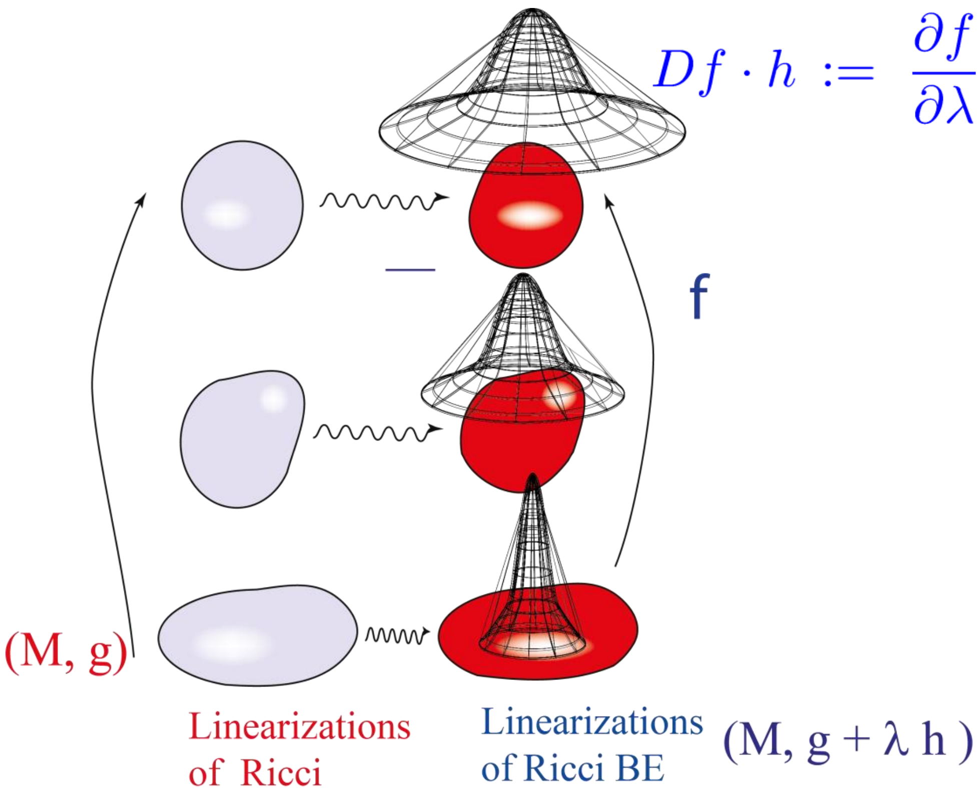
Similarly, the linearization of $\mathrm{Ric}^{BE}(g, f)$, ($d\omega$ preserving: $\frac{d}{d\lambda}\, d\omega = 0$), provides

$$D\mathcal{R}ic^{BE} \cdot h_{ij} \;=\; -\,\frac{1}{2}\,\left(\Delta_L^{(\omega)} h_{ij} \;-\; \mathcal{L}_{div^{(\omega)}h}\, g_{ij}\right)$$

- where $\Delta_L^{(\omega)} h_{ij} \;:=\; \Delta^{(\omega)} h_{ij} + 2\mathcal{R}_{kijl} h^{kl} - (\mathcal{R}^{BE})_i^k h_{jk} - (\mathcal{R}^{BE})_j^k h_{ik}$ is the weighted Lichnerowicz Laplacian operator, and where

- the $d\omega$ preserving constraint yields the induced linearization of f

$$Df \cdot h \;:=\; \left.\frac{\partial f}{\partial \lambda}\right|_{\lambda=0} \;=\; \frac{1}{2}\, tr_g(h)$$

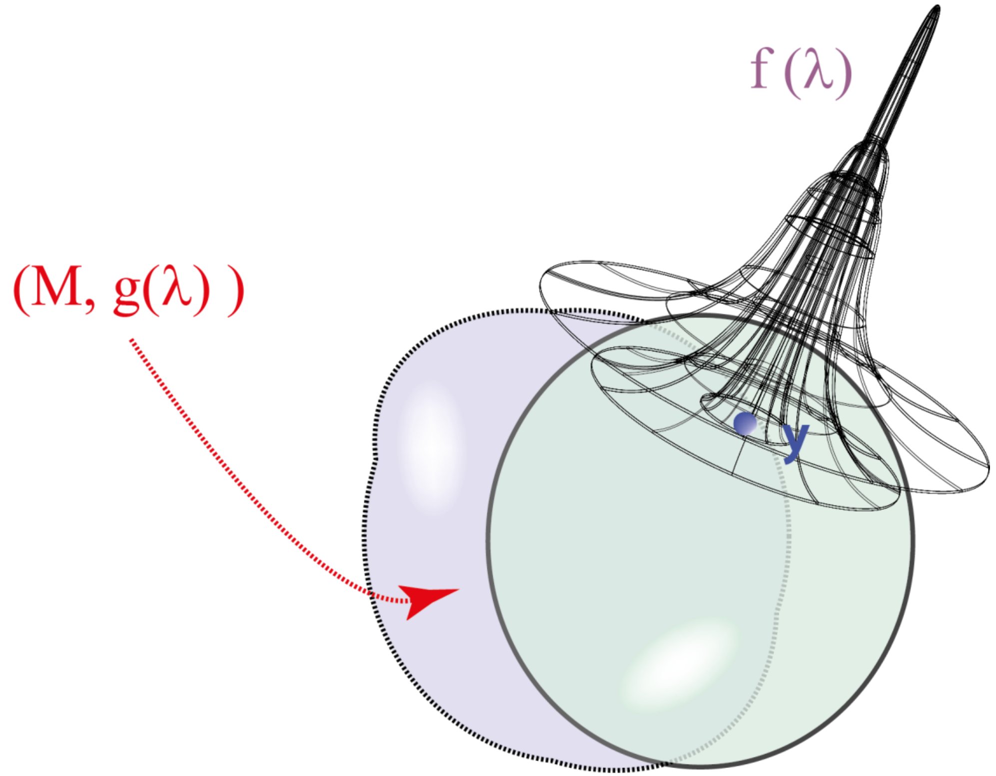


We also have an induced linearization for Perelman's scalar curvature $\mathcal{R}^{Per}$

$$D\mathcal{R}^{Per} \cdot h_{ij} \; := \; \left. \frac{d\mathcal{R}^{Per}(g_\lambda, f_\lambda)}{d\lambda} \right|_{\lambda=0} = \nabla^j_{(\omega)} \nabla^k_{(\omega)} h_{jk} \; - \; (\mathcal{R}ic^{BE})_{jk} h^{jk} \; ,$$

from which we get

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} \int_M \mathcal{R}^{Per}(g_\lambda, f_\lambda) \, d\omega = \; - \int_M \left(\mathcal{R}ic^{BE} \right)_{jk} h^{jk} \, d\omega \; .$$



Ric (and Ric^{BE}) is a collection of (weakly) elliptic operators in disguise

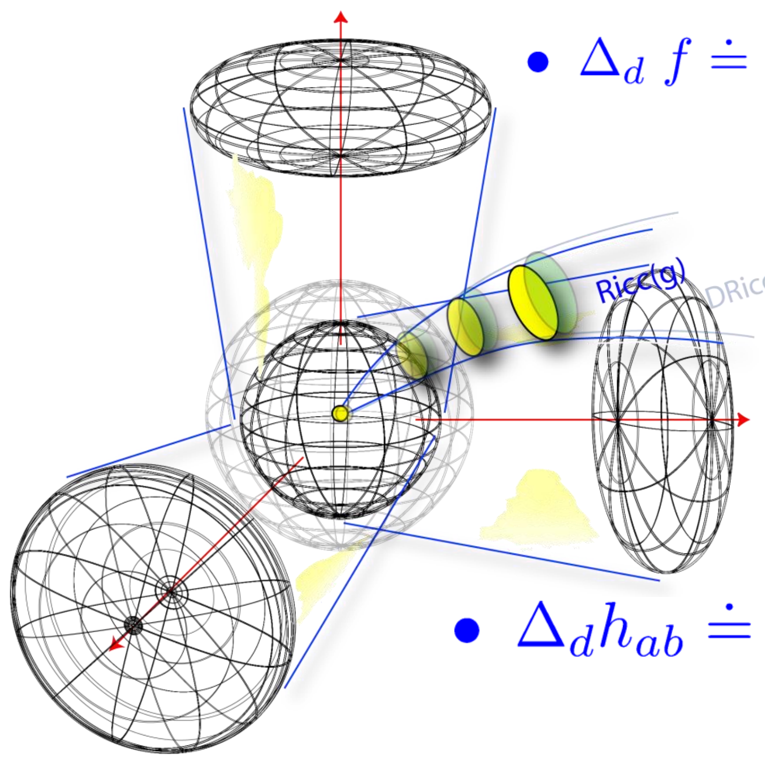
- The Lichnerowicz Laplacian Δ_L acts on symmetric bilinear forms $h_{ik} = h_{ki}$ in the same way the Hodge–DeRham Laplacian $\Delta_d := -(d\delta_g + \delta_g d)$ acts on antisymmetric 2-forms $\omega_{ab} = -\omega_{ba}$, *i.e.* $\Delta_d\, \omega_{ab} = \Delta_L\, \omega_{ab}$, (D. Knopf)

- Natural to associate (m.c.) to the linearization of $\mathcal{R}ic(g)$ the multiplet of elliptic operators $(g;\, f, \xi, h) \longmapsto \Delta_d(f, \xi, h)$

- Laplace–Beltrami:
  $\Delta_d\, f \doteq \triangle_g\, f$

- (co)Vector Laplacian:
  $\Delta_d \xi_a \doteq \triangle_g \xi_a - R^b_a\, \xi_b$

- Lichnerowicz Laplacian:
  $\Delta_d h_{ab} \doteq \triangle_L h_{ab}$



- $\Delta_d\, f \doteq \triangle_g\, f$

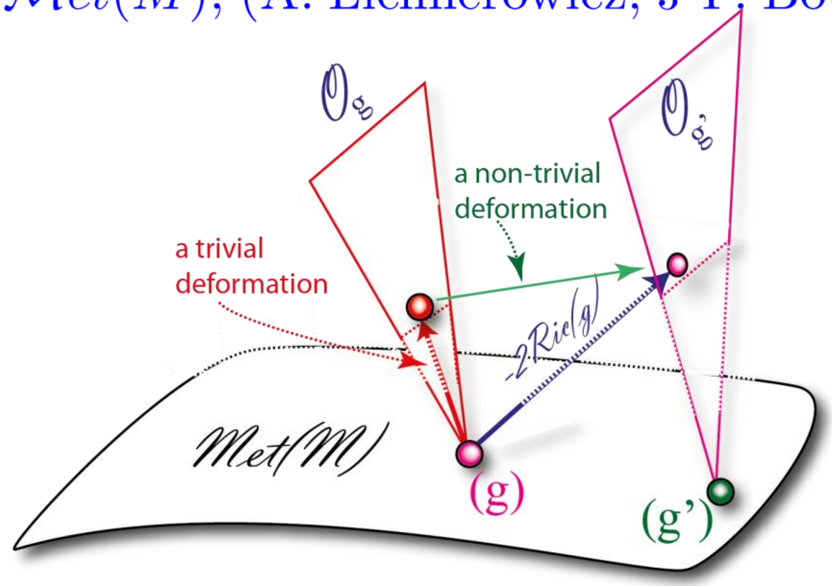
- $\Delta_d h_{ab} \doteq \triangle_L h_{ab}$

- $\Delta_d \xi_a \doteq \triangle_g \xi_a - R^b_a\, \xi_b$

$$\begin{aligned} \mathcal{M}et(M) &\longrightarrow T\mathcal{M}et(M) \\ g &\longmapsto -2\,\mathcal{R}ic(g) \end{aligned}$$

- can be seen as a non–linear *weakly–elliptic* 2nd order operator acting on the metric, (*i.e.* its linearization is elliptic were not for the presence of the Lie derivative term $\mathcal{L}_{Bianchi_g(h)}\, g$ generated by $\mathcal{D}iff(M)$–equivariance).

- This, well before the advent of Ricci flow, raised the question of the existence of a flow on $\mathcal{M}et(M)$ associated with the Ricci tensor thought of as a *tangent vector field* on $\mathcal{M}et(M)$, (A. Lichnerowicz, J-P. Bourguignon,...)



May we interpret $-2\,\mathcal{R}ic(g)$ as a non–trivial Vector Field on $\mathcal{M}et(M)$?

... Yes, but flow me gently: The interplay between SCALING and Diff(M)

Besides diffeomorphisms, the metric g is naturally acted upon also by overall rescalings according to

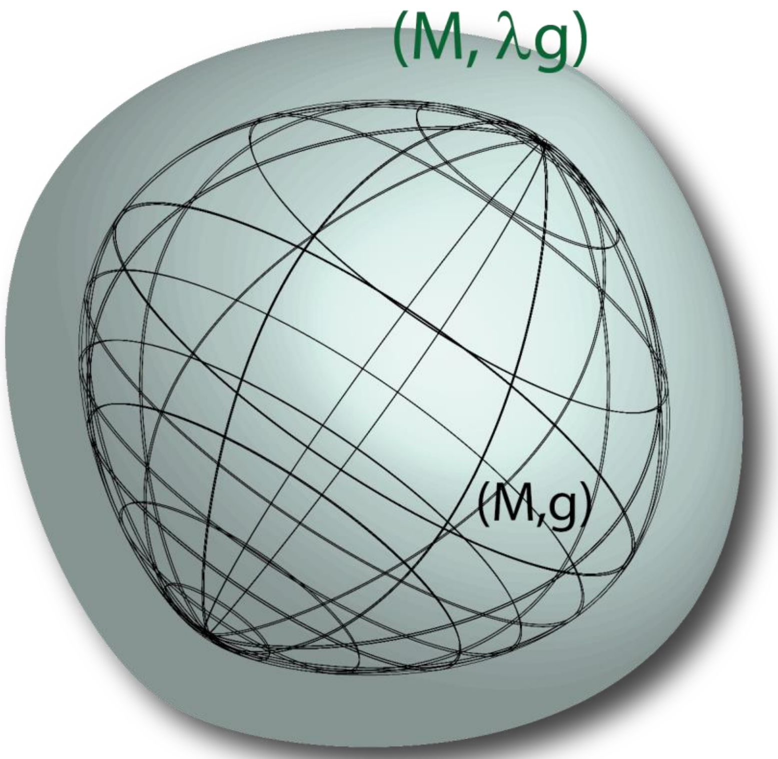
$$g \longmapsto \lambda g\,, \quad \forall\lambda \in R_{>0}\,,$$

(in local coordinates (U, x^i), $g_{ik} \longmapsto \lambda\, g_{ik}$ and $g^{ik} \longmapsto \lambda^{-1}\, g^{ik}$).

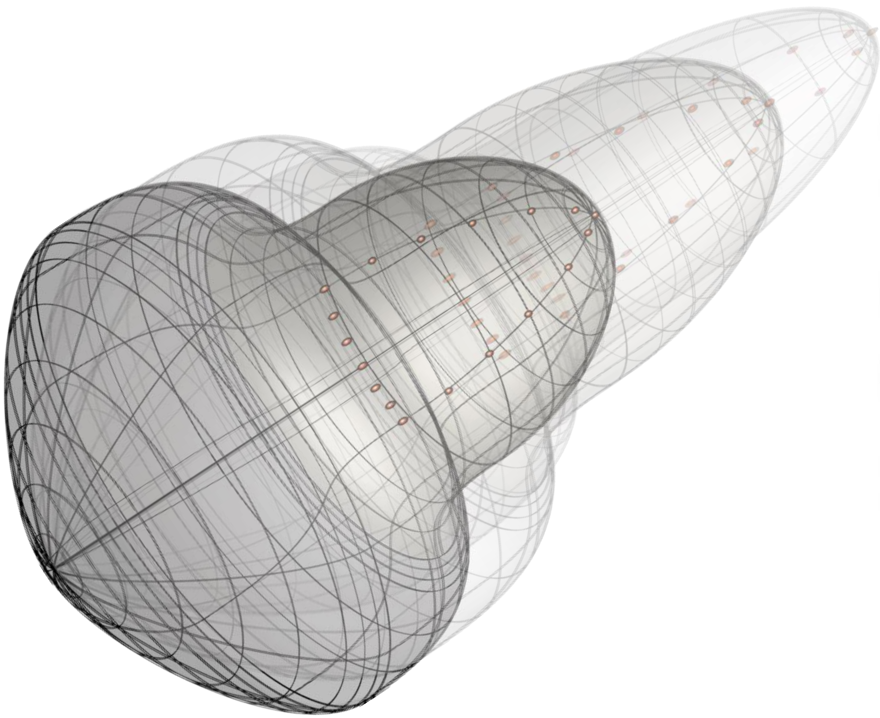
$$\begin{aligned} d_g(p,q) &\longmapsto d_{\lambda g}(p,q) = \lambda^{\frac{1}{2}}\, d_g(p,q) \\ Vol_g(\Sigma) &\longmapsto Vol_{\lambda g}(\Sigma) = \lambda^{\frac{n}{2}}\, Vol_g(\Sigma)\,, \end{aligned}$$

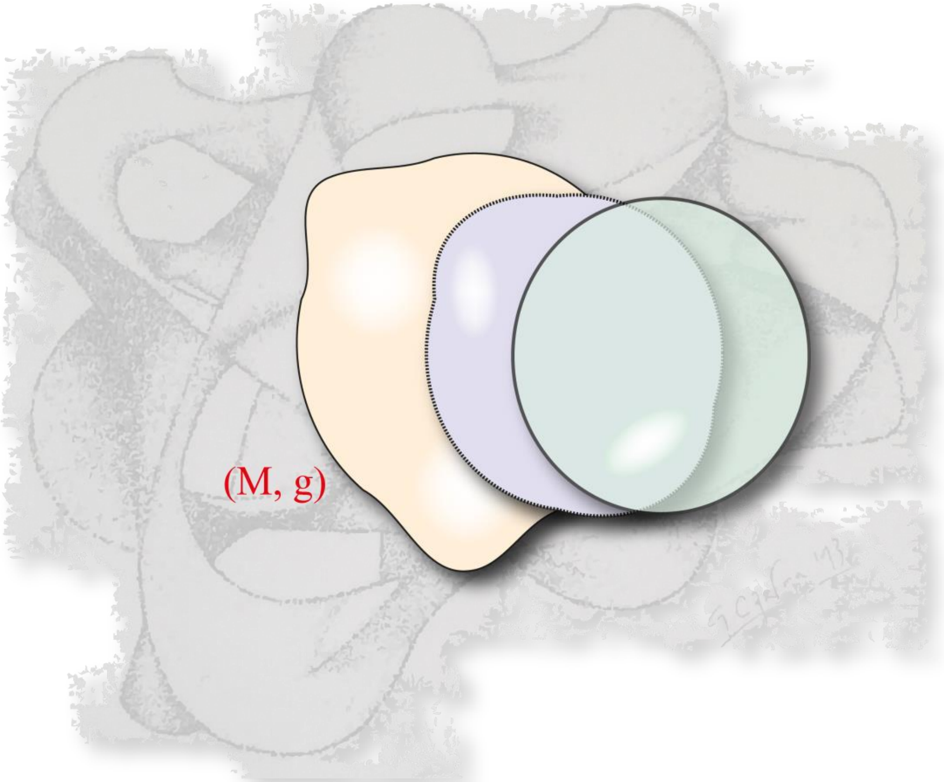
$$\nabla^{(\lambda g)} = \nabla^{(g)},$$

$$\begin{aligned} Hess_{(\lambda g)} &= Hess_{(g)}\,, \\ \triangle_{(\lambda g)} &= \lambda^{-1}\,\triangle_{(g)}\,. \\ \mathcal{R}m(\lambda g) &= \mathcal{R}m(g) \\ \mathcal{S}ec(\lambda g)(X,Y) &= \lambda^{-1}\,\mathcal{S}ec(g)(X,Y) \\ \mathcal{R}ic(\lambda g) &= \mathcal{R}ic(g) \\ \mathcal{R}(\lambda g) &= \lambda^{-1}\,\mathcal{R}(g)\,, \end{aligned}$$



- Subtle interplay between Diffeomorphisms and scaling equivariance
- A further aspect of the Ricci curvature: Not only Einstein, but also Quasi-Einstein Metrics do matter.



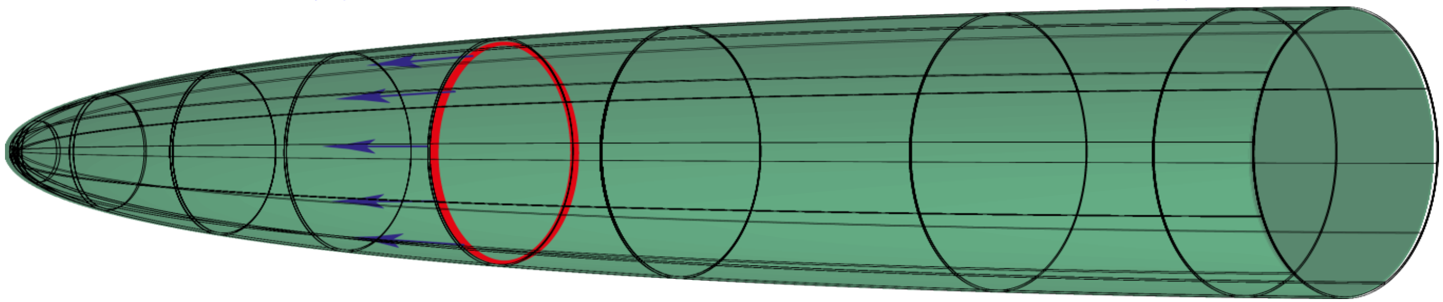


- A Riemannian metric g is Einstein if its Ricci tensor $\mathcal{R}ic(g) = \rho_{(g)}\, g$ for some constant $\rho_{(g)}$.
- The Einstein constant $\rho_{(g)}$ scales non–trivially: Since $\mathcal{R}ic(g)$ is scale–invariant, we must have $\rho_{(\lambda g)} \longmapsto \lambda^{-1}\, \rho_{(g)}$.

Quasi–Einstein metrics are characterized by a Ricci tensor which can be written as

$$\mathcal{R}ic(g) = \rho_{(g)}\, g - \frac{1}{2}\,\mathcal{L}_{V_{(g)}}\, g = \rho_{(g)}\, g - \frac{1}{2}\,(\nabla_i V_k + \nabla_k V_i)\ ,$$

for some constant $\rho_{(g)}$ and some complete vector field $V_{(g)} \in C^{\infty}(M, TM)$.



- If V is a gradient, $V^i = g^{ik}\partial_k f$ for some $f \in C^{\infty}(M, \mathbb{R})$, then the quasi –Einstein condition becomes

$$Ric^{B-E}(g, d\omega) := \mathcal{R}ic(g) + Hess_g\, f = \rho_{(g)}\, g$$

- *i.e.* the isotropy of the Bakry–Emery Ricci curvature of the Riemannian manifold with density

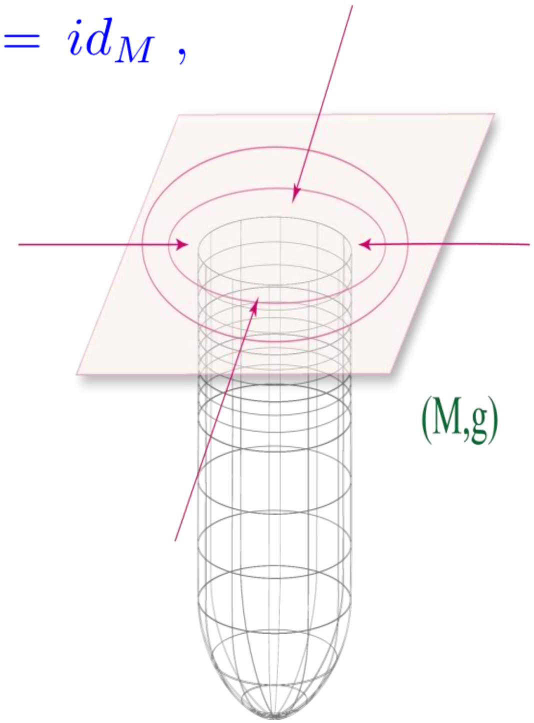
$$(M, g, d\omega := e^{-f} d\mu_g))$$

.

- For $0 \leq \beta < \epsilon < \frac{1}{2\rho_{(g)}}$ define $\lambda(\beta) := (1 - 2\rho_{(g)}\,\beta)$.

- Consider the one–parameter family of diffeomorphisms $\phi_\beta : M \longrightarrow M$ solution of the non–autonomous ordinary differential equation

$$\frac{\partial}{\partial \beta}\phi_\beta(p) = \frac{1}{\lambda(\beta)} V_{(g)}(\phi_\beta(p)), \quad \phi_{\beta=0} = id_M \ ,$$



- and the one–parameter family of metrics defined by

$$g(\beta) := \lambda(\beta)\,\phi^*_\beta g \ .$$

with $g(\beta = 0) = g$.

- By scale invariance and $\mathcal{D}iff(M)$–equivariance we compute

$$\frac{\partial}{\partial\beta} g(\beta) = -2\,\rho_{(g(\beta))}\,g(\beta) + \mathcal{L}_{V_{(g(\beta))}}\,g(\beta) = -2\,\mathcal{R}ic(g(\beta)) \ .$$

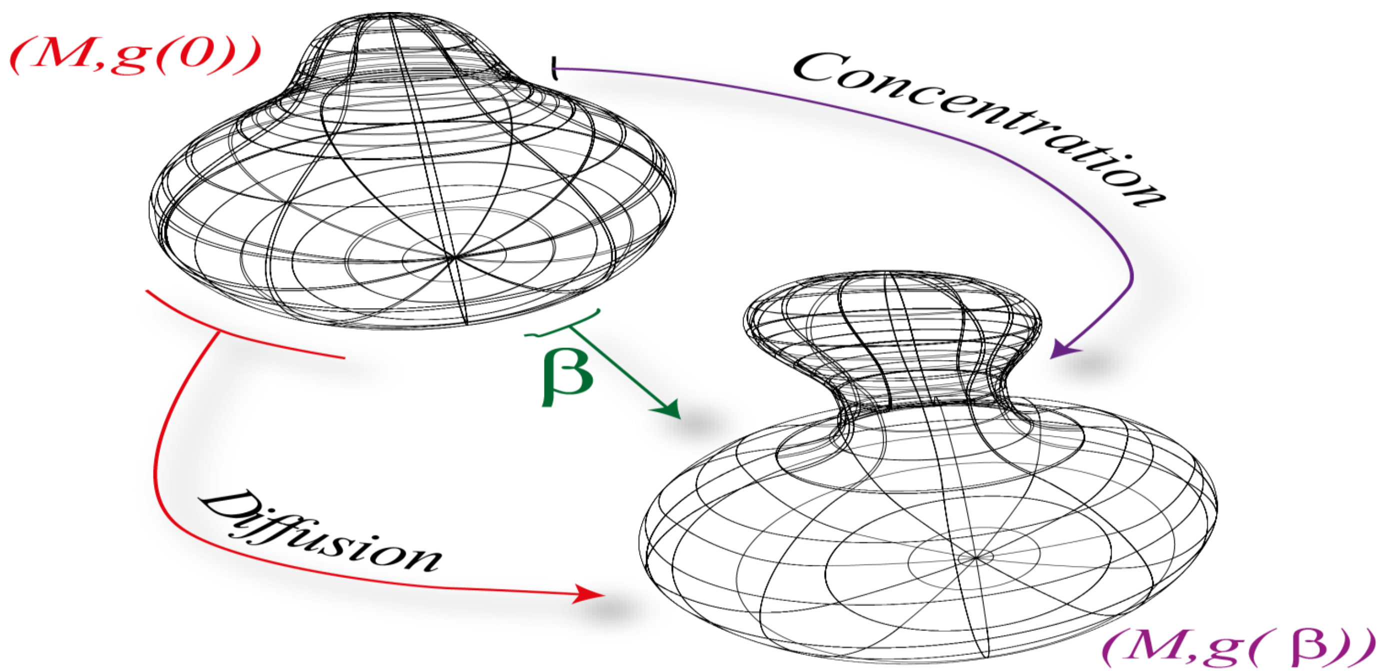
... THE RICCI FLOW EMERGES ...

Hence, under the combined action of this family of diffeomorphisms and of the scaling, the quasi–Einstein metric g generates a self–similar solution $g(\beta) := \lambda(\beta)\,\phi^*_\beta\, g$, $0 \leq \beta < \epsilon$, of the *Ricci flow* , (R. Hamilton, 1982)

$$\frac{\partial}{\partial\beta} g_{ab}(\beta) = -2\,\mathcal{R}_{ab}(\beta),$$

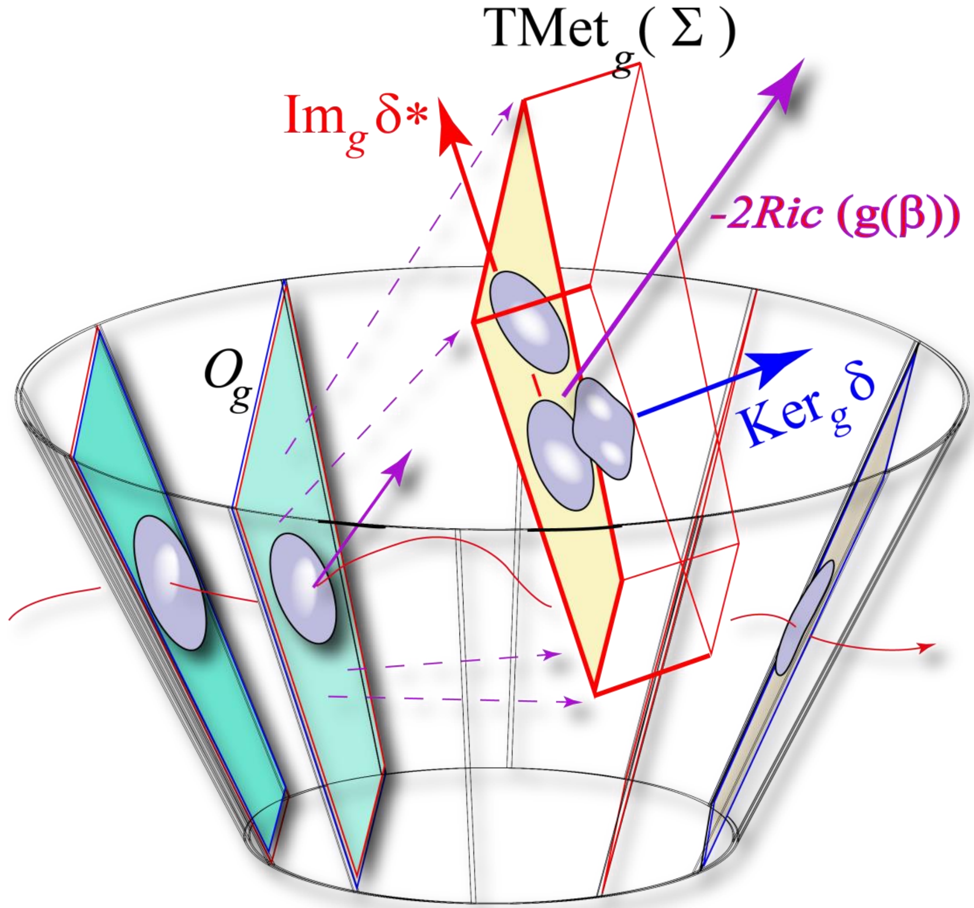
$$g_{ab}(\beta = 0) = g_{ab}\ , \quad 0 \leq \beta < \frac{1}{2\rho_{(g)}}\ . \tag{1}$$

These solutions are known as *Ricci solitons*, (R. Hamilton, 1988).



... AS A NATURAL DYNAMICAL SYSTEM ON $\mathcal{M}et(M)$ ...

The Ricci flow can be thought of as a (weakly–parabolic) dynamical system on $\mathcal{M}et(M)$.



$$\begin{array}{ccc} \mathcal{M}et(M) & \longrightarrow & \mathcal{M}et(M) \\ (M,g) & \mapsto & (M,g(\beta))\,, \end{array}$$

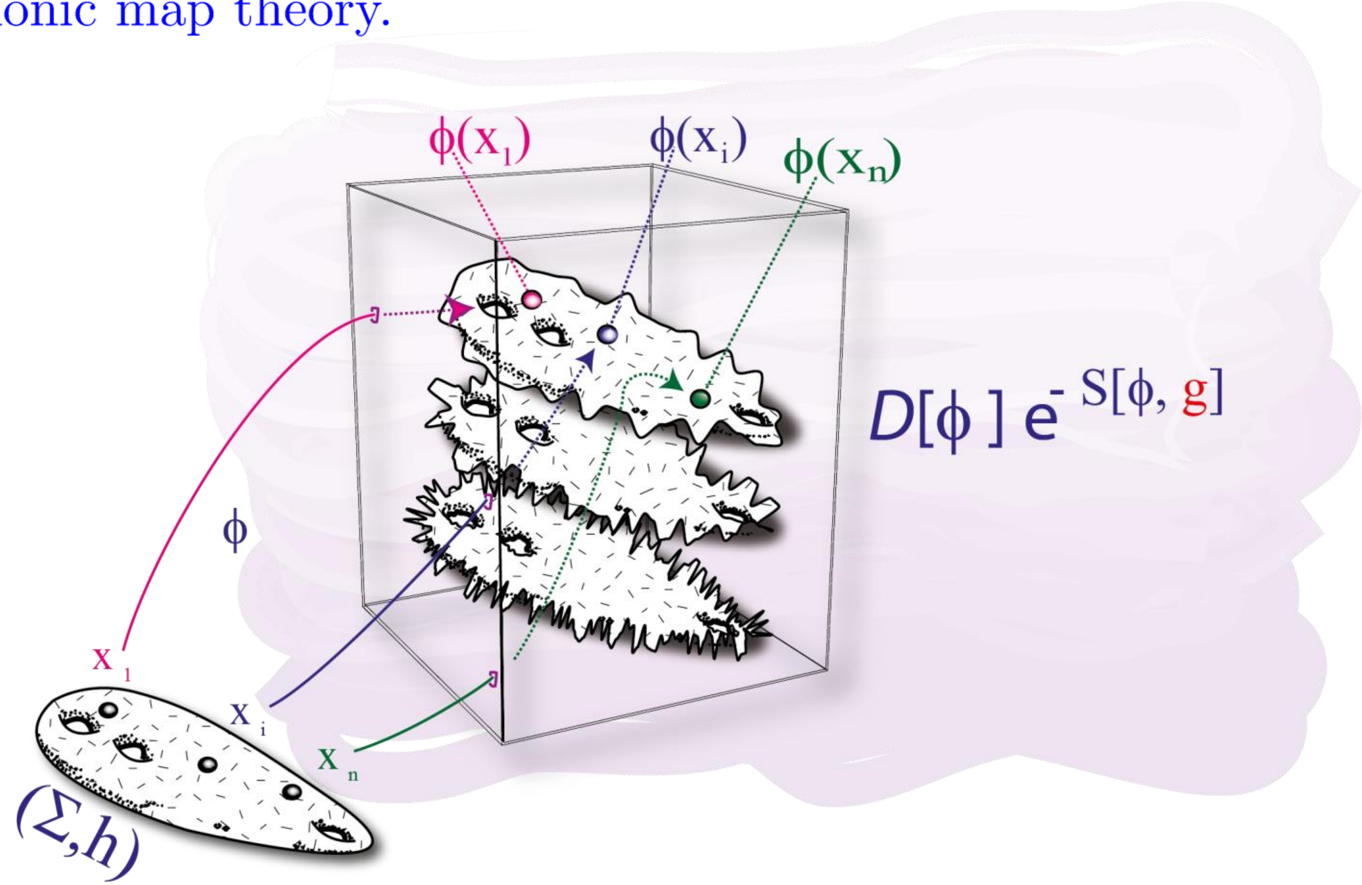
defined by deforming the metric (M,g) in the direction of $-2\mathcal{R}ic(g)$ thought of as a (non–trivial) vector in $T_g\,\mathcal{M}et(M)$, *i.e.*,

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \beta} g_{ab}(\beta) = -2R_{ab}(\beta), \\ \\ g_{ab}(\beta=0) = g_{ab}, \end{array} \right.$$

$$0 \leq \beta \leq \beta^* < T_0$$

A RENORMALIZATION GROUP PERSPECTIVE ...

- Quasi-Einstein metrics originated from theoretical physics (D. Friedan, 1980), in the (perturbative) analysis of the Renormalization Group for (Dilatonic) Non-Linear σ Model (NLσM), the quantum field theory avatar of harmonic map theory.



- A non-trivial point of view on the nature of (Ricci) curvature.

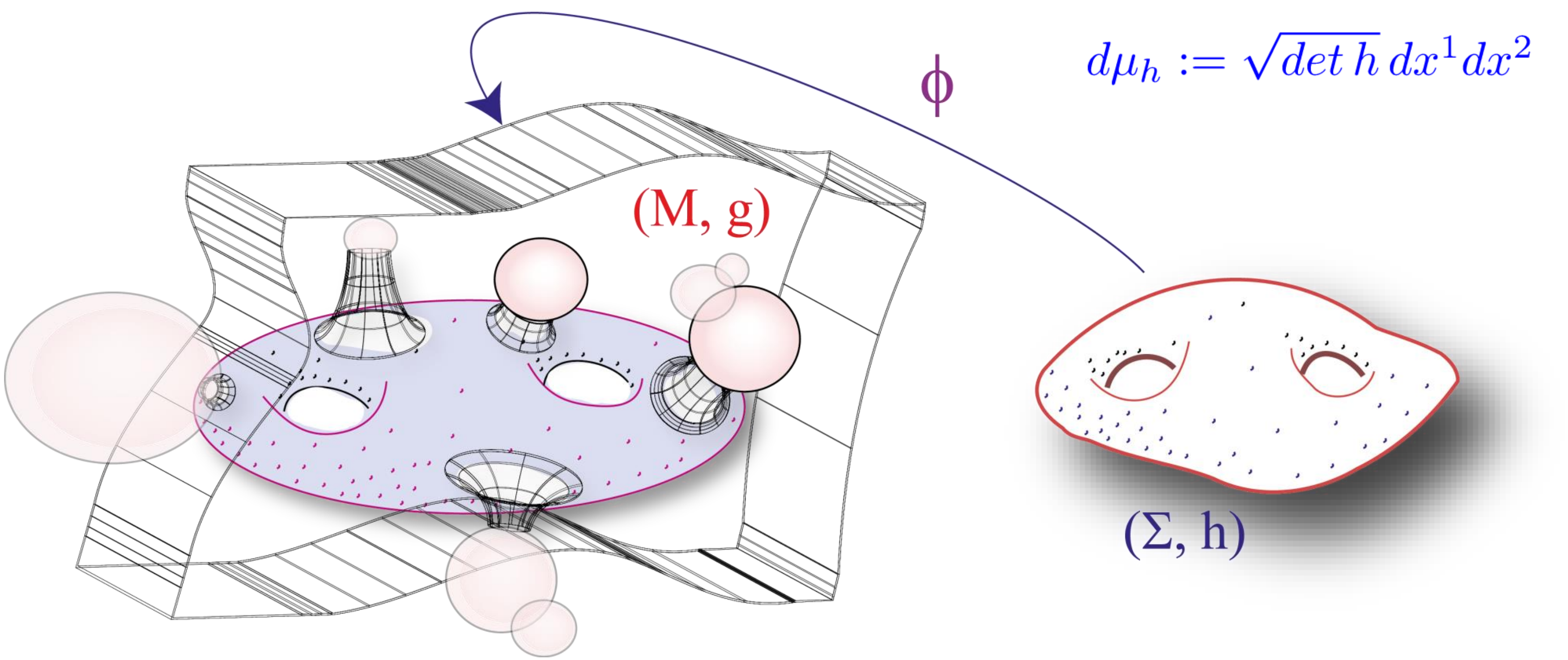
ROOTS IN HARMONIC MAP THEORY

$$E[\phi, g]_{(\Sigma, M)} := \int_\Sigma e(\phi)\, d\mu_h = \frac{1}{2} \int_\Sigma trace_{h(x)} (\phi^* g)\, d\mu_h$$

Conformally invariant Energy density of the map ϕ:

$$e(\phi)(x) := \frac{1}{2} h^{\mu\nu}(x) \frac{\partial \phi^i(x)}{\partial x^\mu} \frac{\partial \phi^k(x)}{\partial x^\nu}\; g_{ik}(\phi(x))\, d\mu_h$$

$$d\mu_h := \sqrt{det\, h}\, dx^1 dx^2$$



IN PHYSICS: NON-LINEAR σ MODEL

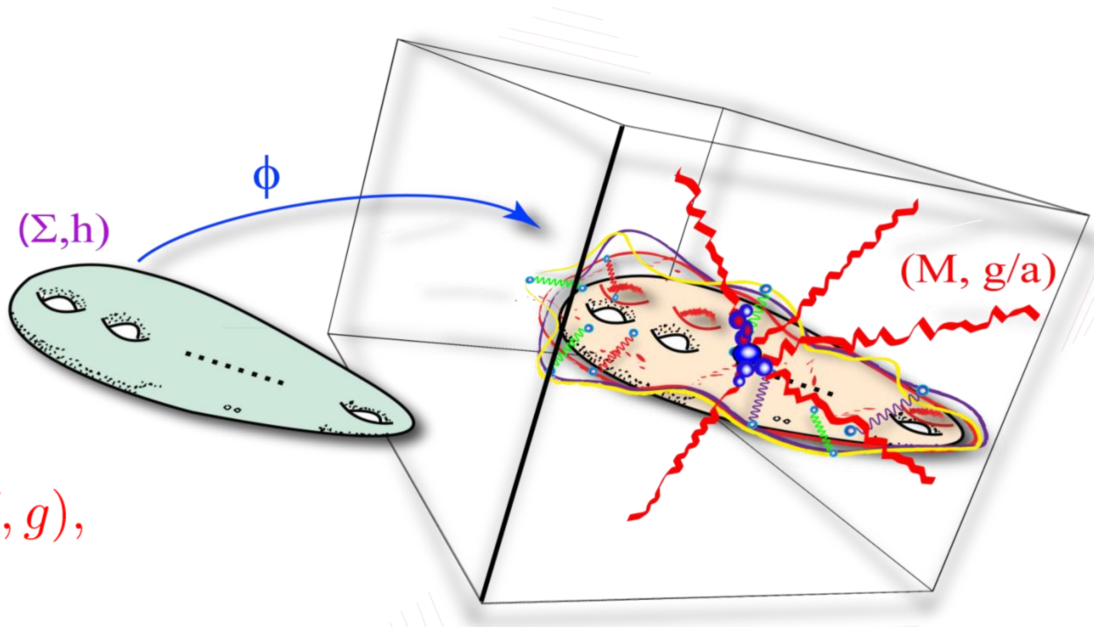
In QFT this translates into the Non-linear σ model action

$$S[\phi;\alpha] := \frac{1}{a}E[\phi,g]_{(\Sigma,M)} = \frac{1}{2a}\int_\Sigma d^2x\sqrt{h}\,h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^k g_{ik}$$

where $a \sim [L]^2$ and the curvature $|Riem(g)|$ of (M,g), (together with the injectivity radius $inj(M,g)$), set the scale at which (Σ,h) perturbatively probes the manifold (M,g) in the point–like limit

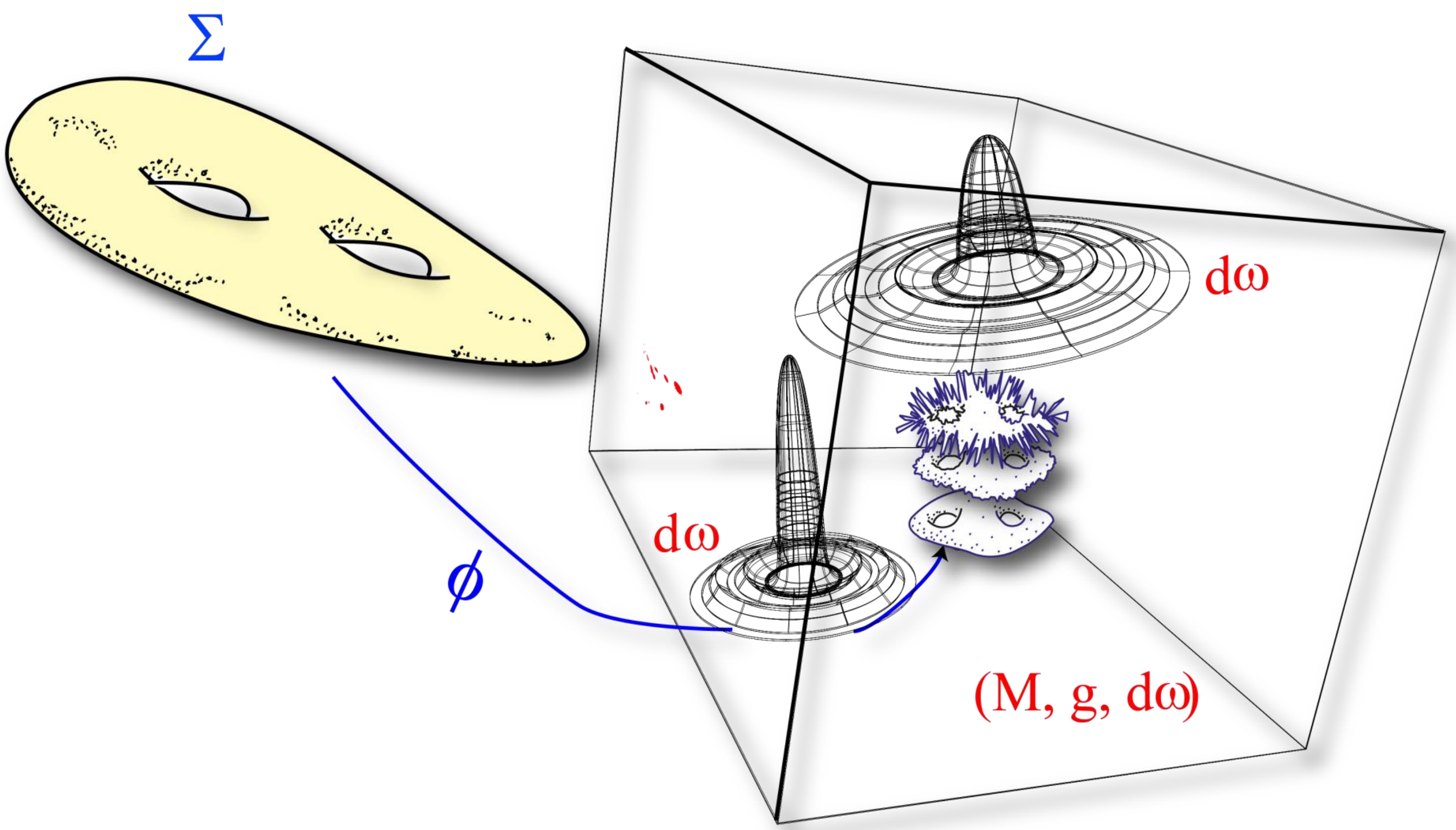
- $a\,r_0^{-2} << 1$

$$r_0 := \min\left\{\frac{1}{3}\,inj\,(M),\ \frac{\pi}{6\sqrt{\kappa}}\right\},$$



κ: the upper bound to the sectional curvature of (M,g), ($1/\sqrt{\kappa} := \infty$, when $\kappa \leq 0$).

If the manifold M carries the structure of a Riemannian metric measure space $(M, g, d\omega)$ with $d\omega = e^{-f}\, d\mu_g$ then we can consider the extension of $E[\phi,\, g]_{(\Sigma, M)}$ associated with the dilatonic coupling $f \in C^\infty(M, \mathbb{R})$ corresponding to the measure $d\omega = e^{-f}\, d\mu_g$.



This extension is provided by the dilatonic non–linear σ–model action

$$S[\phi;\ a,\ f,\ g] \ := \ \frac{1}{a}\,E[\phi,\ g]_{(\Sigma,M)} + \int_\Sigma \mathcal{K}_h\, f(\phi)\, d\mu_h$$

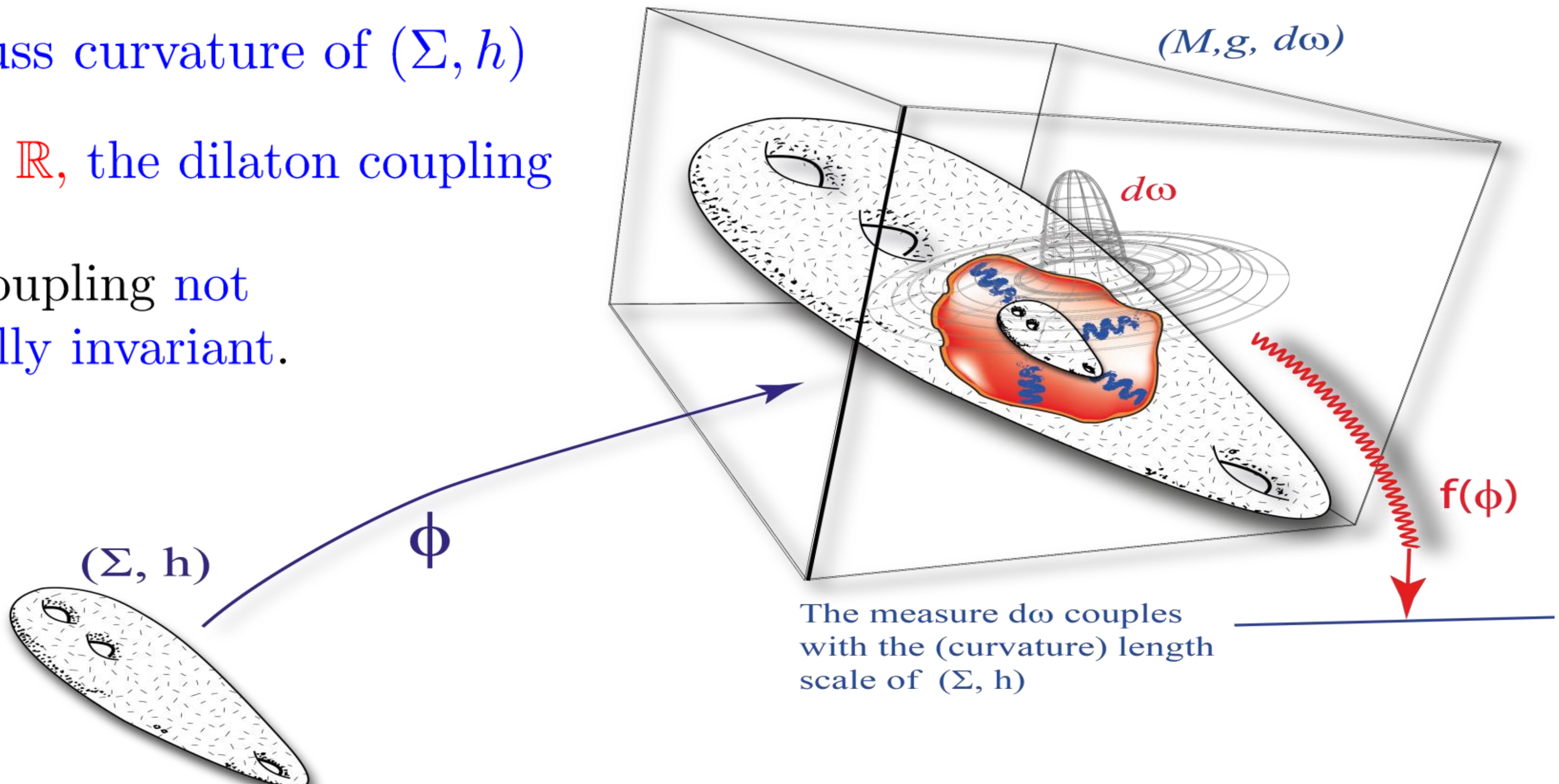
$$:= \ (2a)^{-1}\int_\Sigma \left[tr_{h(x)}\,(\phi^*\, g) - \, 2a\,\mathcal{K}_h\ \ln\ \phi^*\left(\frac{d\omega}{d\mu_g}\right)\right] d\mu_h\ ,$$

$$= \frac{1}{2a}\int_\Sigma d^2x\,\sqrt{h}\ \left[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^k\ \ g_{ik} + 2a\,\mathcal{K}_h(x)\, f(\phi(x))\right]$$

$\mathcal{K}_h$: Gauss curvature of (Σ, h)

$f : M \longrightarrow \mathbb{R}$, the dilaton coupling

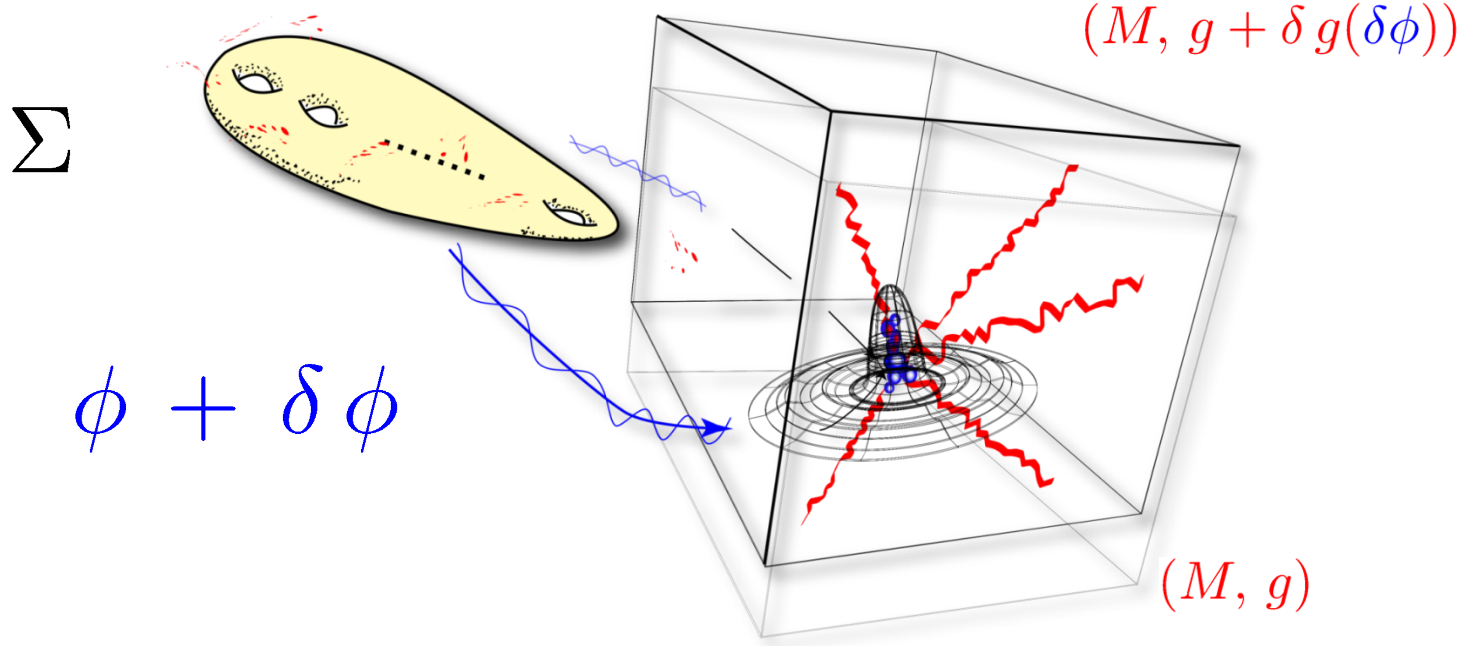
dilaton coupling not conformally invariant.



If, for $dim\Sigma = 2$, we assume the existence of a (typically non existant) reference functional measure $\mathcal{D}[\phi]$ on the non–linear space $Map(\Sigma, M)$, (see however R. Leandre, JMP44, 2003; C. Taubes, JDG70, 2005; J. Weitsman,CMP277, 2008), then

- QFT Heuristics in $dim\,\Sigma = 2$:

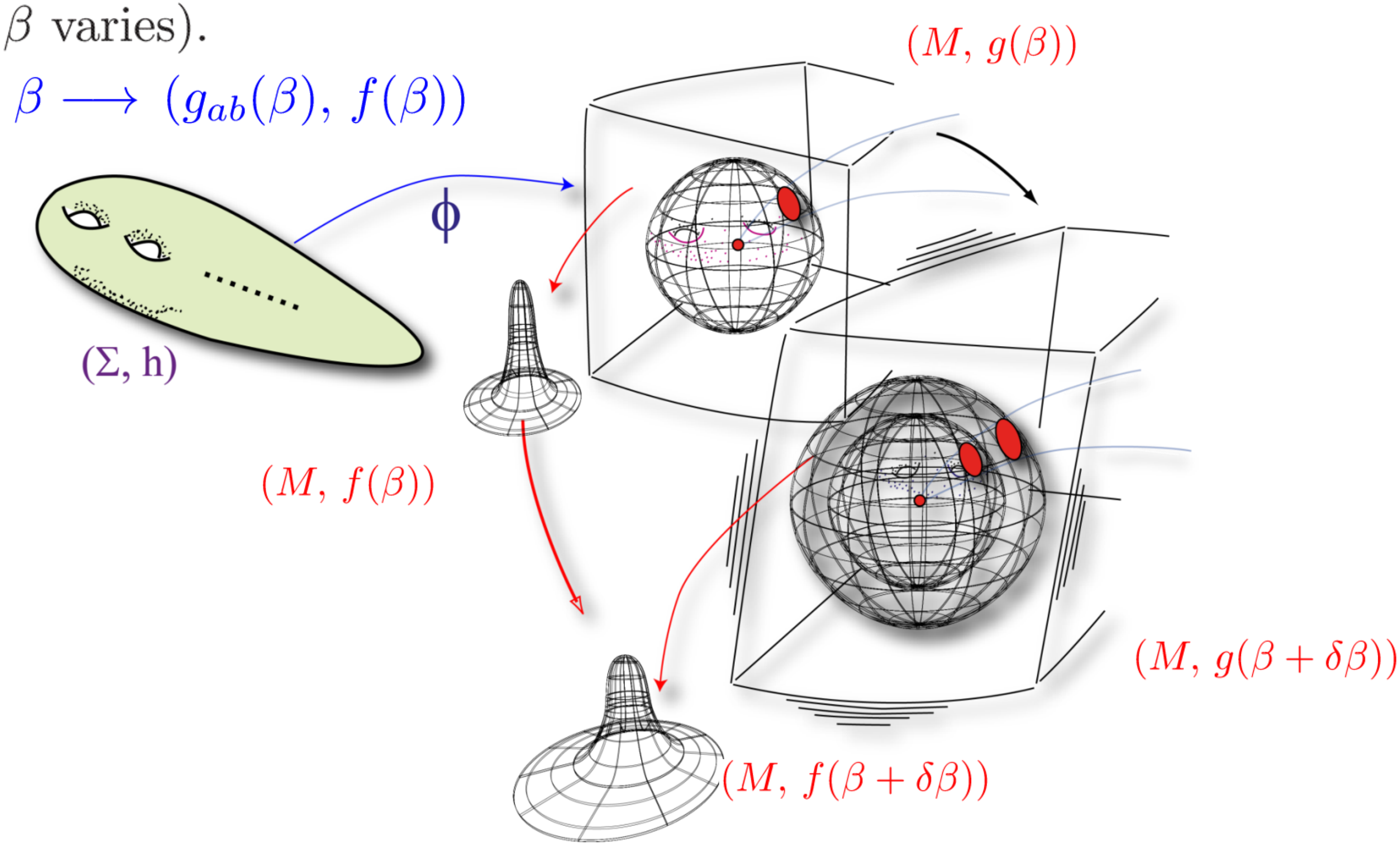
$\mathcal{D}[\phi]$–random fluctuations of $\phi : \Sigma \longrightarrow M$ around a classical configuration ϕ_{cm}, (a center of mass of a large collection ($\to \infty$) of $\mathcal{D}[\phi] - i.i.d.$ constant maps $\{\phi_{(i)}\}$), can modify the geometry of (M, g): they can actually generate the Riemannian structure, (D. Friedan)



... RENORMALIZATION GROUP FLOW (IDEAS) AT WORK...

- via a scale–dependent $\beta := a\,t, t \geq 0$, perturbative renormalization group flow, ($[a] = [L^2]$ a dimensionful coupling of the theory).

The *RG-Flow* is controlled by a large deviation mechanism w.r.t. the Gaussian fluctuations around the (classical) background ϕ_{cm}, (*i.e.* by the control of the exponential decline of large field fluctuations, around ϕ_{cm}, as the energy (= $length^2$) scale β varies).

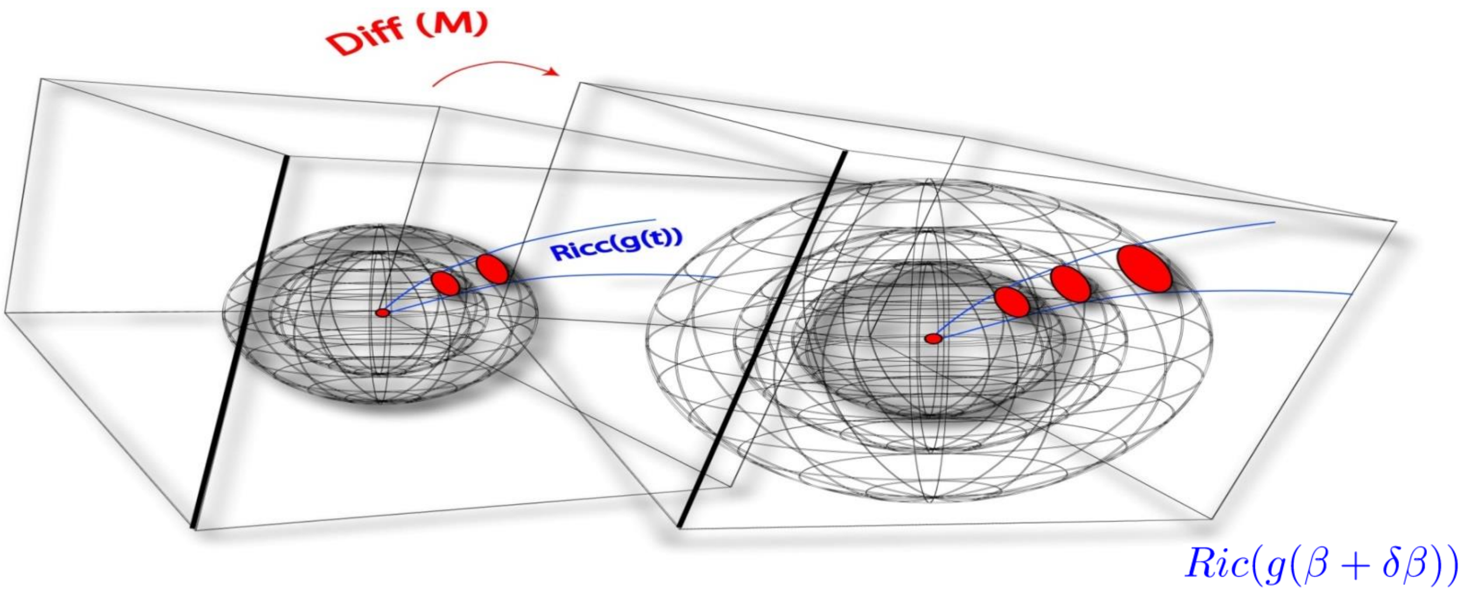


... AN ALTERNATIVE VIEW TO RIEMANNIAN GEOMETRY ...

- This procedure (re)constructs perturbatively the geometry in a ball around ϕ_{cm} as a function of the parameter β according to

$$\frac{\partial}{\partial\beta}\, g_{ik}(\beta) = -2\, R_{ik}(\beta) - 2\nabla_i \nabla_k f - \frac{a}{2}\,(R_{ilmn} R_k^{lmn}) + \mathcal{O}(a^2)$$

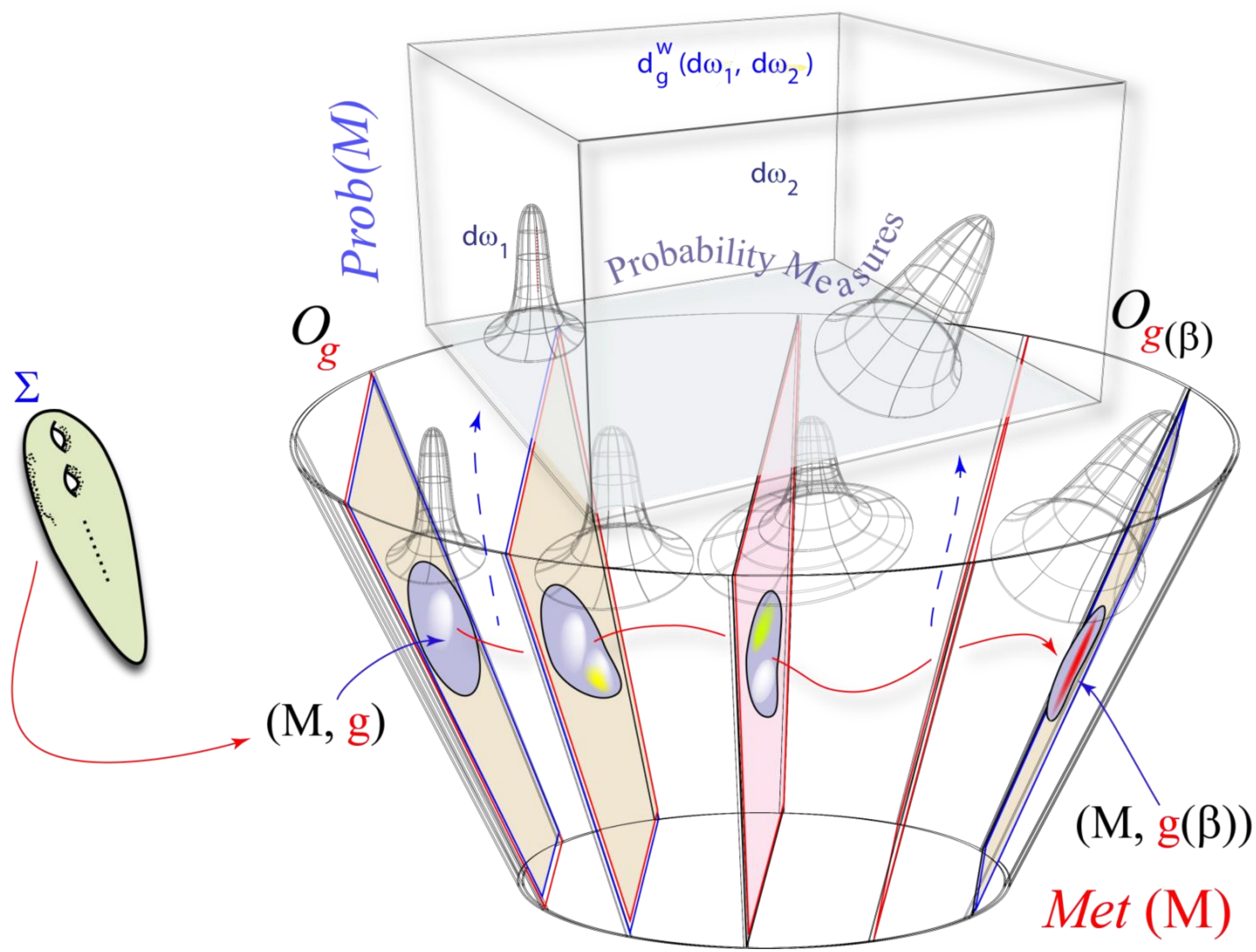
$$\frac{\partial}{\partial\beta}\, f(\beta) = \Delta f(\beta) - |\nabla f(\beta)|^2 + \mathcal{O}(a^2).$$



- Extension of Ricci flow via non–perturbative RG flow arguments difficult to handle. Perhaps, more sophisticated ideas needed.

... (Almost) THE SAME WITHOUT QFT: A WASSERSTEIN GEOMETRY PERSPECTIVE

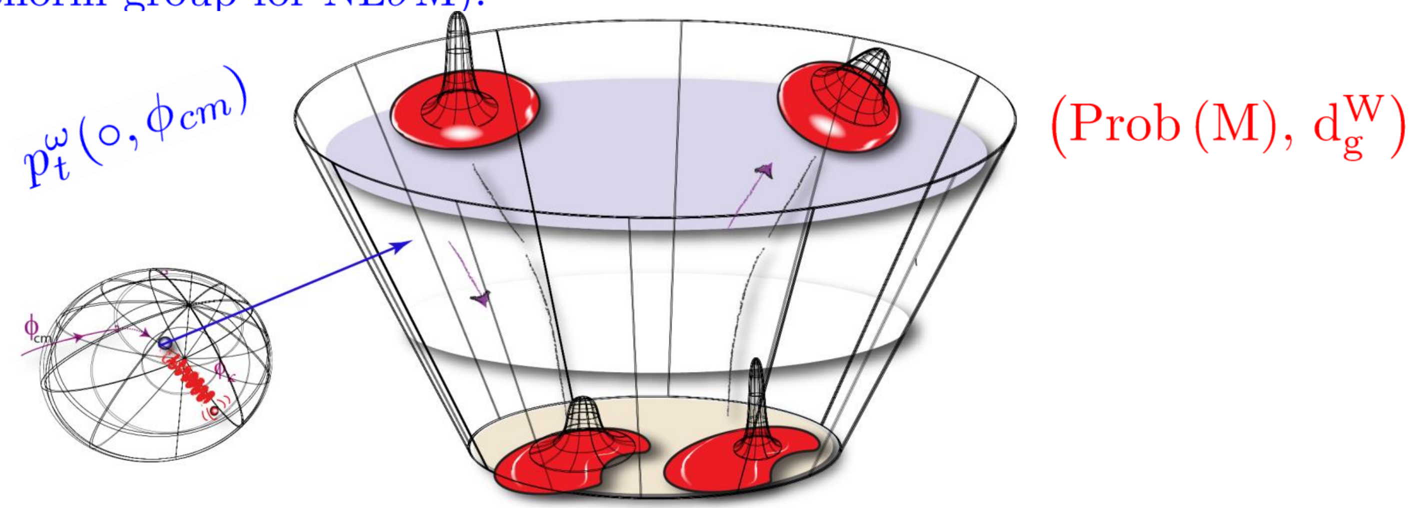
The role of Ricci curvature as a *generator* of a natural RG flow has a rigorous mathematical avatar: Many of its feature follow from a heat kernel localization mechanism by exploiting the Wasserstein geometry of the space of probability measures on (M, g).



The idea is to use the (weighted) heat kernel of $(M, g, d\omega)$, $(t, \delta_x) \longmapsto p_t^\omega(\circ, x)$, with source at $x \in M$, to generate an injective embedding of $(M, g, d\omega)$

$$\begin{aligned} \iota_{p_t^\omega} : (M, g) \quad &\hookrightarrow \quad \left(\mathrm{Prob}\,(\mathrm{M}),\, \mathrm{d}_\mathrm{g}^\mathrm{W}\right) \\ x \quad &\longmapsto \quad \iota_{p_t^\omega}(x) := p_t^\omega(\circ, x)\,. \end{aligned}$$

in the space $\left(\mathrm{Prob}(M),\ d_g^W\right)$ of all probability measures over M endowed with the quadratic Wasserstein distance d_g^W, (N.Gigli–C. Mantegazza, for the pure Riemannian case, M.C. for the general case of $(M, g, d\omega)$ and the relation with Renorm group for NLσM).

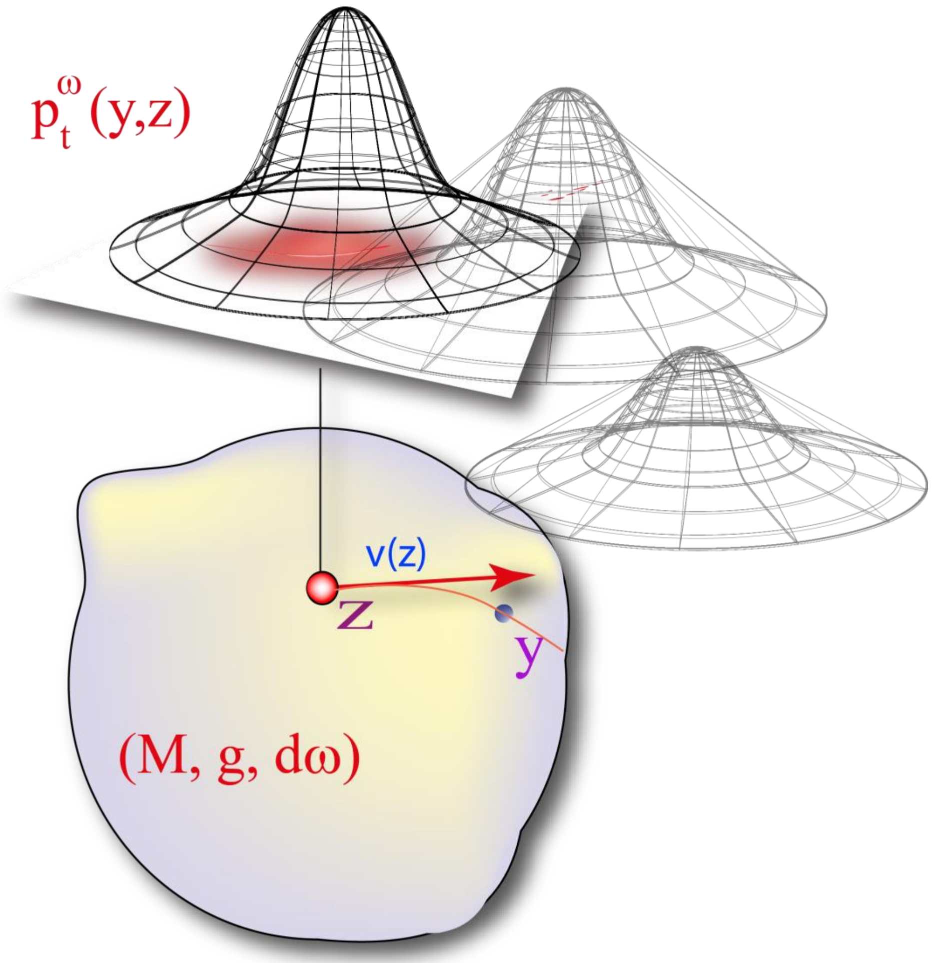


- And *pull–back*, via $\iota_{p_t^\omega}$, the Wasserstein distance to a t–dependent metric tensor g_t on M, (Gigli-Mantegazza for the pure Riemannian case, with a peripheral contribution by M. C. in the Riemannian metric measure space case.)

For any smooth vector field v over M, and $t > 0$, there exists a unique smooth solution $\widehat{\psi}_{(t,z,v)}$, smoothly depending on the data t, z, v, of the elliptic PDE

$$\nabla^{i}_{(y)} \left(p^{\omega}_{t}(y, z)\, \nabla^{(y)}_{i}\, \widehat{\psi}_{(t,z,v)}(y) \right) = -\, v^{i}(z)\, \nabla^{(z)}_{i}\, p^{\omega}_{t}(y, z) \; ,$$

and with $\nabla^{(y)}_{i}\, \widehat{\psi}_{(t,z,v)}(y) \neq 0$ for all $v \neq 0$.



$$\nabla^{(y)}\, \widehat{\psi}_{(t,z,v)}(y)$$

$$T_z(M) \to \mathcal{T}_{p^{\omega}_{t}} Prob(M)$$

$$v(z) \to \nabla^{(y)}_{i}\, \widehat{\psi}_{(t,z,v)}(y)$$

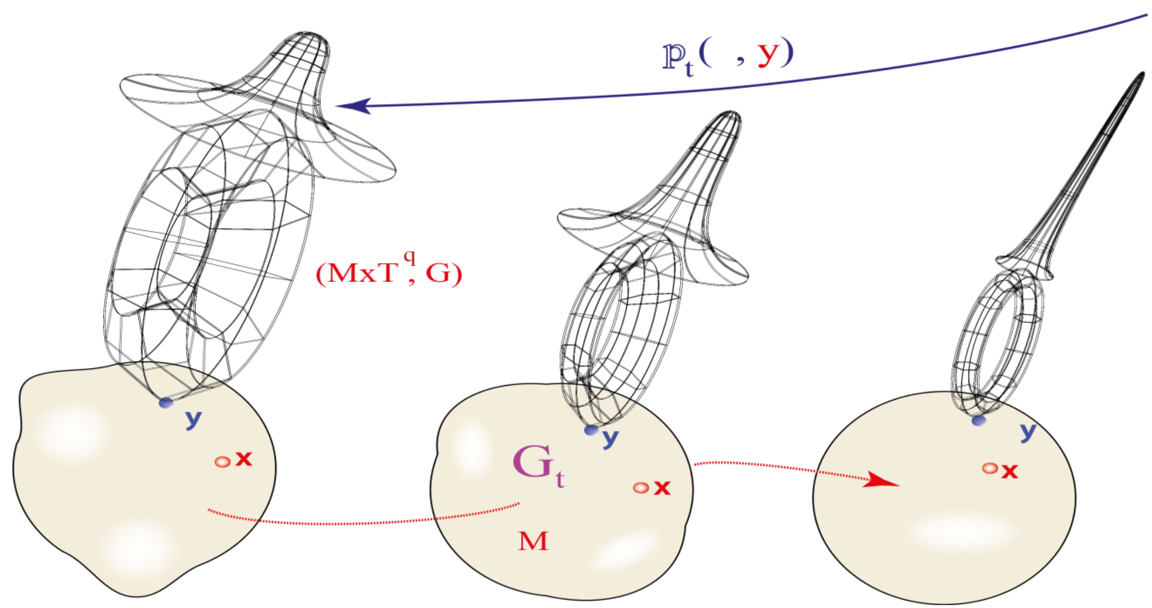
By exploiting this heat kernel parametrization of the vector fields of (M, g), one defines, for all $t \in (0, \infty)$, a t–dependent metric tensor g_t on M according to (Gigli-Mantegazza, m.c.)

$$g_t\,(v(x), w(x)) \;:=\; \int_{M_y} g^{ik}(y)\, \nabla_i^{(y)}\, \widehat{\psi}_{(t,x,v)}(y)\, \nabla_k^{(y)}\, \widehat{\psi}_{(t,x,w)}(y)\, p_t^{\omega}(y, x)\, d\mu_g(y)\;.$$

This is the scale dependent metric induced on M by the heat kernel immersion. As $t \searrow 0$, the metric g_t reduces to g, *i.e.* $\lim_{t \searrow 0} g_t(v, v) \;=\; g(v, v),\;\; v \in T_y\, M$, $y \in M$, and

$$g_t(v, v) \;\leq\; e^{-\,2\,K_g^{B-E}\,t} g(v, v)\;,$$

where K_g^{B-E} denotes the lower bound of the Bakry–Emery Ricci curvature of $(M, g, d\omega))$, $Ric^{B-E}(g, d\omega) :=\; Ric(g) + Hess_g\, f$.



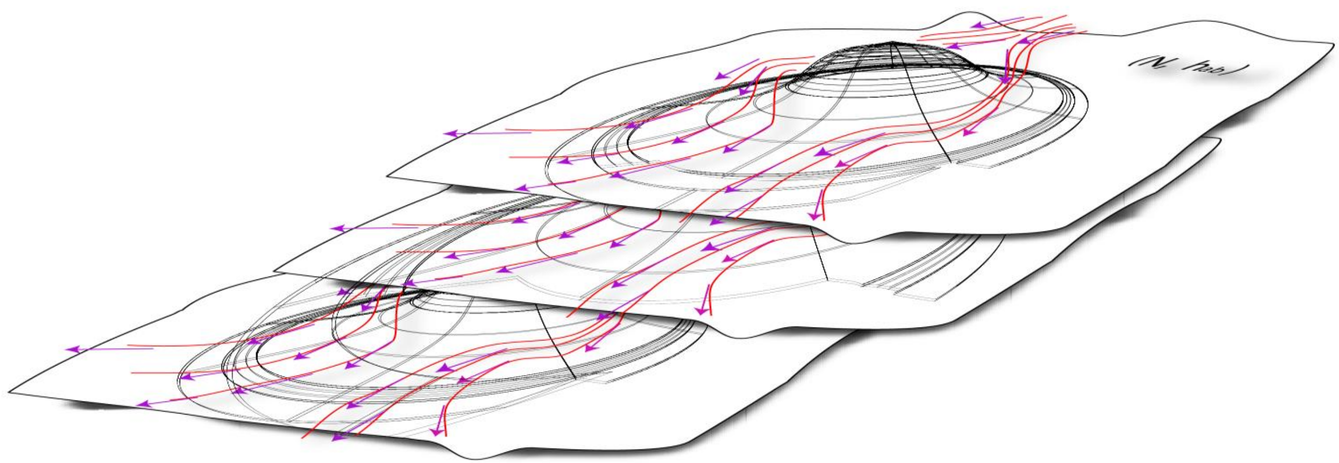
$t \to (M, g_t)$ induces a corresponding flow in the harmonic map energy ,

$$t \mapsto E[\phi,\, g_t]_{(\Sigma,\, M)} + a_t \int_{\Sigma} \mathcal{K}_h\, f_t(\phi)\, d\mu_h$$

$$= a_t\, S[h, \phi;\; a_t,\, f_t,\, g_t]\,,$$

with

$$S[h, \phi;\; a_t,\, f_t,\, g_t] \le e^{-\,2\, K_g^{B-E}\, t}\, S[h, \phi;\; a,\, f,\, g]$$



A rather sophisticated use of the (weak) Riemannian geometry of the Wasserstein space $\left(Prob(M, g),\, d_g^W)\right)$, (related to the Riemannian geometry of the diffeomorphisms group $\mathcal{D}iff(M)$ *a' la Arnold*), allows to compute full–fledged flows for the metric $t \longmapsto g(t)$ and for the dilaton field $t \longmapsto f(t)$:

$$\frac{\partial}{\partial t}\, g_t(u,w) = -2\, Ric^{(t)}(u,w) - 2\, Hess_{g(t)}\, f_t\,(u,v)$$

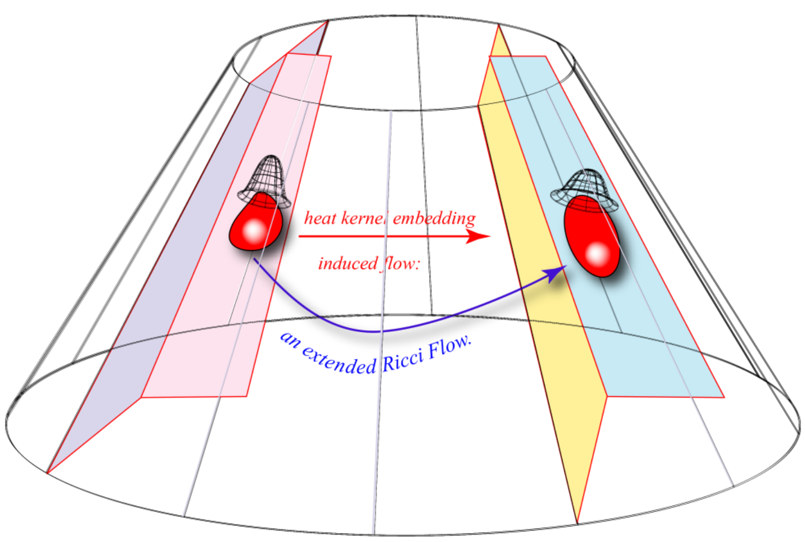
$$-2 \int_M \left(Hess\, \widehat{\psi}_{(t,u)} \cdot Hess\, \widehat{\psi}_{(t,w)} \right) p_t^{(\omega)}(y,z)\, d\omega(y)\ ,$$

where $Ric^{(t)}$ denotes the Ricci curvature of the evolving metric $(M,\, g_t)$

$$Ric^{(t)}(u,v) := \int_M Ric\left(\nabla\psi_{(t,U)},\, \nabla\psi_{(t,W)}\right)\, p_t^{(\omega)}(y,z) d\omega(y)$$

$$\frac{\partial}{\partial t}\, f_t^{(\omega)} = \triangle_{\omega}^{(z)}\, f_t^{(\omega)} - \frac{2}{q}\, |\nabla\, f_t^{(\omega)}|_g^2$$

$$= \triangle_{g}^{(z)}\, f_t^{(\omega)} - \nabla^i f\, \nabla_i\, f_t^{(\omega)} - \frac{2}{q}\, |\nabla\, f_t^{(\omega)}|_g^2\ .$$

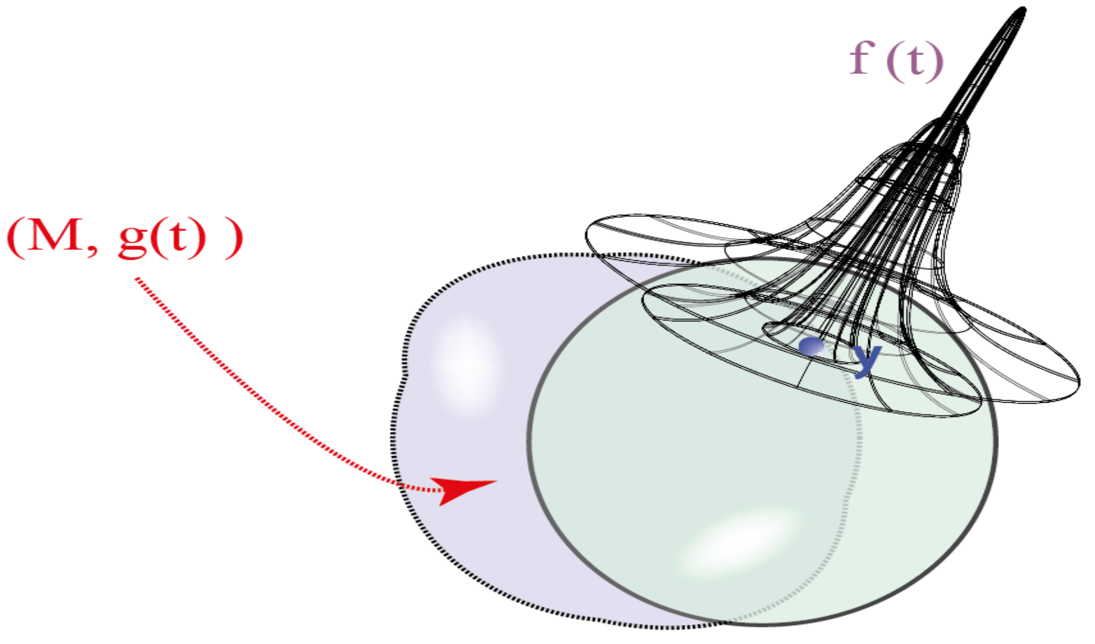


Moreover, for almost every t, one computes

$$\frac{\partial}{\partial t} g_{ab}(t)|_{t=0} = -2R_{ab} - 2\nabla_a \nabla_b f + \frac{2}{q} \nabla_a f \nabla_b f$$

$$\frac{\partial}{\partial t} f(t)\,|_{t=0} = \Delta_g f - \frac{2+q}{q} |\nabla f|^2$$

The $q \to \infty$ corresponds to the point–like limit for the localization. In this case we recover the standard RG computation.



The flow

$$\frac{\partial}{\partial t}\, g_t(u,w) = -\,2\,Ric^{(t)}(u,w) -\, 2\,Hess_{g(t)}\,f_t\,(u,v)$$

$$-\,2\int_M \left(Hess\,\widehat{\psi}_{(t,u)}\cdot Hess\,\widehat{\psi}_{(t,w)}\right) p_t^{(\omega)}(y,z)\,d\omega(y)\ ,$$

has a number of remarkable properties:

- good behaviour under Gromov–Hausdorff convergence
- (potential) better control on singularity generation w.r.t. standard Ricci flow
- it extends naturally to measure metric space not generated by smooth Riemannian metrics
- It is a gradient flow w.r.t. an entropy functional that generalizes Perelman's $\mathcal{F}$ energy
- However: very hard to provide explicit non-trivial examples. Lot of territory to explore.