Some analytical aspects of the Kontsevich matrix model

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Plan of the talk

- The Witten’s conjecture and the Witten–Kontsevich tau function.
- The Painlevé I hierarchy and the string equation.
- Kontsevich’s model.
- The convergence of the Kontsevich model to (some) solutions of the Painlevé I hierarchy.

The Deligne-Mumford moduli space of Riemann surfaces

\[ \overline{M}_{g,n} := \left\{ \text{Riemann surfaces with } n \text{ marked points} \right\} / \sim \]

A point in \( \overline{M}_{g,n} \) is a (possibly singular) Riemann surface with \( n \) marked points (modulo isomorphisms).

\( \overline{M}_{g,n} \) is a complex orbifold of dimension \( 3g - 3 + n \).

We denote with \( \mathcal{L}_j \) the tautological line bundle, their fibers over \([C]\) are given by \( T^*_p C \), \( \psi_j \) will denote the corresponding Chern classes.
Intersection numbers

Intersection numbers are given by the integrals

$$\langle \tau_0^{k_0} \tau_1^{k_1} \cdots \rangle_{g,n} := \int_{\mathcal{M}_{g,n}} \psi_1^{\ell_1} \wedge \cdots \wedge \psi_n^{\ell_n},$$

where $k_j = \# \text{ occurences of } j \text{ as an exponent}.$

Example:

$$< \tau_0^2 \tau_1^3 \tau_2 > = \int \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \psi_4^2 \wedge \psi_5 \wedge \psi_6^0.$$

The numbers $k_i$ satisfy

$$3g - 3 + n = \sum_{k=0}^{\infty} jk_j,$$

$$n = \sum k_j.$$

Let’s define

$$F(T_0, T_1, \ldots, ) := \sum \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \tau_{\ell}^{k_{\ell}} \cdots \rangle \prod_{j} \frac{T_j^{k_j}}{k_j!}.$$
Witten’ conjecture (Kontsevich theorem) :

Let \( \tilde{R}_n[U] \) be the Lenard polynomials defined by the recursion

\[
\tilde{R}_0[U] = U, \quad \frac{\partial \tilde{R}_{n+1}}{\partial T_0} = \frac{1}{2n+1} \left( \frac{\partial U}{\partial T_0} + 2U \frac{\partial}{\partial T_0} + \frac{1}{4} \frac{\partial^3}{\partial T_0^3} \right) \tilde{R}_n.
\]

**Theorem** : The formal series

\[
F(T_0, T_1, \ldots) := \sum \langle \tau_0^{k_0} \tau_1^{k_1} \cdots \tau_\ell^{k_\ell} \cdots \rangle \prod T_j^{k_j} / k_j!
\]

is uniquely determined by the following conditions :

1) \( U := \frac{\partial^2 F}{\partial T_0^2} \) is a solution of the Korteweg de-Vries hierarchy

\[
\frac{\partial U}{\partial T_i} = \frac{\partial}{\partial T_0} \tilde{R}_i[U], \quad i \geq 0.
\]

2) \( F \) satisfies the string equation

\[
\frac{\partial F}{\partial T_0} = \sum_{i \geq 0} T_{i+1} \frac{\partial F}{\partial T_i} + \frac{T_0^2}{2}.
\]

In other words, \( e^F = \tau \) is a tau function for the KdV hierarchy, uniquely determined by the Virasoro constraints or -equivalently- by its initial value \( \ln \tau = \frac{T_0^3}{6} \).
String equation and the Painlevé I hierarchy

\[
\begin{align*}
\frac{\partial^2 F}{\partial T_0^2} &= \sum_{i \geq 0} T_{i+1} \frac{\partial^2 F}{\partial T_i \partial T_0} + T_0 \\
\frac{\partial^2 F}{\partial T_0 \partial T_i} &= \tilde{R}_i \left[ \frac{\partial^2 F}{\partial T_0^2} \right]
\end{align*}
\]

Putting \( T_i = 0 \) for all \( i \neq 0, N \) we get the collection of equations

\[
T_N \tilde{R}_N [U] = U - T_0, \quad N \geq 1
\]

known as Painlevé I hierarchy.

**Remark:**

The same equations can be written as

\[
[L, M] = 1, \quad L := \frac{\partial^2}{\partial T_0^2} - U.
\]

(Douglas, “String in less than one dimension”).
The Kontsevich matrix model

\[ Z_n(x; Y) := \frac{\int_{H_n} dM e^{\text{Tr}\left(i\frac{M^3}{3} - YM^2 + ixM\right)}}{\int_{H_n} dM e^{-\text{Tr}(YM^2)}} , \]

\[ H_n := \left\{ M = M^\dagger \in \text{Mat}_{n\times n}(\mathbb{C}) \right\} \]

\[ Y := \text{diag}(y_1, \ldots, y_n) \]

\( (x \text{ is added for later convenience}) \).

**Theorem** (Kontsevich, 1992) :

When \( n \to \infty \), the following formal identity holds

\[ F(\vec{T}) = \lim_{n \to \infty} \ln Z_n(0; Y), \quad \text{i.e.} \quad \tau_{WK}(\vec{T}) = \lim_{n \to \infty} Z_n(0; Y), \]

under the identification (Miwa's variables)

\[ T_j = T_j(Y) := -(2j - 1)!! \sum_{\ell=1}^{n} \frac{1}{y_\ell ^{2j+1}}, \]

for \( |Y| \to \infty \).
How do we choose $Y$ in such a way that $Z_n(0; Y)$ converges to a solution of the PI hierarchy? What are the properties of such solutions?
The main idea

It's easy to prove that $Z_n(x; Y)$ can be written as a “wronskian” - type determinant

$$Z_n(x; Y) = 2^n \pi \frac{n}{2} e^{\frac{3}{2} \text{Tr} Y^3 + x \text{Tr} Y} \frac{\det \left[ \text{Ai}^{(j-1)}(y_k^2 + x) \right]_{k,j \leq n} \prod_{j=1}^n (y_j) \frac{1}{2}}{\prod_{j<k}(y_j - y_k)} \quad \text{Re} y_j > 0,$$

and this suggest a link with Darboux transformations...

Let's consider the system

$$\begin{cases}
\partial_\lambda \Psi_0(x; \lambda) = \begin{pmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{pmatrix} \Psi_0(x; \lambda), \\
\partial_x \Psi_0(x; \lambda) = \begin{pmatrix} 0 & -i \\ i(\lambda + x) & 0 \end{pmatrix} \Psi_0(x; \lambda),
\end{cases}$$

and let's add poles on the points $\{\lambda_1, \ldots, \lambda_n\}$, $\lambda_k = y_k^2$ to get the new system
The main idea II

\[
\begin{align*}
\partial_{\lambda} \Psi_n &= A \Psi_n, \\
A &= i \sigma_+ - i \left( \lambda + \frac{x}{2} - \partial_x a^{(n)} \right) \sigma - \sum_{j=1}^{n} \frac{A_j}{\lambda - \lambda_j}, \\
\partial_x \Psi_n &= U \Psi_n, \\
U &= i \sigma_+ - i \left( \lambda - 2 \partial_x a^{(n)} \right) \sigma, \\
\partial_{\lambda_k} \Psi_n &= -\frac{A_k}{\lambda - \lambda_k} \Psi_n, \quad k = 1, \ldots, n.
\end{align*}
\]  

(1)

The isomonodromic (Jimbo-Miwa-Ueno) tau function associated to the system above is defined by the equations

\[
\partial_{\lambda_k} \ln \tau_n = \text{res}_{\lambda_k} \text{Tr} A^2 d\lambda, \quad \partial_x \ln \tau_n = a^{(n)}
\]

and we will prove that

\[
\tau_n(x, \{\lambda_k\}) = e^{\frac{x^3}{12}} Z_n(x, Y).
\]

Once this equality is established, one can study the large \( n \) limit of the Riemann–Hilbert problem associated to the system (1)…
An extension of the Kontsevich matrix model I:

Remark:

\[ \text{Ai}(\lambda) = \frac{e^{-\frac{2}{3}\lambda^{\frac{3}{2}}}}{2\sqrt{\pi}\lambda^{\frac{1}{4}}}(1 + \mathcal{O}(\lambda^{-\frac{3}{2}})), \quad \lambda \to \infty, \]

\[ \Rightarrow e^{\frac{2}{3}y^3 + xy}\text{Ai}(y^2 + x) = \begin{cases} 
\frac{1}{2\sqrt{\pi}\sqrt{y}}(1 + \mathcal{O}(y^{-3})) & \text{for } \text{Re}y > 0, \\
\frac{e^{\frac{4}{3}y^3 + 2xy}}{2\sqrt{\pi}\sqrt{y}}(1 + \mathcal{O}(y^{-3})) & \text{for } \text{Re}y < 0,
\end{cases} \]

\[ \Rightarrow Z_n(x; Y) = 2^n \pi^{\frac{n}{2}} e^{\frac{2}{3} \text{Tr}Y^3 + x \text{Tr}Y} \frac{\det \left[ \text{Ai}^{(j-1)}(y_k^2 + x) \right]_{k,j \leq n}}{\prod_{j<k} (y_j - y_k)^{\frac{1}{2}}} \prod_{j=1}^{n} (y_j)^{\frac{1}{2}}, \]

this expression admits a “regular” expansion if \( \text{Re}(y_i) > 0 \) for all \( i \).
An extension of the Kontsevich matrix model II:

\[ Y \mapsto y^{(0)} \sqcup y^{(1)} \sqcup y^{(2)}; \]

\[
Z_n(x; y^{(0)}, y^{(1)}, y^{(2)}) = C_n \frac{e^{\frac{2}{3} X Y + X Y}}{\prod_{j<k} (y_j - y_k)} \det \begin{bmatrix}
\mathbf{A}_{i_0}^{(k-1)}(y_j^2 + x)_{y_j \in y^{(0)}_{1\leq k \leq n}} \\
\mathbf{A}_{i_1}^{(k-1)}(y_j^2 + x)_{y_j \in y^{(1)}_{1\leq k \leq n}} \\
\mathbf{A}_{i_2}^{(k-1)}(y_j^2 + x)_{y_j \in y^{(2)}_{1\leq k \leq n}}
\end{bmatrix}.
\]

\[ \mathbf{A}_s(\lambda) := \mathbf{A}(\omega^s \lambda), \quad \omega := e^{\frac{2i\pi}{3}}. \]

This determinant have a regular expansion if

\[ y^{(a)} \ni y_j \to \infty, \quad y_j \in S_a. \]

\[ S_0 : \quad \; S_1 : \quad S_2 : \]

For what follows let’s introduce the parameters \( \{\lambda_i, \mu_j\} \) such that \( y_i = \sqrt{\lambda_i} \) if \( \text{Re}(y_i) > 0 \) et \( y_j = -\sqrt{\mu_j} \) if \( \text{Re}(y_j) \leq 0 \).

\[ d(\vec{\lambda}, \vec{\mu}) := \prod_{j=1} \frac{\sqrt{\mu_j} + \sqrt{\lambda}}{\sqrt{\mu_j} - \sqrt{\lambda}} \prod_{j=1} \frac{\sqrt{\lambda_j} - \sqrt{\lambda}}{\sqrt{\lambda_j} + \sqrt{\lambda}}. \]
A Riemann-Hilbert problem for $Z_n$

**Theorem** (M. Bertola, M.C.) :

$$Z_n(x; y^{(0)}, y^{(1)}, y^{(2)}) = e^{\frac{3}{12}} \tau_n(x, \tilde{\lambda}, \tilde{\mu}), \tau_n \text{ tau function of the Riemann–Hilbert problem with asymptotics}$$

$$\Gamma_n(\lambda) \sim \lambda^{-\frac{\sigma_3}{4}} \frac{1 + i\sigma_1}{\sqrt{2}} \left(1 + \frac{\alpha^{(n)}(x; \tilde{\lambda}, \tilde{\mu})}{\sqrt{\lambda}} \sigma_3 + O(\lambda^{-1})\right),$$
The matrix

\[
\Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) := \Gamma_n e^{\left(-\frac{2}{3} \lambda^{\frac{3}{2}} - x \sqrt{\lambda}\right) \sigma_3} D^{-1}(\lambda)
\]

\[
D(\lambda) := \begin{bmatrix}
\prod_{j=1}^{n_2} (\sqrt{\lambda_j} + \sqrt{\lambda}) \prod_{j=1}^{n_1} (\sqrt{\mu_j} - \sqrt{\lambda}) & 0 \\
0 & \prod_{j=1}^{n_2} (\sqrt{\lambda_j} - \sqrt{\lambda}) \prod_{j=1}^{n_1} (\sqrt{\mu_j} + \sqrt{\lambda})
\end{bmatrix}
\]

is a solution of the isomonodromic system

\[
\frac{\partial}{\partial \lambda} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = A(\lambda; x, \vec{\lambda}, \vec{\mu}) \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu})
\]

\[
\frac{\partial}{\partial x} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = U(\lambda; x, \vec{\lambda}, \vec{\mu}) \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu})
\]

\[
\frac{\partial}{\partial \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{A_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \lambda_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}),
\]

\[
\frac{\partial}{\partial \mu_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}) = -\frac{B_k(x, \vec{\lambda}, \vec{\mu})}{\lambda - \mu_k} \Psi_n(\lambda; x, \vec{\lambda}, \vec{\mu}).
\]
Riemann-Hilbert for the tronquées solutions of $\text{PI}_{N-1}$

\[ \vartheta(\lambda; t, x) := t\lambda \frac{2N+1}{2} + \frac{2}{3}\lambda^{\frac{3}{2}} + x\lambda^{\frac{1}{2}}, \]

jumps are given by:

\[ 1 + e^{-2\vartheta(\lambda;t,x)} \sigma_+, \lambda \in \mathcal{W}_0 \]
\[ 1 + e^{2\vartheta(\lambda;t,x)} \sigma_-, \lambda \in \mathcal{W}_\pm \]
\[ i\sigma_2, \lambda \in \mathbb{R}_- \]

\[ \Gamma(\lambda) = \lambda^{-\frac{3}{4}} \frac{1 + i\sigma_1}{\sqrt{2}} \left( 1 + a \frac{\sigma_3}{\sqrt{\lambda}} + \vartheta(\lambda^{-1}) \right), \quad \lambda \to \infty. \]

(The matrix $\Psi(\lambda) = \Gamma(\lambda)e^{-\vartheta(\lambda)}$ is the solution of the corresponding RH problem)
Riemann-Hilbert for the tronquées solutions of $PI_{N-1}$

Let's choose three integers $k_+, k_0, k_- \in \left\{ -\left\lfloor \frac{N-1}{2} \right\rfloor, \ldots, \left\lceil \frac{N-1}{2} \right\rceil \right\}$ with $k_+ > k_-, k_+ \geq k_0 \geq k_-$, and

$$
\theta_0 \in \left( -\frac{\pi}{2N+1}, \frac{\pi}{2N+1} \right) + \frac{4k_0 \pi}{2N+1} - \frac{2\arg(t)}{2N+1},
$$

$$
\theta_{\pm} \in \left( -\frac{\pi}{2N+1}, \frac{\pi}{2N+1} \right) + \frac{(4k_{\pm} \pm 2) \pi}{2N+1} - \frac{2\arg(t)}{2N+1}
$$

$$
M(\lambda) = \begin{cases} 
1 + e^{-2\vartheta(\lambda;t,x)} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0} \mathbb{R}_+ \\
1 + e^{2\vartheta(\lambda;t,x)} \sigma_- & \lambda \in \varpi_{\pm} := e^{i\theta_{\pm}} \mathbb{R}_+ \\
i\sigma_2 & \lambda \in \mathbb{R}_-
\end{cases}
$$

$$
\vartheta(\lambda; t, x) := t \lambda \frac{2N+1}{2} + \frac{2}{3} \lambda^\frac{3}{2} + x \lambda^\frac{1}{2}
$$

Find a matrix $\Gamma(\lambda)$ such that

$$
\Gamma_+(\lambda) = \Gamma_-(\lambda) M(\lambda), \ \lambda \in \varpi_{-0,+}, \quad \Gamma_+(\lambda) = \Gamma_-(\lambda) i\sigma_2, \quad \lambda \in \mathbb{R}_-
$$

with asymptotics

$$
\Gamma(\lambda; t) = \lambda^{-\frac{\sigma_3}{4}} \frac{1 + i\sigma_1}{\sqrt{2}} \left( 1 + a(t) \frac{\sigma_3}{\sqrt{\lambda}} + O(\lambda^{-1}) \right), \ \lambda \to \infty.
$$
Which solutions are they?

- For $N = 2$ there is just one solution, the one studied by Boutroux.

- For $N = 3$ there are 4 solutions. One is the *tritronquée* solution $U_0$ of $PI_2$ related to the Dubrovin’s conjecture on universality Dubrovin (with $t$ fixed) (for $t = 0$ it had been studied by Brezin-Marinari-Parisi and Moore).

- The other three belong to the set of “two parameters solutions” studied by Grava-Kapaev-Klein.

- For $N$ générique, the analog of $U_0$ had been used by Claeys-Its-Krasovsky to describe “higher order” Tracy–Widom distributions.
How to go from $Z_n$ to $PI_{N-1}$?

Jumps for $Z_n$:

$$M_n(\lambda) = \begin{cases} 
1 + d(\lambda)e^{-\frac{4}{3}\lambda\frac{3}{2}-2x\lambda\frac{1}{2}} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}R_+ \\
1 + \frac{1}{d(\lambda)}e^{\frac{4}{3}\lambda\frac{3}{2}+2x\lambda\frac{1}{2}} \sigma_- & \lambda \in \varpi_\pm := e^{i\theta \pm}R_+ \\
i\sigma_2 & \lambda \in R_- \end{cases}$$

Jumps for $PI_{N-1}$:

$$M(\lambda) = \begin{cases} 
1 + e^{-2\vartheta(\lambda; t, x)} \sigma_+ & \lambda \in \varpi_0 := e^{i\theta_0}R_+ \\
1 + e^{2\vartheta(\lambda; t, x)} \sigma_- & \lambda \in \varpi_\pm := e^{i\theta \pm}R_+ \\
i\sigma_2 & \lambda \in R_- \end{cases}$$

$$\vartheta(\lambda; t, x) = t\lambda\frac{2N+1}{2} + \frac{2}{3}\lambda\frac{3}{2} + x\lambda\frac{1}{2}$$

So we need to approximate $e^{-t\lambda\frac{2N+1}{2}}$ using the rational function $d(\lambda)...$

Padé approximants!
Let $P_r$ be the $r$-th Padé approximant for $e^{-z}$:

$$e^{-z} = \frac{P_r(z)}{P_r(-z)} + \mathcal{O}(z^{2r+1}), \quad z \to 0.$$ 

The distribution of its zeros is known [SaffVarga78], they are all contained in the region $\text{Re} z > 0$.

$$\mu e^{1+\mu} = 1, \quad \mu \simeq 0, 278...$$
Let’s fix $N \in \mathbb{N}$ and

$$y^{(0)} \sqcup y^{(1)} \sqcup y^{(2)} := y = \left\{ y : \ P_r(2ty^{2N+1}) = 0 \right\}, \ n = r(2N + 1).$$

**IMPORTANT**: There’s an ambiguity on the choice of the location of the zeroes...

- $k_0 = (\#\{y_\kappa\} \text{ in the second quadrant assigned to } y^{(2)}) - (\#\{y_\kappa\} \text{ in the third quadrant assigned to } y^{(1)})$
- $k_- = -\left\lfloor \frac{N}{2} \right\rfloor + (\#\{y_\kappa\} \text{ in the first quadrant assigned to } y^{(1)})$
- $k_+ = \left\lfloor \frac{N}{2} \right\rfloor - (\#\{y_\kappa\} \text{ in the fourth quadrant assigned to } y^{(2)})$

**Figure**: Exemple avec $N = 3$, $k_+ = 1, k_0 = 0, k_- = -1$.

**Figure**: Example with $N = 4$, $k_+ = 1, k_0 = 0, k_- = -1$. 
**Theorem** (M. Bertola, M.C.) :

Let's fix \( N \in \mathbb{N} \) and choose \( \mathcal{Y} = \{y_1, \ldots, y_n\} \) as above.

\[ Z_n(x; \mathcal{Y}) \] converges, for \( n \to \infty \), to the tau function \( \Pi_N \) identified by the corresponding \( k_0, k_\pm \).

In particular, \( u(x, t) := 2\partial_x^2 \tau(x, t) \) satisfies the equation

\[
(2N + 1) t R_N[u(x; t)] + u(x; t) + x = 0,
\]

with \( R_j \) defined by the Lenard's recursion

\[
\frac{\partial}{\partial x} R_{N+1}[u] = \left( \frac{1}{4} \frac{\partial^3}{\partial x^3} + u(x) \frac{\partial}{\partial x} + \frac{1}{2} u_x(x) \right) R_N[u], \quad R_0[u] = 1.
\]

**Example** :

\( N = 2; \quad \frac{5}{8} t \left( u'' + 3u^2 \right) + u + x = 0, \)

\( N = 3; \quad \frac{7}{32} t \left( u^{(4)} + 10uu'' + 5(u')^2 + 10u^3 \right) + u + x = 0. \)
Thanks!