“Integrable” gap probabilities for the Generalized Bessel process

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First result: differential identity for gap probabilities
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  - Painlevé-type equation
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Conclusive remarks and open questions
Consider a system of $n$ independent squared Bessel paths $\text{BESQ}^\alpha$

$$\{X_1(t), \ldots, X_n(t)\}$$

with parameter $\alpha > -1$, conditioned never to collide.

The process $\{\vec{X}(t)\}_{t \geq 0}$ is a diffusion process on $[0, +\infty)^n$. Additionally, we impose initial and final conditions

$$X_j(0) = a > 0 \text{ and } X_j(T) = 0 \quad \forall j = 1, \ldots, n.$$
BESQ$^\alpha$ model

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\[ X_j(0) = a > 0 \text{ and } X_j(T) = 0 \quad \forall j = 1, \ldots, n. \]
The joint probability density is given as

\[
\frac{1}{Z_{n,t}} \det \left[ x_j^{\alpha+1-j(\mod 2)} (a, x_k) \right]_{j,k=1}^n \det \left[ x_j^{k-1} e^{-\frac{x_j}{2(T-t)}} \right]_{j,k=1}^n \, dx_1 \ldots dx_n
\]

\[
= \frac{1}{n!} \det [K_n(x_i, x_j; t)]_{i,j=1}^n \, dx_1 \ldots dx_n
\]

where \( p_t^\alpha(x, y) \) is the transition probability \( p_t^\alpha(x, y) = \frac{1}{2t} \left( \frac{y}{x} \right)^{\alpha/2} e^{-\frac{x+y}{2t}} I_\alpha \left( \frac{\sqrt{xy}}{t} \right) \)

and the correlation kernel \( K_n \) given in terms of MOP with weights depending on the Bessel functions \( I_\alpha \).

Remark (Random Matrix interpretation)

Let \( M(t) \) be a \( p \times n \) matrix with independent complex Brownian entries (with mean zero and variance \( 2t \)).

The set of singular values

\[
\{ \lambda_1(t), \ldots, \lambda_n(t) \}, \quad \lambda_i(t) \geq 0 \ \forall i
\]

i.e. the eigenvalues of the product \( M(t)^* M(t) \), has the same distribution as the above noncolliding particle system \( \text{BESQ}^\alpha \) with \( \alpha = 2(n-p+1) \) (König, O’Connell, ’01).
(Double) Scaling limit

Starting from the kernel $K_n$, one can perform a double scaling limit as $n \to +\infty$ in different parts of the domain of the spectrum: the sine kernel appears in the bulk, the Airy kernel at the soft edges and the Bessel kernel appears at the hard edge $x = 0$ (Kuijlaars et al., ’09).

At a critical time $t^*$, there is a transition between the soft and the hard edges and the local dynamics is described by a new critical kernel.
The Generalized Bessel kernel

Theorem (Kuijlaars, Martinez-Finkelshtein, Wielonsky, ’11)

\[
\lim_{n \to +\infty} \frac{c^*}{n^{3/2}} K_n \left( \frac{c^* x}{n^{3/2}}, \frac{c^* y}{n^{3/2}}; t^* - \frac{c^* \tau}{\sqrt{n}} \right) = K_{\alpha}^{\text{crit}}(x, y; \tau) \quad x, y \in \mathbb{R}_+, \ \tau \in \mathbb{R},
\]

with

\[
K_{\alpha}^{\text{crit}}(x, y; \tau) = \int_{\Gamma} \frac{du}{2\pi i} \int_{\Sigma} \frac{dv}{2\pi i} e^{\frac{xu + u}{2u^2} - yv - \frac{v}{2v^2}} \left( \frac{u}{v} \right)^\alpha.
\]
“Integrable” gap probabilities for the Generalized Bessel process

Introduction: the Generalized Bessel process

Gap probabilities of the Generalized Bessel process

Our object of study are the gap probabilities, meaning the probability of finding no points in a given domain.

For a determinantal process with kernel $K_n$, this boils down to calculating a Fredholm determinant:

$$
\mathbb{P}(X_{\min} > s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det [K_n(x_i, x_j)]_{i,j=1,\ldots,k} \, dx_1 \ldots dx_k
$$

$$
= \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_n \bigg|_{[0,s]} \right)
$$

and in the scaling limit regime

$$
\det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_n \bigg|_{\left[0, \frac{c^* s}{n^{3/2}} \right]} \right) \to \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_{\alpha}^{\text{crit}} \bigg|_{[0,s]} \right) \quad \text{as } n \nearrow +\infty.
$$
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First result: differential identity for gap probabilities

Differential identity

Theorem (Girotti, ’14)

Let $s > 0$ and $K_{\alpha}^{\text{crit}}$ be the integral operator acting on $L^2(\mathbb{R}_+)$ with kernel defined above. Then, the following differential formula for gap probabilities holds

$$d_{s, \tau} \ln \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_{\alpha}^{\text{crit}} \right) |_{[0, s]} = (Y_1)_{2, 2} ds - \left( \hat{Y}_0^{-1} \hat{Y}_1 \right)_{2, 2} d\tau$$

where $Y$ is the solution to a suitable RH problem and $Y_1$ and $\hat{Y}_j$ are the coefficients appearing in the asymptotic expansion of $Y$ at infinity and in a neighbourhood of zero, respectively.
The Riemann-Hilbert problem for $Y$

Find a $2 \times 2$ matrix-valued function $Y = Y(\lambda; s, \tau)$ such that

- $Y$ is analytic on $\mathbb{C}\setminus(\Gamma \cup \Sigma)$
- $Y$ admits a limit when approaching the contours from the left $Y_+$ or from the right $Y_-$ (according to their orientation), and the following jump condition holds

$$
Y_+(\lambda) = Y_-(\lambda) \begin{cases}
1 & -\lambda - \alpha e^{-\lambda s - \frac{\tau}{x} - \frac{1}{2\lambda^2}} \\
0 & 1 \\
-\lambda \alpha e^{\lambda s + \frac{\tau}{x} + \frac{1}{2\lambda^2}} & 1
\end{cases}
\begin{cases}
\lambda \in \Sigma \\
\lambda \in \Gamma
\end{cases}
$$

- $Y$ has the following (normalized) behaviour at $\infty$:

$$
Y(\lambda) = I + \frac{Y_1(s, \tau)}{\lambda} + O\left(\frac{1}{\lambda^2}\right) \quad \lambda \to \infty.
$$
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Proposition

The following identity holds

$$\det \left( \operatorname{Id}_{L^2(\mathbb{R}_+)} - K^\text{crit}_\alpha \bigg|_{[0,s]} \right) = \det \left( \operatorname{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right)$$

where $\mathbb{H}$ is an Its-Izergin-Korepin-Slavnov ('90) integral operator with kernel

$$\mathbb{H} = \frac{f(\lambda)^T g(\mu)}{\lambda - \mu}$$

$$f(\lambda) = \frac{1}{2\pi i} \left[ e^{-\frac{\lambda s}{2}} \chi_\Sigma(\lambda) \right] \quad \quad g(\mu) = \left[ \begin{array}{c} \mu^\alpha e^{\mu s + \frac{\tau}{\mu} + \frac{1}{2\mu^2}} \chi_\Gamma(\mu) \\ -\alpha e^{-\frac{\mu s}{2} - \frac{\tau}{2\mu} - \frac{1}{2\mu^2}} \chi_\Sigma(\mu) \end{array} \right].$$

The result can be proved by noticing that $K^\text{crit}_\alpha \bigg|_{[0,s]}$ is unitarily equivalent (via Fourier transform) to a certain integral operator that can be decomposed as the above operator $\mathbb{H}$. 
IIKS operators naturally carry an associated RH problem, whose solution $Y$ is tied to the invertibility of their resolvent operator.

Given such RH problem, we make use of a major (and more general) result due to Bertola (’10) and Bertola-Cafasso (’11) which, if applied to our case, reads as follows

**Theorem (Bertola-Cafasso, ’11)**

Define the quantity for $\rho = s, \tau$

$$\omega(\partial \rho) := \int_{\Sigma \cup \Gamma} \text{Tr} \left[ Y^{-1} Y' (\partial \rho J) J^{-1} \right] \frac{d\lambda}{2\pi i}.$$  

Then, we have the equality

$$\omega(\partial \rho) = \partial \rho \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right).$$

By expanding the solution $Y$ at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

$$d_{s, \tau} \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2, 2} \ ds - \left( \hat{Y}_0^{-1} \hat{Y}_1 \right)_{2, 2} \ d\tau.$$
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$$d_{s,\tau} \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - \left( \hat{Y}_0^{-1}\hat{Y}_1 \right)_{2,2} \, d\tau.$$
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By expanding the solution \( Y \) at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

\[
ds,\tau \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - \left( \hat{Y}_0^{-1}\hat{Y}_1 \right)_{2,2} \, d\tau.
\]
A few more words on $\omega(\partial)$

The solution to the RH problem $Y$ solves a rational ODE (up to a gauge transformation)

$$\frac{dY}{d\lambda} = A(\lambda)Y(\lambda)$$

With this extra property, it turns out that (Bertola, '10) given

$$\omega(\partial) = \int_{\Sigma \cup \Gamma} \text{Tr} \left[ Y^{-1} Y' (\partial J) J^{-1} \right] \frac{d\lambda}{2\pi i},$$

then $\omega$ is the logarithmic total differential of the isomonodromic $\tau$ function:

$$d\omega = 0 \quad \text{and} \quad e^{\int \omega} = \tau_{\text{JMU}}.$$

Conclusion

*We give a specific geometrical meaning to a probabilistic quantity:

$$\tau_{\text{JMU}} = \det \left( \text{Id}_{L^2(\mathbb{R}_+)} - K_{\alpha}^{\text{crit}} \right)_{[0,s]} = \begin{cases} \text{infinitesimal fluctuation of smallest path of BESQ}^\alpha \\ \text{at the critical time } t^* \end{cases}$$

(up to a normalization constant).
Given
\[ d_{s,\tau} \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - \left( \tilde{Y}_0^{-1} \tilde{Y}_1 \right)_{2,2} \, d\tau \]

we can further study our RH problem to draw some interesting conclusions:

- asymptotic behaviour of gap probability (large/small gap, degeneration regimes) → Deift-Zhou steepest descent method
- integrability and differential equations (Tracy-Widom) → Lax pair, hamiltonian formalism
What now?

Given
\[ ds, \tau \ln \det \left( \text{Id}_{L^2(\Sigma \cup \Gamma)} - \mathbb{H} \right) = (Y_1)_{2,2} \, ds - \left( \tilde{Y}_0^{-1} \tilde{Y}_1 \right)_{2,2} \, d\tau \]

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- asymptotic behaviour of gap probability (large/small gap, degeneration regimes) \( \rightarrow \) Deift-Zhou steepest descent method

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4 Conclusive remarks and open questions
The Lax triplet

From the RH problem $Y$ associated to our critical kernel $K_{\alpha}^{\text{crit}}$

$$Y_{+}(\lambda) = Y_{-}(\lambda) \begin{cases} 
1 & -\lambda^{-\alpha}e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2}} \\
0 & 1 \\
-\lambda^{\alpha}e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2}} & 1 
\end{cases} \begin{cases} \lambda \in \Sigma \\
\lambda \in \Gamma 
\end{cases}$$

we can derive the following Lax triplet:

$$A = A^{(\lambda)} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3},$$

$$B = B^{(s)} = \lambda B_1 + B_0,$$

$$C = C^{(\tau)} = \frac{C_{-1}}{\lambda}.$$
Up to a change of variables $\lambda \mapsto \frac{1}{\lambda}$, the Lax pair $\{A, C\}$ is

$$A = \frac{\lambda}{2} \sigma_3 + A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \quad C = \frac{\lambda}{2} \sigma_3 + C_0$$

with coefficients

$$A_0 = \begin{bmatrix} \frac{\tau}{2} & uw \\ -\frac{1}{w} [v \tau + u (v^2 - \Theta)] & -\frac{\tau}{2} \end{bmatrix}, \quad A_{-2} = \begin{bmatrix} v & w \\ -\frac{1}{w} (v^2 - \Theta) & -v \end{bmatrix},$$

$$A_{-1} = \begin{bmatrix} u [v \tau + u (v^2 - \Theta)] + \frac{\alpha}{2} & w [u \tau - 2u^2v + \tau u] \\ \frac{1}{w} [(u \tau - 4u^2v + \tau u) (v^2 - \Theta) - 2uvv_\tau - \alpha v + \tilde{\Theta}] & -u [v \tau + u (v^2 - \Theta)] - \frac{\alpha}{2} \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & uw \\ -\frac{1}{w} [v \tau + u (v^2 - \Theta)] & 0 \end{bmatrix}.$$

We can recognize the Lax pair associated to the second member of the Painlevé III hierarchy defined by Sakka ('09):

$$\begin{cases} u_{\tau\tau} = (6uv - \tau)u_\tau - 6u^3v^2 + 2\tau u^2v + 2\Theta u^3 - (\alpha + 1)u + 1 \\ v_{\tau\tau} = -(6uv - \tau)v_\tau - 2u(3uv - \tau)(v^2 - \Theta) - \alpha v + \tilde{\Theta}. \end{cases}$$
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The quest for a Garnier system...

As in the classical Painlevé theory (Jimbo, Miwa, Ueno, ’81), we would like to find a completely integrable (Hamiltonian) system associated with the Lax triplet \( \{A, B, C\} \).

In this case, we have two independent parameters that describe the flow, the time \( \tau \) and the space \( s \), therefore we need a 2-D version of Hamiltonian system (Garnier system, ’26) for the canonical coordinates \((\mu_1, \mu_2; \lambda_1, \lambda_2)\):

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial \tau} &= \frac{\partial H_\tau}{\partial \mu_j} \\
\frac{\partial \lambda_j}{\partial \mu_j} &= -\frac{\partial H_\tau}{\partial \lambda_j} \\
\frac{\partial \lambda_j}{\partial s} &= \frac{\partial H_s}{\partial \mu_j} \\
\frac{\partial \lambda_j}{\partial \mu_j} &= -\frac{\partial H_s}{\partial \lambda_j}
\end{align*}
\]

with rational Hamiltonians \( H_\tau = H_t(\lambda_j, \mu_j; s, \tau) \) and \( H_s = H_s(\lambda_j, \mu_j; s, \tau) \).
The quest for a Garnier system...

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\frac{\partial \mu_j}{\partial \tau} &= -\frac{\partial H_\tau}{\partial \lambda_j} & \frac{\partial \mu_j}{\partial \lambda_j} &= \frac{\partial H_s}{\partial \lambda_j}
\end{align*}
\]

with rational Hamiltonians \( H_\tau = H_t(\lambda_j, \mu_j; s, \tau) \) and \( H_s = H_s(\lambda_j, \mu_j; s, \tau) \).
**Action plan**

**Step 1:** we identify the canonical coordinates in our system

\[
\{\lambda_j\}_{j=1,2} \text{ as the solutions of the equation } (\mathcal{A}(\lambda; s, \tau))_{1,2} = 0
\]

\[
\{\mu_j\}_{j=1,2} \text{ as } \mu_j = (\mathcal{A}(\lambda_j; s, \tau))_{1,1}
\]

**Step 2:** the compatibility equations of the Lax triplet yield a system of 8 differential equations (4 for the variable \(s\), 4 for the variable \(\tau\)) which can be represented as a Garnier system

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial \tau} &= \frac{\partial H_\tau}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial \tau} &= - \frac{\partial H_\tau}{\partial \lambda_j}
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \lambda_j}{\partial s} &= \frac{\partial H_s}{\partial \mu_j} \\
\frac{\partial \mu_j}{\partial s} &= - \frac{\partial H_s}{\partial \lambda_j}
\end{align*}
\]

with rational Hamiltonians \(H_\tau = H_\tau(\lambda_j, \mu_j; s, \tau)\) and \(H_s = H_s(\lambda_j, \mu_j; s, \tau)\).
\[ H_\tau = -\frac{\lambda_1^2 \mu_1^2}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2 \mu_2^2}{\lambda_1 - \lambda_2} - \frac{s^2 (\lambda_1 + \lambda_2)}{4 \lambda_1^2 \lambda_2^2} + \frac{\tau^2 (\lambda_1 + \lambda_2)}{4} - \frac{ks}{\lambda_1 \lambda_2} - \frac{\tau (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)}{2} + \frac{\lambda_1^3}{4} + \frac{\lambda_1^2 \lambda_2}{4} + \frac{\lambda_1 \lambda_2^2}{4} + \frac{\lambda_2^3}{4} - \frac{(\alpha + 1) \lambda_1 + 2\alpha \lambda_2}{2} \]

\[ H_s = -\frac{\lambda_1 \lambda_2 (\lambda_1 \mu_1^2 + \mu_1)}{s (\lambda_1 - \lambda_2)} + \frac{\lambda_1 \lambda_2 (\lambda_2 \mu_2^2 + \mu_2)}{s (\lambda_1 - \lambda_2)} + \frac{\tau^2 \lambda_1 \lambda_2}{4s} - \frac{k (\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2} - \frac{\alpha \lambda_1 \lambda_2}{2s} - \frac{s (\lambda_1 + \lambda_2)}{4 \lambda_1^2 \lambda_2} - \frac{\tau \lambda_2 (\lambda_1^2 + \lambda_1 \lambda_2)}{2s} + \frac{\lambda_1 \lambda_2 (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 - 2)}{4s} - \frac{s}{4 \lambda_2^2} \]

Remark

These Hamiltonians are different from the Hamiltonians of the K(2 + 3) system defined in Okamoto-Kimura, ’86. The identification process is on-going...
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New horizons

Explicit connection between Hamiltonians and gap probabilities/RH problem for $K_{\alpha}^{\text{crit}}$?

\[
d_{s,\tau} \ln \det \left( \left. \text{Id}_{L^2(R_+)} - K_{\alpha}^{\text{crit}} \right|_{[0,s]} \right) = \mathcal{L}_1 (H_\tau, H_s) \, ds + \mathcal{L}_2 (H_\tau, H_s) \, d\tau
\]
Find a suitable canonical transformation of variables \((\mu_j, \lambda_j) \mapsto (\tilde{\mu}_i, \tilde{\lambda}_i)\) such that the Hamiltonians become polynomials or of the form \(p^2 + V(q)\).

Via the classical substitution of the operators \(\{x_j, \hbar \frac{\partial}{\partial x_j}\}\) into the canonical coordinates \((\tilde{\lambda}_j, \tilde{\mu}_j)\), study the Schrödinger system

\[
\hbar \frac{\partial}{\partial \tau} \Phi(x; s, \tau) = \hat{H}_\tau \left(x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau\right) \Phi(x; s, \tau)
\]

\[
\hbar \frac{\partial}{\partial s} \Phi(x; s, \tau) = \hat{H}_s \left(x_j, \hbar \frac{\partial}{\partial x_j}; s, \tau\right) \Phi(x; s, \tau)
\]
Further work:

- what will the Lax pair \(\{A, B\}\) yield?

\[
A = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3}, \quad B = \lambda B_1 + B_0;
\]

- asymptotic behaviour?

**Conjecture**: degeneration of the gap probabilities of \(K_{\alpha}^{\text{crit}}\) into gap probabilities of the Airy process (for \(\tau \searrow -\infty\)) or the Bessel process (for \(\tau \nearrow +\infty\)).
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References


Thanks for your attention!