On the DR/DZ equivalence conjecture

Paolo Rossi

Institut de Mathématiques de Bourgogne
Integrable hierarchies from CohFTs

CohFT: $V$ $\mathbb{C}$-vector space, $\eta$ symm. nondeg. bilinear form, $e_1 \in V$

$c_{g,n} : V^\otimes n \to H^*(\overline{M}_{g,n}, \mathbb{C})$ linear maps
Integrable hierarchies from CohFTs

CohFT:

\[ V \] \( \mathbb{C} \)-vector space,
\[ \eta \] symm. nondeg. bilinear form,
\[ e_1 \in V \]

\[ c_{g,n} : V^\otimes n \rightarrow H^* (\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \]
linear maps
Integrable hierarchies from CohFTs

Dependent variables:

\[ w^\alpha = w^\alpha(x), \quad w_k^\alpha := \partial_x w^\alpha, \quad 1 \leq \alpha \leq N = \dim V \]

Hamiltonian densities:

\[ \{ h_{\beta,d}(w^*_\bullet; \varepsilon) \}_{1 \leq \beta \leq N} \quad h_{\beta,d} \in A^0_w := \mathbb{C}[[w^*_\bullet, \varepsilon]]^0 \]

\[ (\deg \varepsilon = -1, \quad \deg w_k^* = k) \]

Hamiltonians:

\[ \overline{h}_{\alpha,d} := \int h_{\alpha,d} \, dx \quad \in \Lambda_w := A_w/\text{Im} \partial_x \oplus \mathbb{C} \]
Integrable hierarchies from CohFTs

Poisson structure:

\[ \{ \cdot, \cdot \}_K : \Lambda_w \times \Lambda_w \to \Lambda_w \]

\[ (\overline{f}, \overline{h}) \mapsto \{ \overline{f}, \overline{h} \}_K = \int \left( \frac{\delta \overline{f}}{\delta w^\mu} K^{\mu \nu} \frac{\delta \overline{h}}{\delta w^\nu} \right) \, dx \]

\[ K^{\mu \nu} = \sum_{j \geq 0} K_j^{\mu \nu} \partial^j = \eta^{\mu \nu} \partial_x + O(\varepsilon^2), \quad K_j^{\mu \nu} \in A_w^{[-j+1]} \]

Equations of the hierarchy:

\[ \frac{\partial w^\alpha}{\partial t^\beta_d} = \{ w^\alpha, \overline{h}_{\beta,d} \}_K, \quad \{ \overline{h}_{\beta_1,d_1}, \overline{h}_{\beta_2,d_2} \}_K = 0 \]
Integrable hierarchies from CohFTs

**Tau structure:**
\[ \{ f, h_{1,0} \}_K = \partial_x f \]
\[ \bar{h}_{\alpha,-1} \text{ are } N \text{ independent Casimirs of } \{ \cdot, \cdot \}_K, \]
\[ \{ h_{\alpha,p-1}, \bar{h}_{\beta,q} \}_K = \{ h_{\beta,q-1}, \bar{h}_{\alpha,p} \}_K. \]

**Tau functions:**
For any solution \( u^*(x, t, \varepsilon) \) there exists \( F(t^*, \varepsilon) \)
\[ \frac{\partial h_{\alpha,p-1}}{\partial t^\beta} \bigg|_{w^*=w^*(x,t^*,\varepsilon)|x=0} = \frac{\partial^3 F}{\partial t_0^1 \partial t_p^\alpha \partial t_q^\beta} \]

**Normal coordinates:**
\[ w^\alpha = \eta^{\alpha \mu} h_{\mu,-1} \]
Integrable systems

Normal Miura transf.: (a Miura that preserves the tau structure)

\[ \tilde{u}^\alpha = w^\alpha + \eta^{\alpha\mu} \partial_x \partial_{t^\mu} P(w_*; \varepsilon) \]

where \( w^\alpha \) are normal coordinates

and \( P(w_*; \varepsilon) \in A_w^{[-2]} \)

Effect on tau functions: \[ \tilde{F}(t_*; \varepsilon) = F(t_*; \varepsilon) + P(w_*; \varepsilon) \big|_{w_* = w_*(x,t_*; \varepsilon)} \big|_{x=0} \]
Dubrovin-Zhang hierarchy of a semisimple CohFT

\[ F(t^*; \varepsilon) = \sum_{g \geq 0} F_g(t^*) \varepsilon^{2g}, \quad F_g(t^*) := \sum_{n,d_i} \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_n}(e_{\alpha_n}) \rangle_g t_{d_1}^{\alpha_1} \cdots t_{d_n}^{\alpha_n} \frac{1}{n!}. \]

Genus 0: \[ \begin{cases} \bar{h}_{\alpha,p}^{[0]}(v^*) = \frac{\partial^2}{\partial t_0^{p+1} \partial t_{\alpha}^{p+1}} F_0(t_0^* = v^*, t_{>0}^* = 0) \\ (K_{DZ}^v)_{\alpha\beta} = \eta_{\alpha\beta} \partial_x \\ v^\alpha(x, t^*) \text{ solution with initial datum } v^\alpha(x, t^*_0 = 0) = x \delta_1^\alpha \end{cases} \]
Dubrovin-Zhang hierarchy of a semisimple CohFT

\[ F(t^*; \varepsilon) = \sum_{g \geq 0} F_g(t^*) \varepsilon^{2g}, \quad F_g(t^*) := \sum_{n,d_i} \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_n}(e_{\alpha_n}) \rangle_g \frac{t_{\alpha_1}^{d_1} \cdots t_{\alpha_n}^{d_n}}{n!}. \]

**Genus 0:**

\[
\begin{aligned}
&\{ \bar{h}^{[0]}_{\alpha,p}(v^*) = \frac{\partial^2}{\partial t_0^1 \partial t_{p+1}^\alpha} F_0(t^*_0 = v^*, t^*_0 = 0) \\
&\quad (K_{v}^{DZ})^{\alpha \beta} = \eta^{\alpha \beta} \partial_x \\
&v^\alpha(x, t^*_0) \text{ solution with initial datum } v^\alpha(x, t^*_0 = 0) = x \delta^\alpha_1
\end{aligned}
\]

**Genus g:**

**Proposition (Eguchi-Getzler-Xiong, DZ, Buryak-Posthuma-Shadrin)**

\[ F_g(t^*) = F_g \left( t^*_{\leq 3g-2} = P^*_{\leq 3g-2}(v^*_{\leq 3g-2}), t^*_{> 3g-2} = 0 \right) \mid_{x=0}, \quad g > 0 \]

\[ w^\alpha(v^*_0; \varepsilon) = v^\alpha + \eta^{\alpha \mu} \partial_x \partial_{t_0^\mu} \left( \sum_{g \geq 1} \varepsilon^{2g} F_g(v^*_{\leq 3g-2}) \right) \quad \text{not a Miura transf.}! \]

\[
\begin{aligned}
&\{ \bar{h}^{[0]}_{\alpha,p}(v^*) \mapsto \bar{h}^{DZ}_{\alpha,p}(w^*_0; \varepsilon) \\
&\quad (K_{v}^{DZ}) \mapsto (K_{w}^{DZ}) \quad \text{polynomial anyway!} \\
&\quad [\text{Buryak-Posthuma-Shadrin}]\end{aligned}
\]
Double ramification hierarchy of any CohFT

From Sasha’s talk:

\[
\left\{ \begin{align*}
\text{Intersection numbers} \\
\text{of the CohFT and the DR cycle}
\end{align*} \right\} \quad \mapsto \quad g_{\alpha,d}(u^*; \varepsilon) \in \mathcal{A}_u^{[0]}
\]

Poisson structure: \[ K^\text{DR}_u = \eta \partial_x \quad \text{(Standard Poisson } \{\cdot, \cdot\} := \{\cdot, \cdot\} \eta \partial_x \text{)} \]

Nice properties:

Recursion:
\[ \partial_x(D - 1)g_{\alpha,p+1} = \{g_{\alpha,p}, \overline{g}_{1,1}\} \]

Quantization:
\[ \{\cdot, \cdot\} \mapsto \frac{1}{\hbar}[\cdot, \cdot] \]
\[ g_{\alpha,p}(u^*; \varepsilon) \mapsto G_{\alpha,p}(u^*; \varepsilon, \hbar) \]

Tau-structure:
\[ h_{\alpha,p}^\text{DR}(u^*; \varepsilon) = \frac{\delta g_{\alpha,p+1}}{\delta u_1^1} \]
\[ \tilde{u}^\alpha = \eta^{\alpha\mu} h_{\mu,-1}^\text{DR}(u^*, \varepsilon) \quad \eta \partial_x \mapsto K^\text{DR}_{\tilde{u}} \]
\[ \tilde{u}^\alpha(x, t^* = 0) = x\delta_1^{\alpha} \quad \Longrightarrow \quad F^\text{DR}(t^*; \varepsilon) \]
Comparing the DZ and DR tau-functions

\[
F^{\text{DZ}}(t^*; \epsilon) = \sum \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_n}(e_{\alpha_n}) \rangle^\text{DZ}_g \frac{t^{\alpha_1}_{d_1} \cdots t^{\alpha_n}_{d_n}}{n!} \epsilon^{2g}
\]

\[
F^{\text{DR}}(t^*; \epsilon) = \sum \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_n}(e_{\alpha_n}) \rangle^\text{DR}_g \frac{t^{\alpha_1}_{d_1} \cdots t^{\alpha_n}_{d_n}}{n!} \epsilon^{2g}
\]

**Proposition (Buryak-Dubrovin-Guéré-R. '16)**

\[
\langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_m}(e_{\alpha_m}) \rangle^\text{DR}_g = 0 \quad \text{unless} \quad 2g - 1 \leq \sum_{i=1}^{m} d_i \leq 3g - 3 + n.
\]

Compare:

\[
\langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_m}(e_{\alpha_m}) \rangle^\text{DZ}_g = 0 \quad \text{unless} \quad \sum_{i=1}^{m} d_i \leq 3g - 3 + n.
\]
Strong DR/DZ equivalence conjecture

\[
\left\langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_m}(e_{\alpha_m}) \right\rangle^{\text{DR}}_g = 0 \quad \text{for} \quad \sum_{i=1}^{m} d_i < 2g - 1.
\]

Remark: This does not hold for \( F(t_*^*; \varepsilon) \).

Idea: We know that normal Miura transformations of the form
\[
\tilde{u}^\alpha = w^\alpha + \eta^{\alpha \mu} \partial_x \partial_{t^*_0} P(w_*^*; \varepsilon), \quad P \in \mathcal{A}_w^{-2}
\]
changes tau functions by
\[
\tilde{F}(t_*^*; \varepsilon) = F(t_*^*; \varepsilon) + P(w_*^*(x, t_*^*; \varepsilon); \varepsilon)|_{x=0}
\]
Can we find \( P(w_*^*; \varepsilon) \) so that \( \tilde{F}(t_*^*; \varepsilon) \) has no small correlators?
Strong DR/DZ equivalence conjecture

$$\langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_m}(e_{\alpha_m}) \rangle^{\text{DR}}_g = 0 \quad \text{for} \quad \sum_{i=1}^{m} d_i < 2g - 1.$$

Remark: This does not hold for $F(t^*; \varepsilon)$.

**Theorem (BDGR '16)**

$$\exists! \ P(w^*_\ast; \varepsilon) \in A_w^{-2} \ such \ that \ F^{\text{red}} := F + P(w^*_\ast(x, t^*_\ast; \varepsilon); \varepsilon)|_{x=0} \ satisfies \ the \ above \ selction \ rules.$$  

**Conjecture (Strong DR/DZ equivalence)**

The DR and DZ hierarchies are equivalent via the normal Miura transformation generated by the unique $P(w^*_\ast; \varepsilon)$ found above ($\iff F^{\text{red}} = F^{\text{DR}}$).
Strong DR/DZ equivalence conjecture

\[ \langle \tau_{d_1}(e_{\alpha_1}) \cdots \tau_{d_m}(e_{\alpha_m}) \rangle^{\text{DR}}_g = 0 \quad \text{for} \quad \sum_{i=1}^{m} d_i < 2g - 1. \]

Remark: This does not hold for \( F(t^*_x; \varepsilon). \)

Theorem (BDGR '16)

\[ \exists! \ P(w^*_x; \varepsilon) \in A_w^{[-2]} \text{ such that } F_{\text{red}} := F + P(w^*_x(x, t^*_x; \varepsilon); \varepsilon)|_{x=0} \text{ satisfies the above selection rules.} \]

Conjecture (Strong DR/DZ equivalence)

The DR and DZ hierarchies are equivalent via the normal Miura transformation generated by the unique \( P(w^*_x; \varepsilon) \) found above \( (\iff F_{\text{red}} = F^{\text{DR}}). \)

Conjecture (Strong DR/DZ equivalence, generalization)

For any (even non semisimple) CohFT, \( F_{\text{red}} = F^{\text{DR}}. \)
Another characterization of the reduced DZ hierarchy

\[ \sum_{g \geq 1} \varepsilon^{2g} F_{g}^{DZ}(t^*) = \sum_{g \geq 1} \varepsilon^{2g} F_{g,k}(v^*) (v_1^1)^k \bigg|_{v^* = v^*(x, t^*)|x=0} \]

where:

\[ v^\alpha(x, t^* = 0) = x \delta^\alpha_1 \implies v_{k}^\alpha(x = 0, t^*) = t_k^\alpha + \delta^\alpha_1 \delta_{k,1} + O((t^*)^2) \]
Another characterization of the reduced DZ hierarchy

$$\sum_{g \geq 1} \epsilon^{2g} F^\text{DZ}_g (t^*) = \left. F^\text{DZ}_{\geq 1} (v^*; \epsilon) \right|_{v^* = v^*_*(x, t^*_*)} |_{x = 0}$$
Another characterization of the reduced DZ hierarchy

\[ \sum_{g \geq 1} \varepsilon^{2g} F_\varepsilon^{DZ}(t^*) = F_\varepsilon^{DZ}(v_*; \varepsilon) \bigg|_{v_* = v_*(x, t^*)} \bigg|_{x = 0} \]

\[ \left\{ \begin{array}{l}
F^{DZ}(t^*; \varepsilon) = F_0(t^*) + F_\varepsilon^{DZ} \bigg|_{v_* = v_*(x, t^*)} \bigg|_{x = 0} \\
F^{\text{red}}(t^*; \varepsilon) = F_0(t^*) + (F_\varepsilon^{DZ})^{\text{sing}} \bigg|_{v_* = v_*(x, t^*)} \bigg|_{x = 0}
\end{array} \right. \]

\[ w^\alpha = v^\alpha + \eta^{\alpha \mu} \partial_x \partial_{t^\mu} F_\varepsilon^{DZ} \]

\[ v^\alpha + \sum_{g \geq 1 \atop k \leq 2g} \varepsilon^{2g} w_{g, k}(v_*) (v_1^1)^k, \quad w_{g, k}(v_*) \in \mathbb{C}[[v_*, \varepsilon]][2g-k] \]

\[ \tilde{u}^\alpha = v^\alpha + \eta^{\alpha \mu} \partial_x \partial_{t^\mu} (F_\varepsilon^{DZ})^{\text{sing}} \]

\[ v^\alpha + \sum_{g \geq 1 \atop k \leq 0} \varepsilon^{2g} \tilde{u}_{g, k}(v_*) (v_1^1)^k, \quad \tilde{u}_{g, k}(v_*) \in \mathbb{C}[[v_*, \varepsilon]][2g-k] \]

\[ u^\alpha = v^\alpha + \sum_{g \geq 1 \atop k \leq 0} \varepsilon^{2g} u_{g, k}(v_*) (v_1^1)^k, \quad u_{g, k}(v_*) \in \mathbb{C}[[v_*, \varepsilon]][2g-k] \]
What we already proved

**Theorem (BDGR’16)**

For any semisimple CohFT, the Miura transformation \( u^\alpha = u^\alpha(w^*_\epsilon) \), transforms the DZ Poisson structure to the DR Poisson structure.

**Theorem (B ’14, BR ’14, BG’15, BDGR ’16, BGDR’17, BGR’17)**

The strong DR/DZ equivalence conjecture holds for:
- the trivial CohFT,
- the full Hodge class,
- Witten’s 3-, 4- and 5-spin classes,
- the Fan-Jarvis-Ruan-Witten \( D_4 \) CohFT,
- the GW theory of \( \mathbb{P}^1 \).

It holds up to genus 5 for any rank 1 CohFT.
It holds up to genus 2 for any semisimple CohFT.
Why is this interesting?

- The $DZ$ hierarchies contain basically all known examples of $(1+1)$ integrable Hamiltonian systems of evolutionary PDEs possessing a tau-symmetry (KdV, Toda, ILW, Gelfand-Dickey, Drinfeld-Sokolov, Ablowitz-Ladik, etc.).

- $DZ \mapsto DR$ is the quintessential normal form for such systems:
  - the Poisson structure is reduced to $\eta \partial_x$,
  - the Hamiltonian densities satisfy a universal recursion,
  - the topological tau-function is reduced to the simplest possible form.

- The route $DZ \mapsto DR$ leads to systematic quantization of all systems of topological type.

- The fact that the $DR$ hierarchy exists for non semisimple CohFTs suggests that it might be possible to construct the $DZ$ hierarchy in that case too.

- Makes the relation with geometry of $\overline{M}_{g,n}$ much more transparent.

- Might be the road to finally prove polynomiality of the second Poisson structure for the $DZ$ hierarchy.
Why is this interesting?

- The $DZ$ hierarchies contain basically all known examples of (1+1) integrable Hamiltonian systems of evolutionary PDEs possessing a tau-symmetry (KdV, Toda, ILW, Gelfand-Dickey, Drinfeld-Sokolov, Ablowitz-Ladik, etc.).
- $DZ \mapsto DR$ is the quintessential normal form for such systems:
  - the Poisson structure is reduced to $\eta \partial_x$,
  - the Hamiltonian densities satisfy a universal recursion,
  - the topological tau-function is reduced to the simplest possible form.
- The route $DZ \mapsto DR$ leads to systematic quantization of all systems of topological type.
- The fact that the $DR$ hierarchy exists for non semisimple CohFTs suggests that it might be possible to construct the $DZ$ hierarchy in that case too.
- Makes the relation with geometry of $\overline{M}_{g,n}$ much more transparent.
- Might be the road to finally prove polynomiality of the second Poisson structure for the $DZ$ hierarchy.
The DR/DZ conjecture holds iff, for $2g - 1 \leq \sum d_i \leq 3g - 3 + n$:

$$\int_{A_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}) = \int_{B_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n})$$

where

$$A_{d_1,\ldots,d_n}^g, B_{d_1,\ldots,d_n}^g \in R^*(\overline{M}_{g,n}) \subset H^*(\overline{M}_{g,n})$$

represent certain cycles in $\overline{M}_{g,n}$.
**DR/DZ equivalence and tautological relations**

**Theorem (BGR'17)**

The DR/DZ conjecture holds iff, for $2g - 1 \leq \sum d_i \leq 3g - 3 + n$:

$$
\int_{A_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}) = \int_{B_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n})
$$

$$
A_{d_1,\ldots,d_n}^g = \sum \text{Coeff}_{a_1^{d_1} \ldots a_n^{d_n}} \frac{1}{\sum a_i}
$$

$$
B_{d_1,\ldots,d_n}^g = \sum (-1)^{L-1} \pi_*
$$
DR/DZ equivalence and tautological relations

Theorem (BGR’17)

The DR/DZ conjecture holds iff, for $2g - 1 \leq \sum d_i \leq 3g - 3 + n$:

$$\int_{A_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}) = \int_{B_{d_1,\ldots,d_n}^g} c_{g,n}(e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n})$$

Conjecture (BDGR’17, sufficient for the DR/DZ equivalence)

$$A_{d_1,\ldots,d_n}^g = B_{d_1,\ldots,d_n}^g \in R^d(\overline{M}_{g,n})$$

$$2g - 1 \leq d = \sum d_i \leq 3g - 3 + n$$
Results towards proving the conjecture

**Theorem (BGR'17)**

- \( A = B \) holds in \( \overline{M}_{0,n} \) and \( \overline{M}_{1,n} \).
- \( A \) and \( B \) classes behave the same way upon pullback and pushforward along \( \pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n} \).
- \( i^* \pi_* A = i^* \pi_* B \) with \( \overline{M}_{g,n+m} \xrightarrow{\pi} \overline{M}_{g,n} \xleftarrow{i} \mathcal{M}_{g,n} \).
- For semisimple CohFTs, checking \( A = B \) in degree \( \sum d_i = 2g \) and with \( d_i > 0 \) is enough to prove the DR/DZ equivalence (this is how we proved it for \( g \leq 2 \)).
Theorem (BGR'17)

- \( A = B \) holds in \( \overline{M}_{0,n} \) and \( \overline{M}_{1,n} \).
- \( A \) and \( B \) classes behave the same way upon pullback and pushforward along \( \pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n} \).
- \( i^* \pi_* A = i^* \pi_* B \) with \( \overline{M}_{g,n+m} \overset{\pi}{\to} \overline{M}_{g,n} \leftarrow \overline{M}_{g,n} \).
- For semisimple CohFTs, checking \( A = B \) in degree \( \sum d_i = 2g \) and with \( d_i > 0 \) is enough to prove the DR/DZ equivalence (this is how we proved it for \( g \leq 2 \)).