Methods of tangent and cotangent coverings for Dubrovin-Novikov integrability operators

R. Vitolo

Joint work with E. Ferapontov, M. Pavlov
arXiv:2017

Geometry of Integrable systems
SISSA, 5-7 June 2017
Hamiltonian PDEs

An evolutionary system of PDEs

\[ F = u_t^i - f^i(t, x, u^j, u_x^j, u_{xx}^j, \ldots) = 0 \]

admits a Hamiltonian formulation if

\[ u_t^i = A^{ij} \left( \frac{\delta H}{\delta u^j} \right) \]

where \( A = (A^{ij}) \) is a Hamiltonian operator, i.e. a differential operator

\[ A^{ij} = a^{ij\sigma} D_\sigma \quad \text{such that} \quad A^* = -A \quad \text{and} \quad [A, A] = 0 \]

\( D_\sigma = D_x \circ \cdots \circ D_x \) (total \( x \)-derivatives \( \sigma \) times).

Finding Hamiltonian operators/PDEs is hard.
A Hamiltonian equation shows a \textit{correspondence} between conservation laws and symmetries.

Generalized \textit{symmetries} are vector functions \( \varphi^i = \varphi^i(u^j, u'^j, u''^j, \ldots) \) such that

\[
\begin{cases}
\ell_F(\varphi^i) = D_t \varphi^i - \frac{\partial f^i}{\partial u^j} D_\sigma \varphi^j = 0, \\
F^k = 0
\end{cases}
\]

where \( \ell_F \) is the Fréchet derivative of \( F \).
Conservation laws

A conservation law is a one-form \( \omega = A \, dx + B \, dt \) which is closed modulo the equation:

\[
\bar{d} \omega = \nabla F
\]

where \( \nabla = a_\tau^\sigma D_\tau F^k \). The vector function

\[
\psi_k = \psi_k(u^j, u^j_x, u^j_{xx}, \ldots) = (-1)^{|\tau\sigma|} D_\tau a_\tau^\sigma |F=0
\]

represents uniquely the conservation law and fulfills the equation

\[
\begin{cases}
\ell^*_F (\varphi^i) = -D_t \psi_i + (-1)^{|\sigma|} D_\sigma \left( \frac{\partial f^j}{\partial u^i_\sigma} \psi_j \right) = 0 \\
F^k = 0
\end{cases}
\]

where \( \ell^*_F \) is the formal adjoint of \( \ell_F \);
A necessary condition

If an equation admits a Hamiltonian formulation, this implies that $A$ maps conservation laws into symmetries:

$$\ell_F \circ A = (A')^* \circ \ell_F$$

$A$: almost-Hamiltonian op.

The condition can be extended to all integrability operators:

$$\ell_F^* \circ S = S' \circ \ell_F$$

$S$: almost-symplectic op.

$$\ell_F \circ R = R' \circ \ell_F$$

$R$: recursion operator

$$\ell_F^* \circ C = (C')^* \circ \ell_F^*$$

$C$: co-recursion operator

Note that $A'$, $S'$, $R'$, $C'$ are arbitrary.

*Almost:* it is a necessary condition . . .
Cotangent covering

Kersten, Krasil’shchik, Verbovetsky, JGP 2003.

Introducing new variables $p_k, p_{kx}, p_{kxx}, \ldots$ we can represent operators by linear functions:

$$A(\psi) = a^{ij\sigma} D_{\sigma}^i \psi_j \iff A = a^{ij\sigma} p_{j\sigma}$$

Then a Hamiltonian operator fulfills the equations

$$\mathcal{T}^*:\left\{ \begin{align*}
\ell_F^*(p) &= -p_{i,t} + (-1)^{|\sigma|} D_{\sigma} \left( \frac{\partial f_j}{\partial u_{\sigma}} p_j \right) = 0 \\
F &= u_t^i - f^i = 0
\end{align*} \right.$$  

and $\ell_F(A) = 0$.

The system $\mathcal{T}^*$ is the **cotangent covering**. It is **invariant**.
Introducing new variables $q^k, q^k_x, q^k_{xx}, \ldots$ we can represent operators by linear functions:

$$R(\varphi) = a^i_\sigma D_\sigma \varphi^j \quad \Leftrightarrow \quad R = a^i_\sigma q^j_\sigma$$

Then a recursion operator fulfills the equations

$$\ell_F(q) = q_t^i - \frac{\partial f^i}{\partial u^j_\sigma} q^j_\sigma = 0 \quad \text{and} \quad \ell_F(R) = 0.$$  

The system $\mathcal{T}$ is the tangent covering. It is invariant.
The KdV equation: $u_t = uu_x + u_{xxx}$
The linearization: $\ell_F = D_t - u D_x - u_x - D_{xxx}$
The adjoint linearization: $\ell^*_F = -D_t + u D_x + D_{xxx}$
The cotangent covering for the KdV equation:

\[
\begin{aligned}
  p_t &= p_{xxx} + up_x \\
  u_t &= u_{xxx} + uu_x
\end{aligned}
\]

The equation $\ell_F(A) = 0$ has the two solutions:

\[
\begin{aligned}
  A_1 &= p_x \quad \text{or} \quad A_1 = D_x \\
  A_2 &= \frac{1}{3}(3p_{3x} + 2up_x + u_x p) \quad \text{or} \quad A_2 = \frac{1}{3}(3D_{xxx} + 2uD_x + u_x)
\end{aligned}
\]

For example, $\ell_F(A_1) = D_t p_x - u D_x p_x - u_x p_x - D_{xxx} p_x$. 
Example: recursion operator for KdV

The tangent covering of KdV:

\[ \mathcal{T}: \begin{cases} 
q_t = u_x q + u q_x + q_{xxx} \\
u_t = u_{xxx} + u u_x 
\end{cases} \]

Unfortunately, the equation for recursion operators \( \ell_F(R) = 0 \) has the only trivial solution \( R = q \).

However, there is a conservation law on \( \mathcal{T} \):
\[ \omega = q dx + (u q + q_{xx}) dt. \]

We can introduce a new non-local variable \( w \) such that
\[ w_x = q, \quad w_t = u q + q_{xx}. \]

Then we have the non-local recursion operator
\[ R = q_{xx} + \frac{2}{3} u q + \frac{1}{3} u_x w \quad \text{or} \quad R = D_{xx} + \frac{2}{3} u + \frac{1}{3} u_x D_x^{-1} \]
Applications to Dubrovin–Novikov operators

The cotangent covering of a hydrodynamic-type system is:

\[
\begin{align*}
  p_{i,t} &= (V_{i,j}^k - V_{j,i}^k)u_{x}^j p_k + V_{i}^k p_{k,x} \\
  u_{t}^i &= V_{j}^i (u) u_{x}^j
\end{align*}
\]

A first-order Dubrovin–Novikov Hamiltonian operator:

\[
A^i = g^{ij} p_{jx} + \Gamma_{k}^{ij} u_{x}^k p_j.
\]

Tsarev’s compatibility conditions are the coefficients of the linear equation in \( p_{k,\sigma}, \ell_F(A) = 0:\)

\[
D_t A^i - V_{j,k}^i u_{x}^j A^k - V_{j}^i D_x A^j = 0 \iff \left\{ \begin{array}{l}
  g^{ik} V_{j}^i = g^{jk} V_{i}^i \\
  \nabla_i V_{j}^k = \nabla_j V_{i}^k
\end{array} \right.\]
Further applications

- Higher order Dubrovin–Novikov Hamiltonian operators.
- Symplectic Dubrovin–Novikov operators.
- Recursion operators for cosymmetries.
- Nonlocal Dubrovin–Novikov first-order operators, also known as Ferapontov–Mokhov operators. Higher order analogue.
Dubrovin–Novikov operators can be defined for arbitrary orders. Here we consider the third order ones:

\[
A_{ij}^3 = g_{ij}^k (u) D_x^3 + b_{ij}^k (u) u_x^k D_x^2 \\
+ [c_{ij}^k (u) u_{xx}^k + c_{km}^i (u) u_x^k u_x^m] D_x \\
+ d_{ij}^k (u) u_{xxx}^k + d_{km}^i (u) u_x^k u_{xx}^m + d_{kmn}^i (u) u_x^k u_x^m u_x^n,
\]

Potemin’s canonical form in Casimirs:

\[
A_{ij}^3 = D_x (g_{ij}^k D_x + c_{ij}^k u_x^k) D_x
\]

**Remark:** $g_{ij}$ is the Monge form of a **quadratic line complex** (Ferapontov, Pavlov, V. JGP 2014, IMRN 2016).

We restrict our consideration to hydrodynamic-type systems in these Casimirs. Then they can be written in conservative form: $V_j^i = (V^i)_j$
Compatibility conditions (Ferapontov, Pavlov, V. 2017)

**Theorem** Let $A_3$ be a Hamiltonian operator. Then $u^i_t = V^i_j u^j_x = (V^i)_j u^j_x$ admits a Hamiltonian formulation with $A_3$ if and only if

$$
\begin{align*}
&g_{im} V^m_j = g_{jm} V^m_i \\
&c_{mkj} V^m_i + c_{mik} V^m_j + c_{mji} V^m_k = 0, \\
&V^k_{i,j} = g^{ks} c_{smj} V^m_i + g^{ks} c_{smi} V^m_j
\end{align*}
$$

**Theorem.** The above system is in involution. Its solution depends on at most $(1/2)n(n + 3)$ parameters. The solution is reduced to a linear algebra problem either if the unknown is $g_{ij}$ or if the unknown is $V^i$.

**Remark.** No Hamiltonian needed at this stage!
Properties of the systems of conservation laws

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system $u_t = (V)_j u_x^j$ a congruence of lines in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

Theorem.

- The congruence is linear: there are $n$ linear relations between $u^i$, $V^i$, $u^i V^j - u^j V^i$.
- The system is linearly degenerate, and non diagonalizable.
- $V^i = \psi^i_\alpha w^\alpha$, where $\psi^i_\alpha$ is determined by $g_{ij} = \varphi_{\alpha \beta} \psi^\alpha_i \psi^\beta_j$ and $w^\alpha$ are linear functions. This means that $V^i = p(u^j)/q(u^j)$ where $p(u^j)$, $q(u^j)$ are polynomials of degree $n$, $n - 1$. 
The above systems of conservation laws all admit non-local Hamiltonian, momentum and Casimirs. They all are new non-local conserved quantities.

Let us set $\psi_k^\gamma = \psi_{km}^\gamma u^m + \omega_k^\gamma$, and $w^\gamma = \eta_m^\gamma u^m + \xi^\gamma$.

Let us set $u^i = b^i_x$; the system becomes $b^i_t = V^i(b_x)$.

Theorem.

- Hamiltonian op. $A_3 = -g^{ij}(b_x)D_x - c^{ij}_k(b_x)b^{k}_{xx}$

- Hamiltonian $H = -\int \varphi_{\beta\gamma} \left[ \left( \frac{1}{3} \eta_p^\gamma \psi_{q m}^\beta b^m_x + \frac{1}{2} \omega_p^\beta \eta_q^\gamma \right) b^p b^q + x \left( \frac{1}{2} \psi_{pq}^\gamma \xi^\gamma b^p b^q_x + \xi^\gamma \omega_q^\beta b^q \right) \right] dx$

- $n$ Casimirs $C^\alpha = \int \left( \frac{1}{2} \psi_{mk}^\alpha b^k_x + \omega^\alpha_m \right) b^m dx$

- momentum $P = -\int \left( \frac{1}{3} \varphi_{\beta\gamma} \omega_q^\beta \psi_{pm}^\gamma b^m_x + \frac{1}{2} \varphi_{\beta\gamma} \omega_p^\beta \omega_q^\gamma \right) b^p b^q dx$
**Theorem.** The class of conservative systems of hydrodynamic type possessing third-order Hamiltonian formulation is invariant under reciprocal transformations of the form

\[
\begin{align*}
    d\tilde{x} &= (a_i u^i + a)dx + (a_i V^i + b)dt \\
    d\tilde{t} &= (b_i u^i + c)dx + (b_i V^i + d)dt
\end{align*}
\]
Theorem. Let \( u^i_t = (V^i)_x \) be a hydrodynamic-type system, and suppose that it admits a Hamiltonian formulation via a third-order Dubrovin-Novikov operator whose Casimirs are \( u^i \). Then:

- \( n = 2 \) The system is linearisable.
- \( n = 3 \) The system is either linearisable, or equivalent to the system of WDVV equations (to be discussed); from Castelnuovo’s classification of linear line congruences.
- \( n = 4 \) Far more complicated: there exists no classification of linear congruences in \( \mathbb{P}^5 \). There exist one generic nontrivial integrable example.
Example: WDVV equations in 3-comp.

From $f_{ttt} = f_{xxt}^2 - f_{xxx} f_{xxt}$ setting $u^1 = f_{xxx}$, $u^2 = f_{xxt}$, $u^3 = f_{xtt}$ we have

\[
\begin{align*}
    u^1_t &= u^2_x, \\
    u^2_t &= u^3_x, \\
    u^3_t &= ((u^2)^2 - u^1 u^3)_x,
\end{align*}
\]

endowed with a third-order Hamiltonian operators with nonlocal Hamiltonian

\[
H = - \int \left( \frac{1}{2} u^1 \left( \partial_x^{-1} u^2 \right)^2 + \partial_x^{-1} u^2 \partial_x^{-1} u^3 \right) dx.
\]

(Ferapontov, Galvao, Mokhov, Nutku, 1995). It is bi-Hamiltonian and up to a non-trivial transformation is the 3-wave equation (Zakharov, Manakov, ~1970).
Example: WDVV system in 6-comp.

Dubrovin 1996; Ferapontov-Mokhov 1998; Pavlov-V. 2015. We have a pair of hydrodynamic type systems in conservative form:

\[ a^i_y = (v^i(a))_x, \quad a^i_z = (w^i(a))_x, \]

where

\[ v^1 = a^2, \quad w^1 = a^3, \quad v^2 = a^4, \quad v^3 = w^2 = a^5, \quad w^3 = a^6, \]

\[ v^4 = f_{yyy} = \frac{2a^5 + a^2a^4}{a^1}, \quad v^5 = w^4 = f_{yyz} = \frac{a^3a^4 + a^6}{a^1}, \]

\[ v^6 = w^5 = f_{yzz} = \frac{2a^3a^5 - a^2a^6}{a^1}, \]

\[ w^6 = f_{zzz} = (a^5)^2 - a^4a^6 + \frac{(a^3)^2a^4 + a^3a^6 - 2a^2a^3a^5 + (a^2)^2a^6}{a^1}. \]
Monge metric for 6-components WDVV

\[ g_{ik}(a) = \begin{pmatrix}
(a^4)^2 & -2a^5 & 2a^4 & -(a^1 a^4 + a^3) & a^2 & 1 \\
-2a^5 & -2a^3 & a^2 & 0 & a^1 & 0 \\
2a^4 & a^2 & 2 & -a^1 & 0 & 0 \\
-(a^1 a^4 + a^3) & 0 & -a^1 & (a^1)^2 & 0 & 0 \\
a^2 & a^1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \]

**Remark:** the metric can be found in few seconds by computer.
Example: generic value of $n$

The system of conservation laws:

$$u_t^1 = u_x^2, \ u_t^2 = u_x^3, \ldots, \ u_t^{n-1} = u_x^n, \ u_t^n = [u^1 u^3 - (u^2)^2]_x.$$ 

The third-order Hamiltonian operator’s Monge metric:

$$g_{ij} = \begin{pmatrix} 2a^2 & -a^1 & 0 & 1 \\ -a^1 & 0 & 1 \\ 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and the Hamiltonian is

$$H = -\frac{1}{2}a^1(D^{-1}a^2)^2 + \frac{1}{2} \sum_{m=2}^{N} (D^{-1}a^m)(D^{-1}a^{N+2-m}).$$

**Problem:** integrability for $n \geq 4$?
Open problems

- Integrability for $n \geq 4$ of the systems of conservation laws.
- Geometry of WDVV equations. All of them have a third-order H.o. and all of them are linear line congruences.
- What happens when the fluxes are functions of first or second order derivatives?
- Non-local Hamiltonian operators of second and third order, and their compatibility with hydrodynamic-type systems.
- Extension to symplectic operators, local and non-local.
Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at http://gdeq.org.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE’s kernel.

Thank you!

Contacts: raffaele.vitolo@unisalento.it