Calculating the PBH abundance

Young, Musco - tbd

Van Laak, Young 2023. 2303.05248

Young 2022. 2201.13345

Gow, Byrnes, Cole, Young 2021. 2008.03289

Young, Musso 2020. 2001.06469

Young 2019. 1905.01230

Young, Musco, Byrnes 2019. 1904.00984

See also Young 2024, 2405.13259

Choosing the right window function

Sam Young, 03/11/2025. Primordial black holes in the multi-messenger era.

UNIVERSITY OF SUSSEX

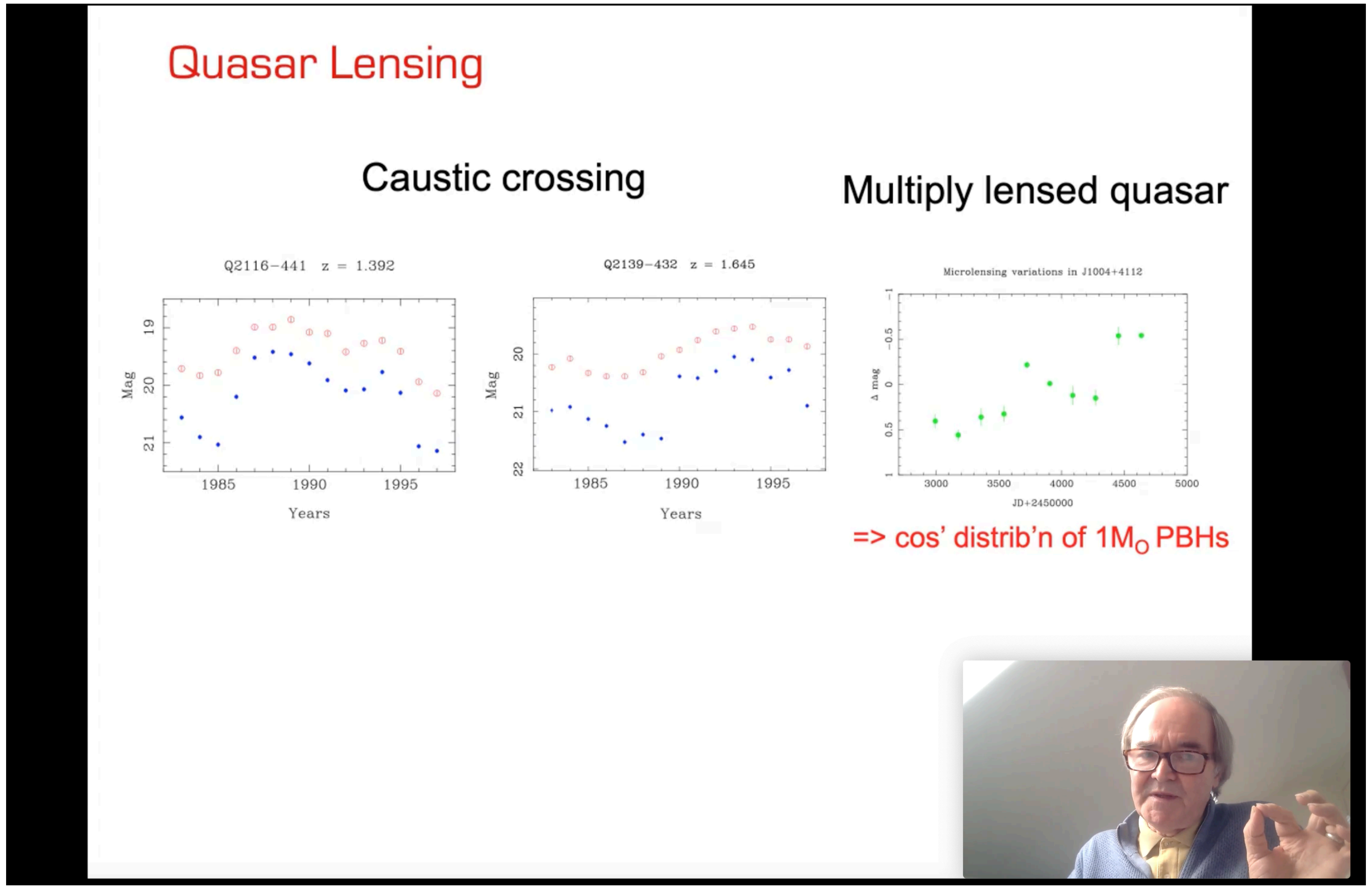
ZOOMing in on PBHs

- Monthly online seminar series, via Zoom
- (Normally) first Monday of every month, 3pm EU time
- Email me:
  - Get added to the mailing list
  - Volunteer to speak (or nominate someone else)
  - sam.young@sussex.ac.uk

Next meeting:

Time: 3pm CEST, 01/12/2025

Speaker: Anish Ghoshal

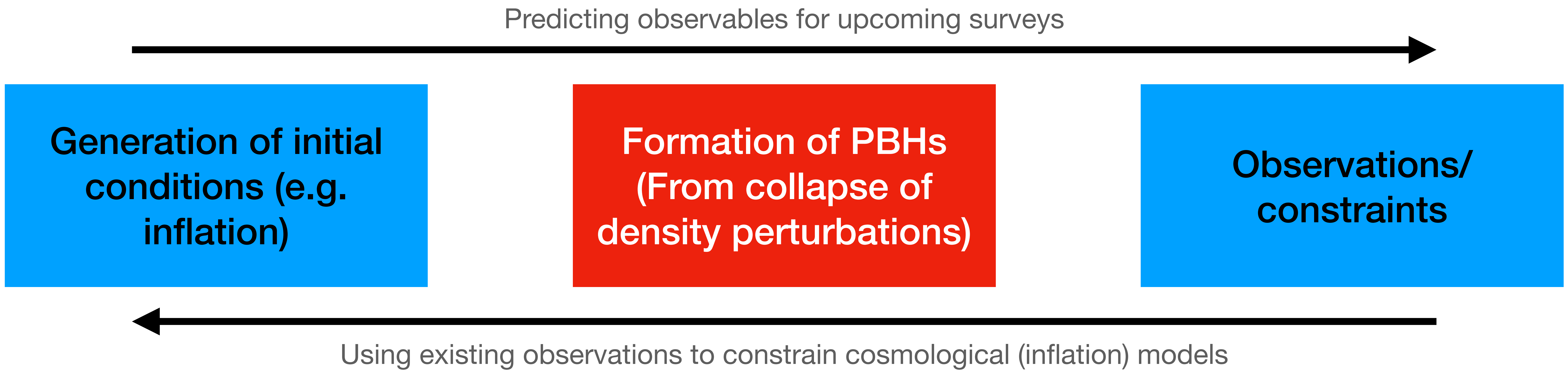


Prof Bernard Carr, Inaugural meeting October 2022

Talk overview

1. Methods to calculate the abundance
2. How to describe perturbations
3. Smoothing
4. Choosing the right window (ongoing work)

Its more important than ever to be able to accurately predict the PBH abundance and mass function, to compare with current and upcoming surveys. One of the largest uncertainties in the calculation comes from the choice of window function.



Calculating the PBH abundance

PBH abundance at formation: $\beta\left(M_H\right)$

The PBH density parameter at t_{eq}: $\Omega_{\mathrm{PBH}} = \int \frac{\mathrm{d}M_H}{M_H}\left(\frac{M_{\mathrm{eq}}}{M_H}\right)^{1/2} \beta\left(M_H\right)$

Fraction of dark matter composed of PBHs: $f_{pbh} = \frac{\Omega_{PBH}}{\Omega_{DM}}$

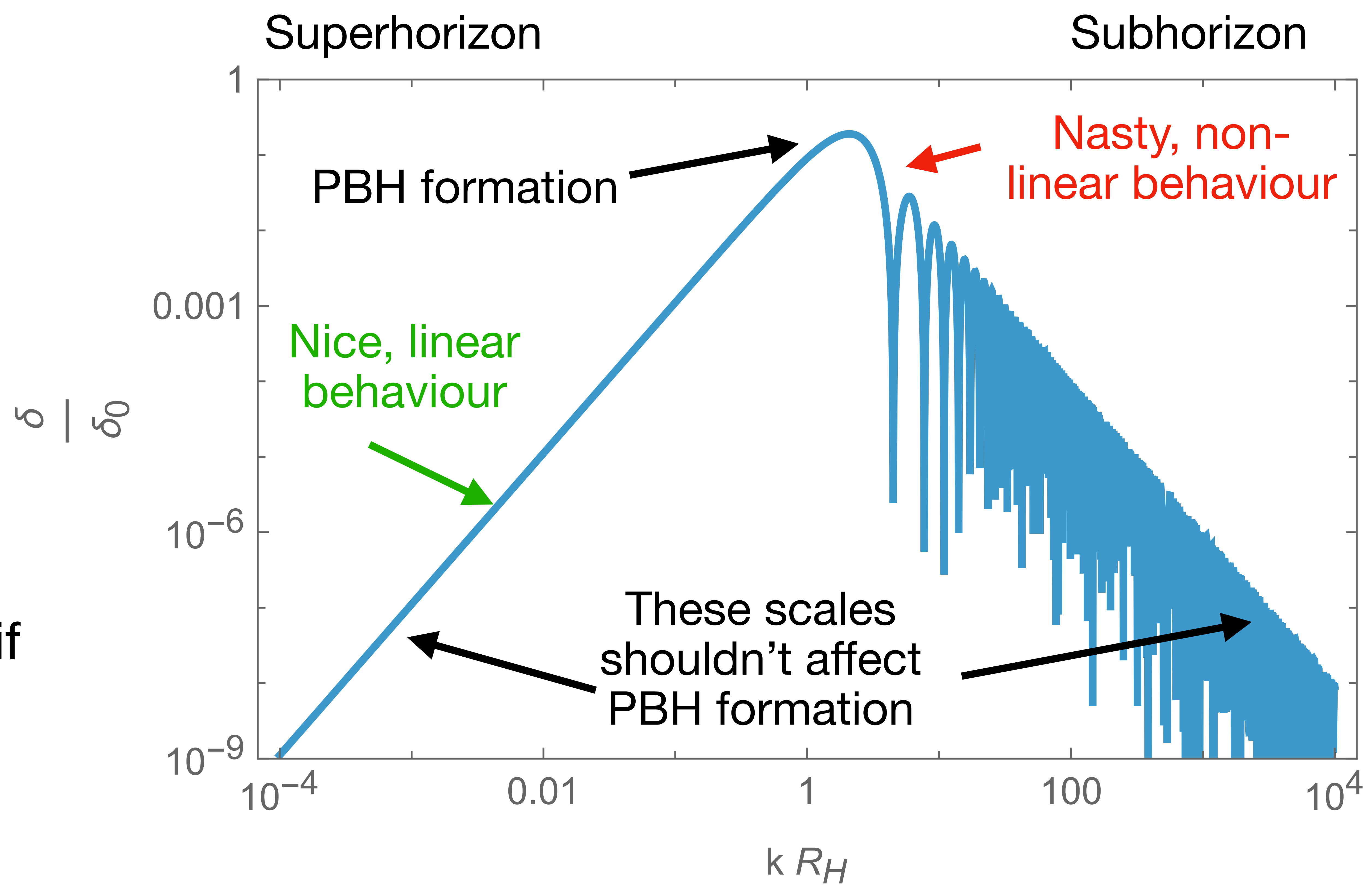
PBH mass function: $f\left(M_{\mathrm{PBH}}\right) = \frac{1}{\Omega_{CDM}} \frac{\mathrm{d}\Omega_{\mathrm{PBH}}}{\mathrm{d}ln(M_{\mathrm{PBH}})}, \psi\left(M_{\mathrm{PBH}}\right) = \frac{1}{\Omega_{\mathrm{PBH}}} \frac{\mathrm{d}\Omega_{\mathrm{PBH}}}{\mathrm{d}M_{\mathrm{PBH}}}$

PBHs from density perturbations

Super- and sub-horizon modes

$$\rho = \bar{\rho}(1 + \delta)$$

- Density modes are strongly suppressed in the super-horizon regime
- Perturbations grow over time
- Reaches a peak around horizon entry
  - PBH formation can occur at (about) this time if a region is dense enough
  - Numerical simulations needed to model the non-linear evolution
- On sub-horizon scales, modes are damped by pressure forces (assuming no PBH formation)

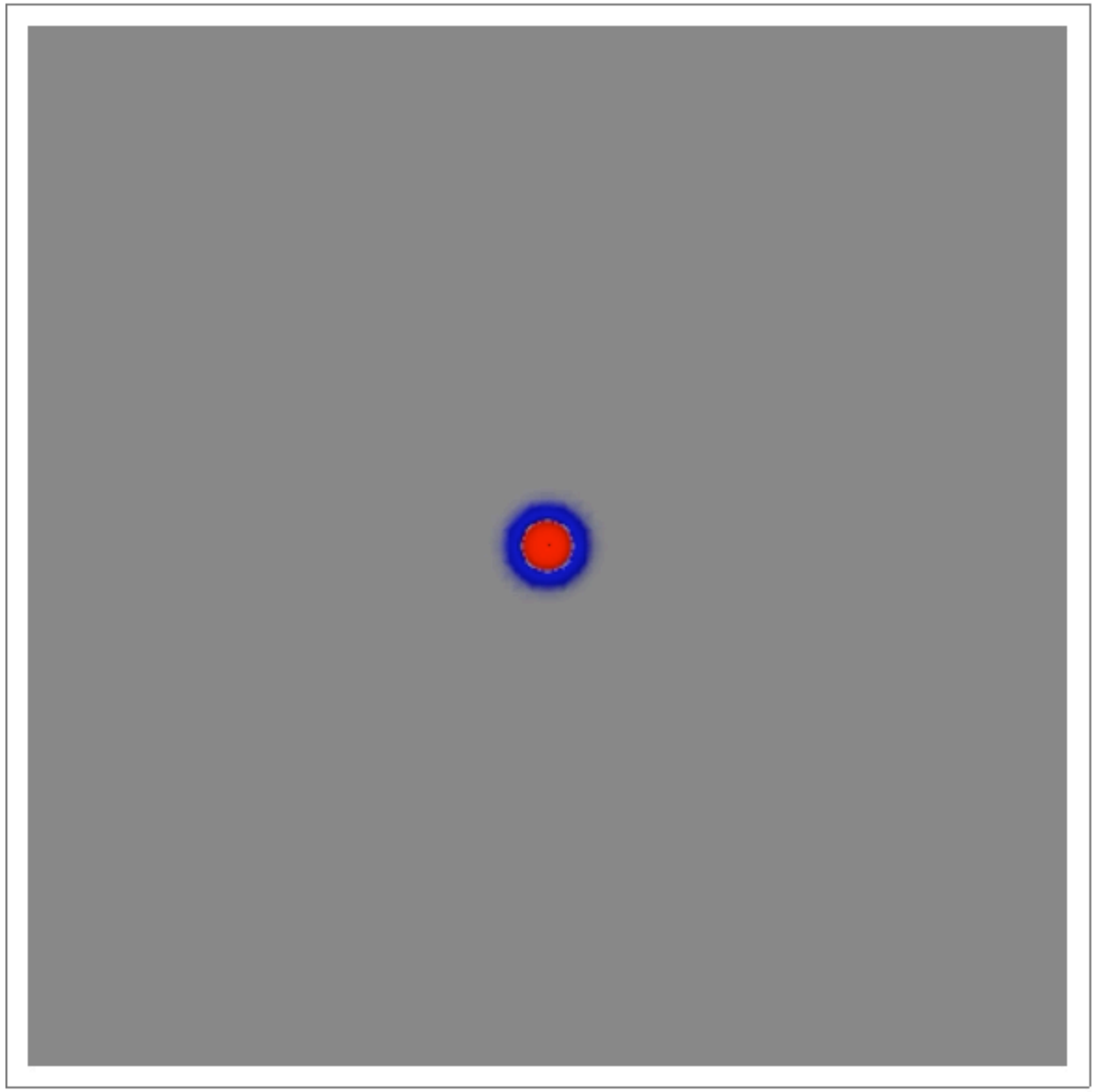


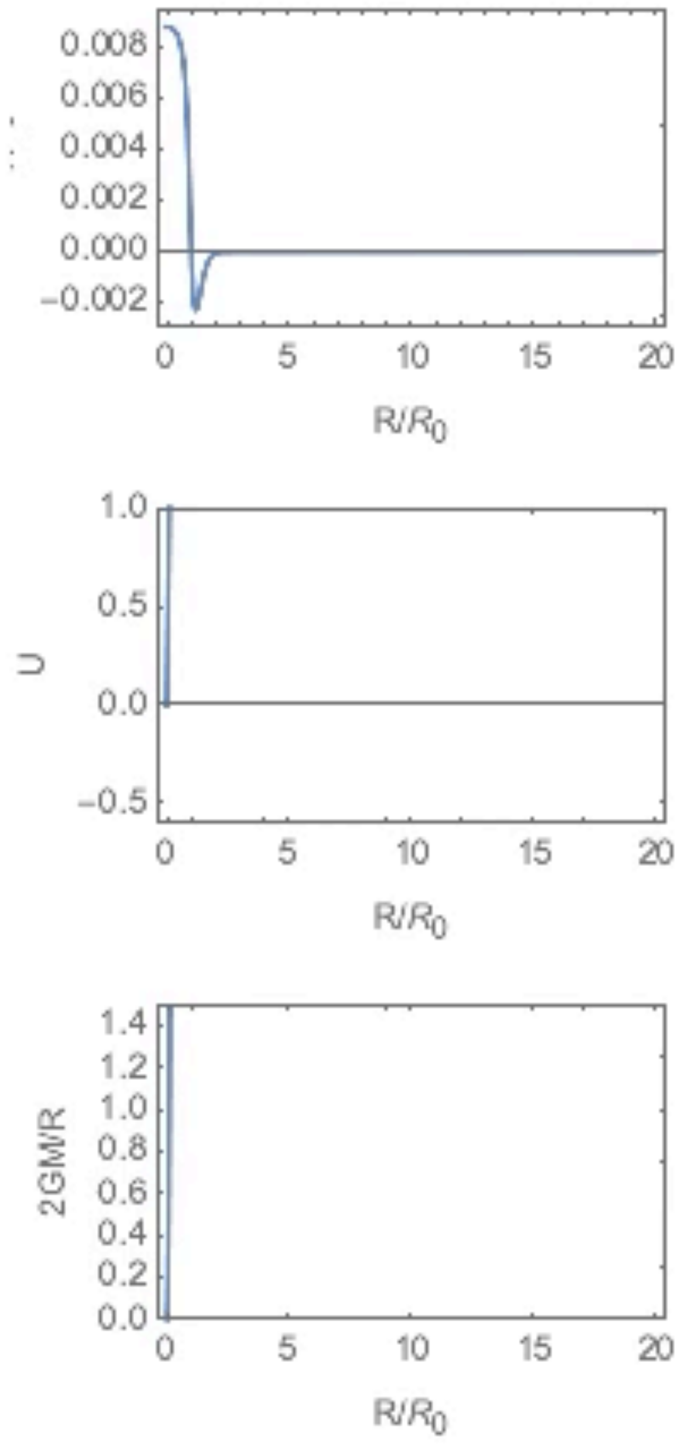
The x-axis can be read as either:

1. Time evolution of a single mode
2. Relative amplitude of different scales at a specific time

Created in collaboration with Ilia Musco

$\frac{t}{t_0}=1$





PBH formation

Threshold statistics

The Press-Schechter approach

- Basic idea: PBHs form in regions where the density takes a value $\bar{\delta}$ above the critical value δ_c

$$n(\delta) = \delta_D\left(\bar{\delta} - \delta\right)\theta_H(\delta - \delta_c)$$

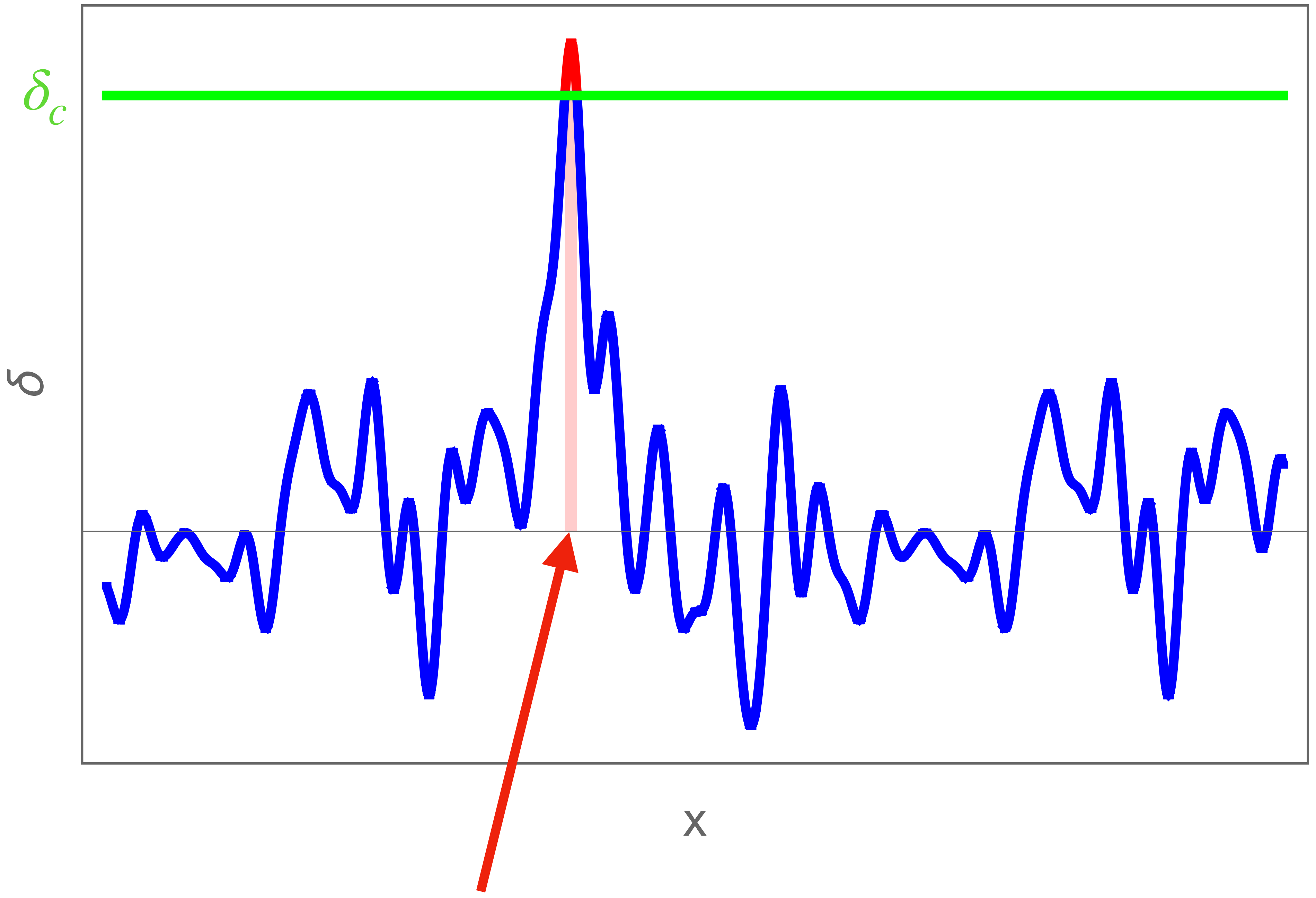
- To calculate the abundance, we want the expectation value:

$$\mathcal{N}\left(\bar{\delta}\right) = \langle n \rangle = \int \mathrm{d}\delta n(\delta) P(\delta)$$

- The fraction of the universe composed of regions with density $\bar{\delta}$ (above δ_c) is given by the PDF:

$$\mathcal{N}(\bar{\delta}) = P(\bar{\delta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\bar{\delta}^2}{2\sigma^2}\right)$$

(Where I'm assuming Gaussian statistics)



The PS approach assumes this volume will collapse to form PBHs

Threshold statistics

The Press-Schechter approach

- To calculate the abundance, integrate this over the range of values which form PBHs:

$$\beta = \int_{\delta_c}^{\infty} \mathrm{d}\delta \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\delta^2}{2\sigma^2}\right)$$

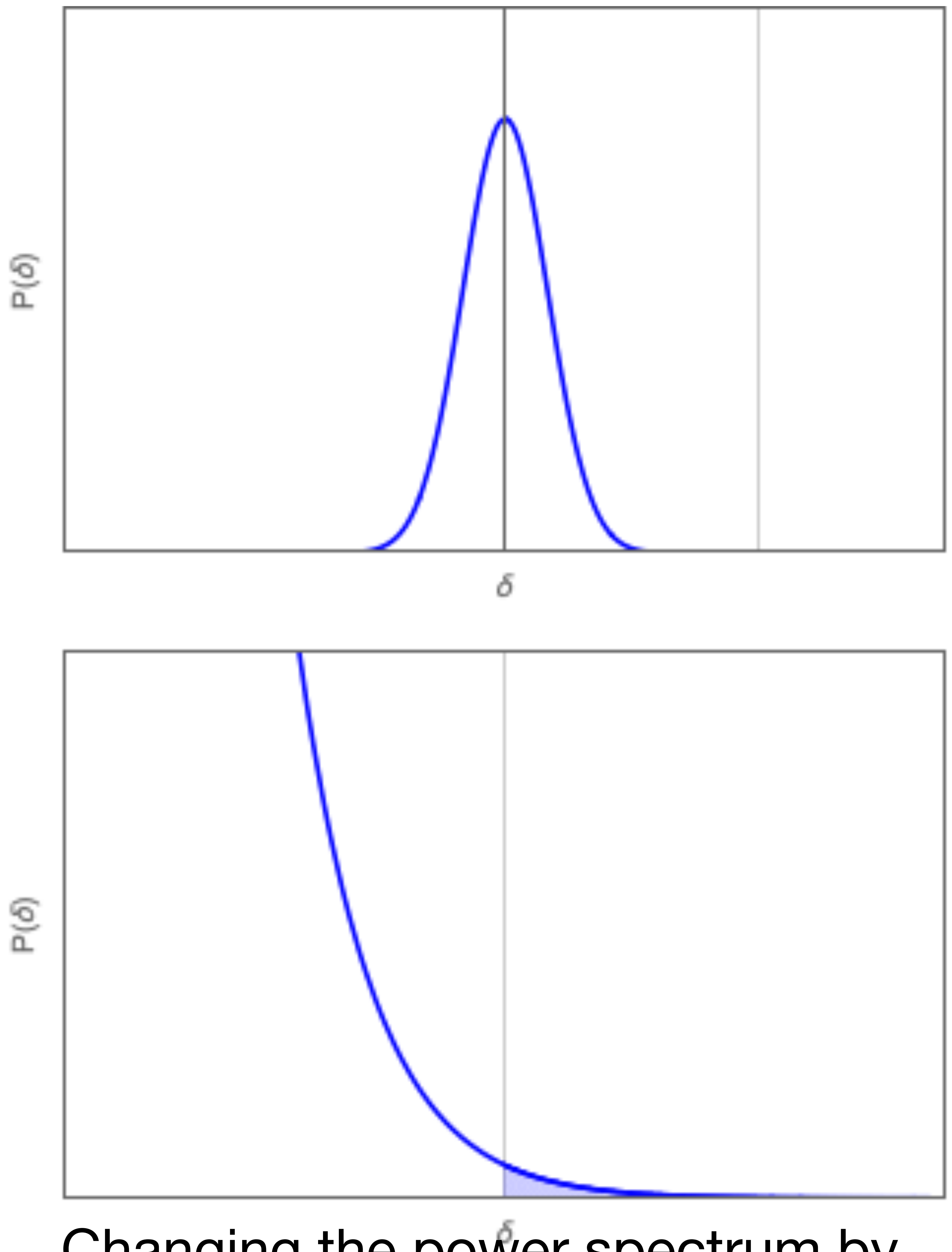
(We can also account for critical scaling here by including a term $M_{PBH}/M_H = K\left(\delta - \delta_c\right)^{\gamma}$)

- The variance is given by an integral over the power spectrum

$$\sigma^2 = \int_0^{\infty} \frac{\mathrm{d}k}{k} \mathcal{P}_{\delta}(k)$$

We'll come back to this later

- PBHs are very rare in the early universe:
    - even small changes to the power spectrum have a big impact on the abundance



Changing the power spectrum by 10% has a big impact on the abundance

The theory of peaks

A BBKS approach

- Basic idea: PBHs form at peaks (in space) in the density above the critical value

$$n = \frac{\sigma_2^3}{\sigma_1^3} \left| \left| \zeta_{ij} \right| \right| \delta_D^{(3)}\left(\eta_i\right) \theta_H\left(\lambda_3\right) \delta_D\left(\bar{\nu} - \nu\right) \theta_H(\delta - \delta_c)$$

(η_i are first derivatives, ζ_{ij} are second derivatives, and ν is the peak height. All normalised to have unit variance. λ_3 is the smallest eigenvalue of the ζ_{ij} matrix)

- We again want the expectation value of n, which this time involves at 10D integral:

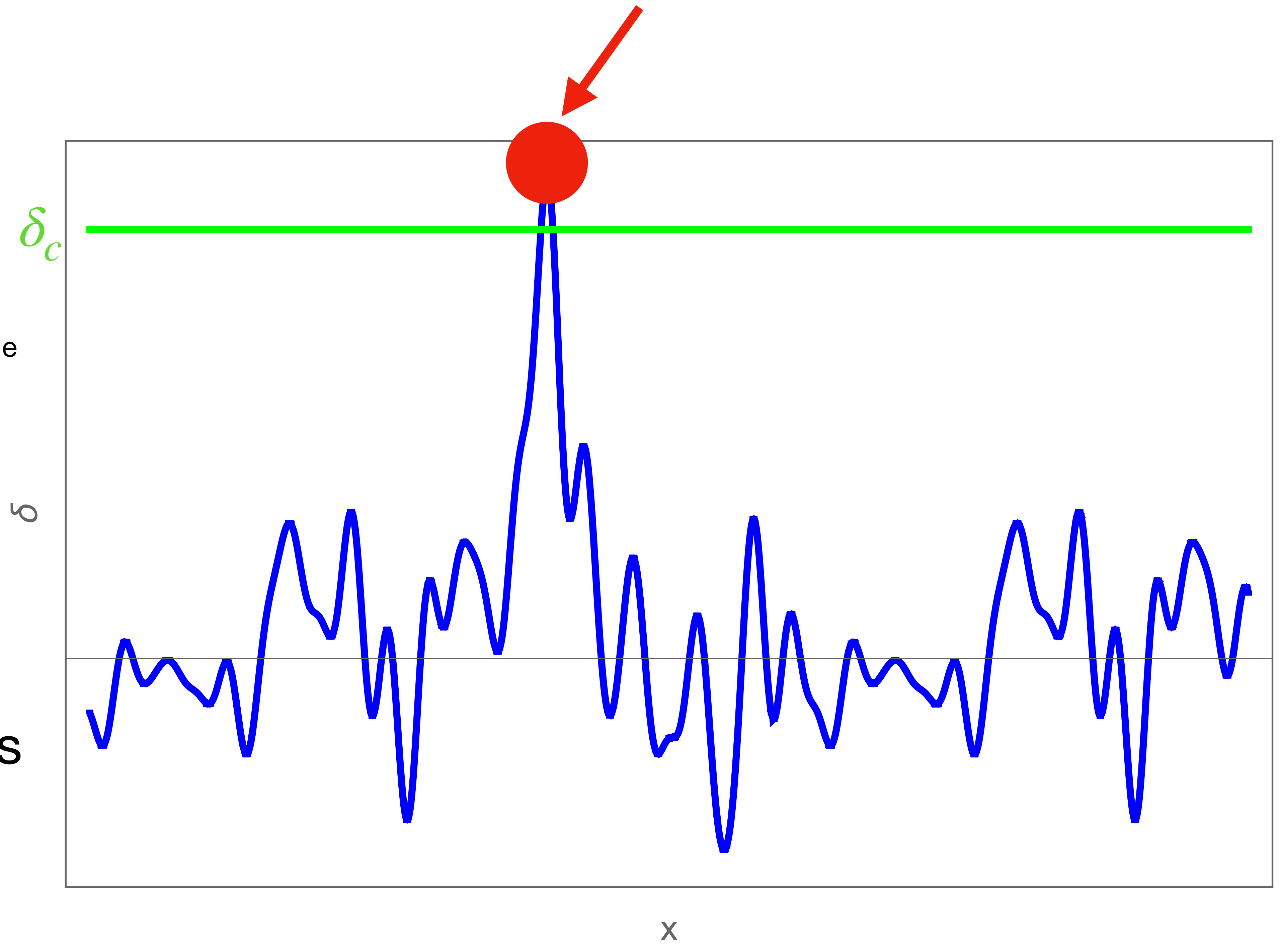
$$\mathcal{N} = \langle n \rangle = \int \mathrm{d}\nu \mathrm{d}^3\eta_i \mathrm{d}^6\zeta_{ij} n P(\nu, \eta_i, \zeta_{ij})$$

- Skipping over some non-trivial algebra, we arrive at the number density of peaks in the high peak limit:

$$\mathcal{N} = \frac{1}{3^{3/2}(2\pi)^2} \frac{\sigma_1^3}{\sigma_0^3} \nu^3 \exp\left(-\frac{\nu^2}{2}\right)$$

- (We are interested in the high-peak limit due to the rarity of PBHs)
- This generally predicts more PBHs, due to sampling efficiency

The BBKS approach assumes 1 PBH will form at this point in space



Extended peaks

e.g. Young-Musso, or Germani-Sheth (2020)

*technically, since different scale perturbations enter the horizon at different times, the calculation looks for the specific scale at which a perturbation is the largest, rather than specific times

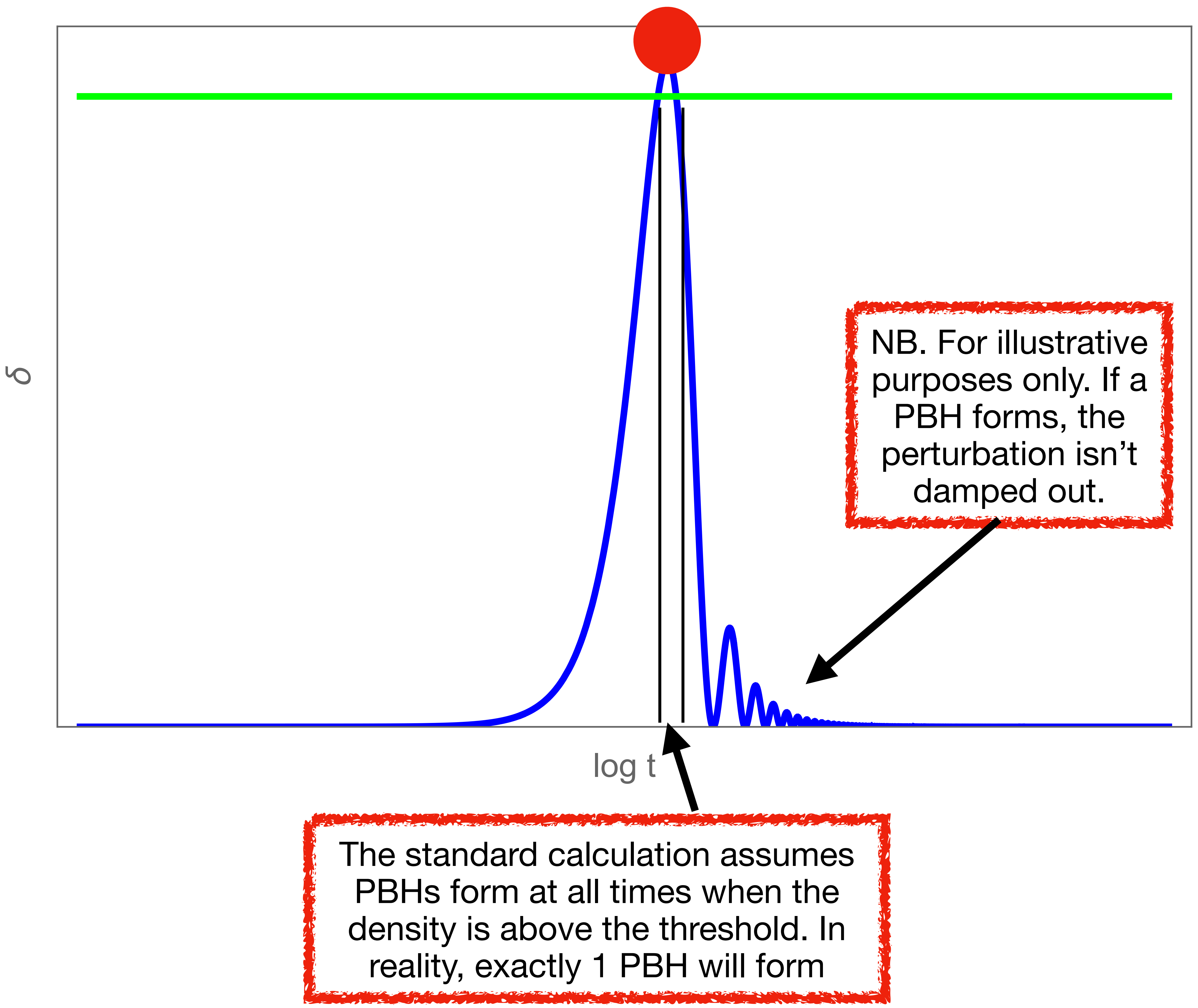
- Basic idea: PBHs form at peaks in the density above the critical value, at the time of horizon entry*

$$n_{\rm pk} = \frac{\sigma_2^3 \sigma_{RR}}{\sigma_1^3 \sigma_R} \left\| \begin{vmatrix} \zeta_{00} & \frac{\sigma_{1R}}{\sigma_{RR}}\phi_j \\ \frac{\sigma_{1R}}{\sigma_2}\phi_i & \zeta_{ij} \end{vmatrix} \right\| \delta_{\rm D}(\eta_0)\theta_{\rm H}(\zeta_{00})\delta_{\rm D}^{(3)}(\eta_i)\theta_{\rm H}(\lambda_3)\delta_{\rm D}(\nu - \bar{\nu})$$

- Again, we want the expectation value, which is

$$\mathcal{N} = \frac{16\sqrt{2}}{3^{3/2}\pi^{5/2}} \frac{\sigma_{RR}\sigma_0^3}{\sigma_2\sigma_1^3 R^7\sqrt{1-\gamma_{0,2}^2}} \alpha\nu^4 \exp\left(-\frac{1 + \frac{16\sigma_0^2}{R^4\sigma_2^2} - \frac{8\sigma_0\gamma_{0,2}}{R^2\sigma_2}}{1-\gamma_{0,2}^2}\frac{\nu^2}{2}\right)$$

- This also solves the PBH cloud-in-cloud problem (see extra slides)
- Perturbation scale is typically defined as the scale at which the compaction peaks (we'll discuss the compaction later)
    - (But can also use curvature at centre of perturbation)



Additional complications

- Critical scaling relationship

$$M_{PBH} = M_H K \left(\delta - \delta_c\right)^\gamma$$

- Non-linearity between ζ and δ
- Primordial non-Gaussianity
- Non-linear evolution of modes prior to PBH formation
- Phase transitions
- What is the appropriate parameter to determine where PBHs form?
- Smoothed vs unsmoothed quantities
- Depending on how the calculation is done, constraints on the power spectrum can vary by nearly an order of magnitude

PBH abundance is exponentially sensitive to the variance and the threshold for collapse:

$$\beta \sim \frac{\sigma}{\sqrt{2\pi}\delta_c} \exp\left(-\frac{\delta_c^2}{2\sigma^2}\right),$$

$$\sigma^2 = \int_0^\infty \frac{\mathrm{d}k}{k} \mathscr{P}(k)$$

Describing perturbations

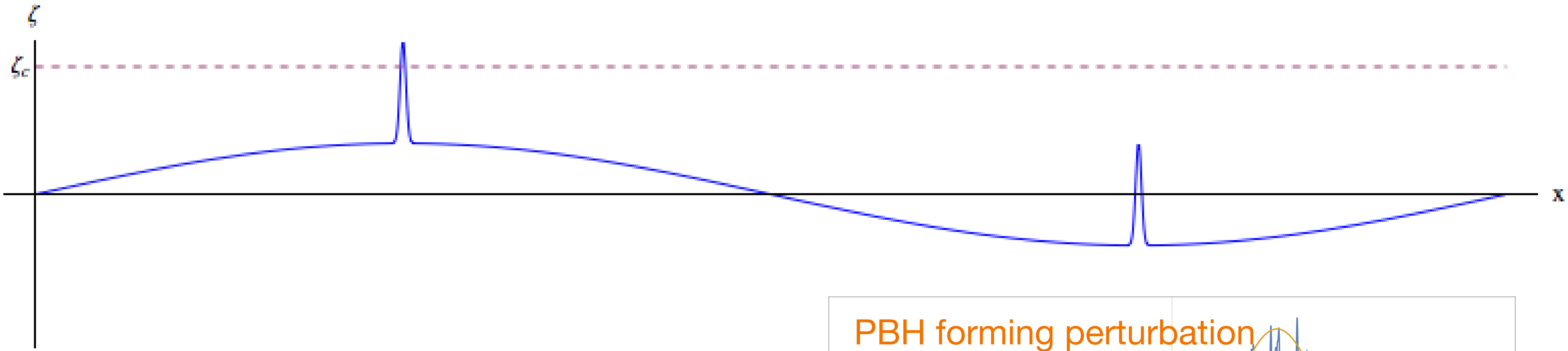
Describing the amplitude of perturbations

Curvature perturbation ζ? Density contrast δ? Compaction C? Smoothing?

- Many different parameterisations describe the amplitude of cosmological perturbations
- Simulations describe a single, isolated perturbation
- Many methods to calculate the PBH abundance
  - One of the main differences is the collapse criterion used
- Is there a difference when we consider calculating the PBH abundance?

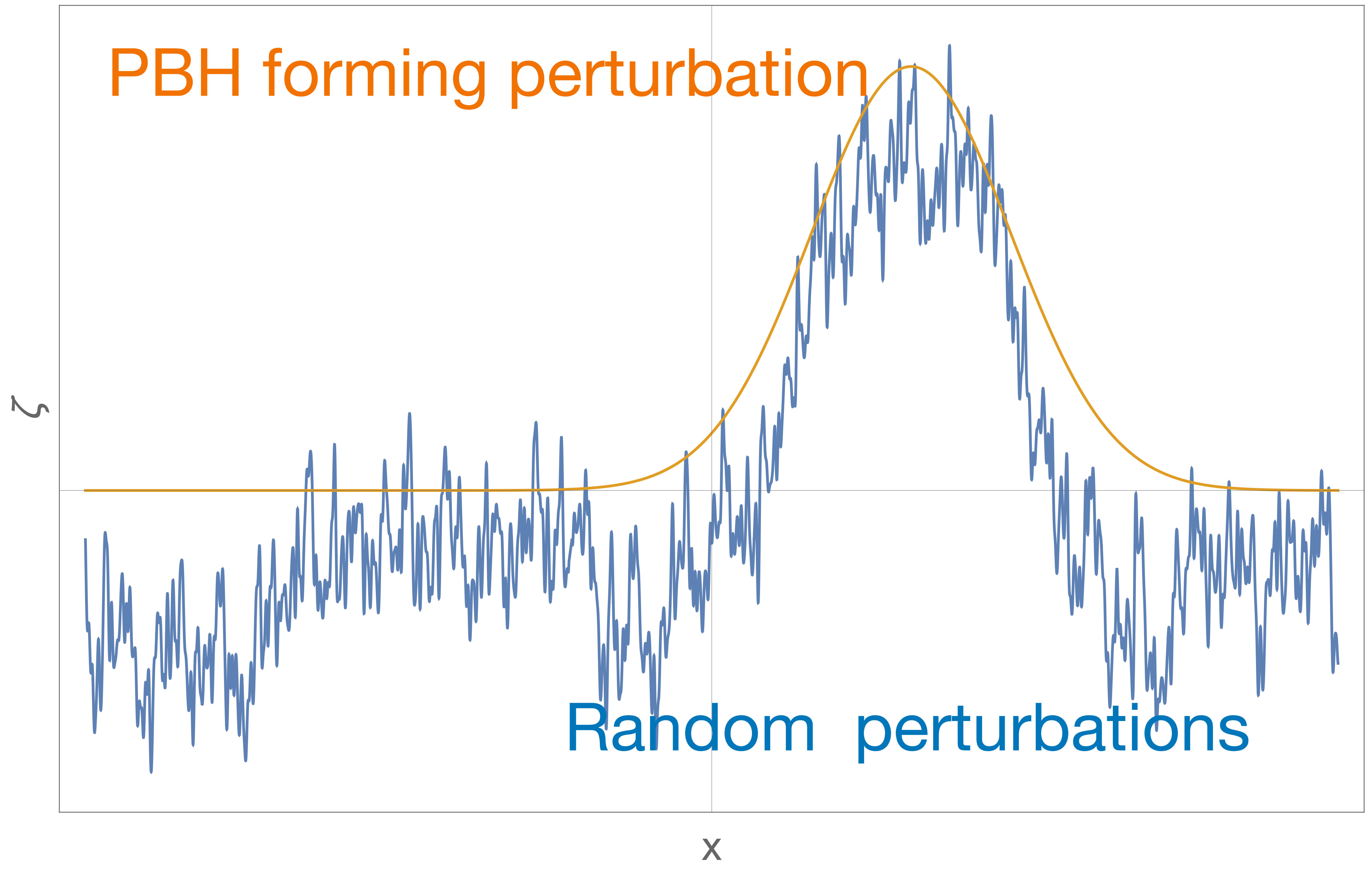
- Generally, we want to relate PBH abundance (which we can observe/constrain) to $\mathscr{P}_\zeta(k)$ to constrain the early universe - so lets first consider ζ

The curvature perturbation ζ



$$dS^2 = -dt^2 + a^2(t)e^{2\zeta}d\mathbf{X}^2$$

- Can be useful for narrow power spectra
- Simple to use
    - Inflationary models typically predict $\mathscr{P}_\zeta(k)$ and its statistics
- Suffers from "background contamination"/ "environmental effects"



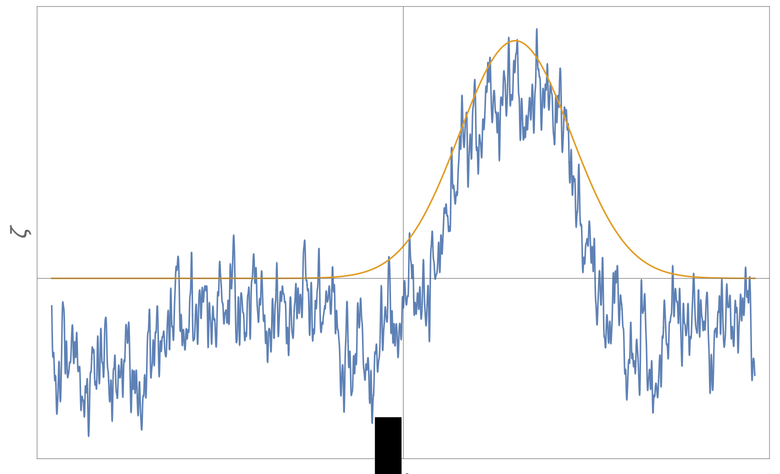
The density contrast δ

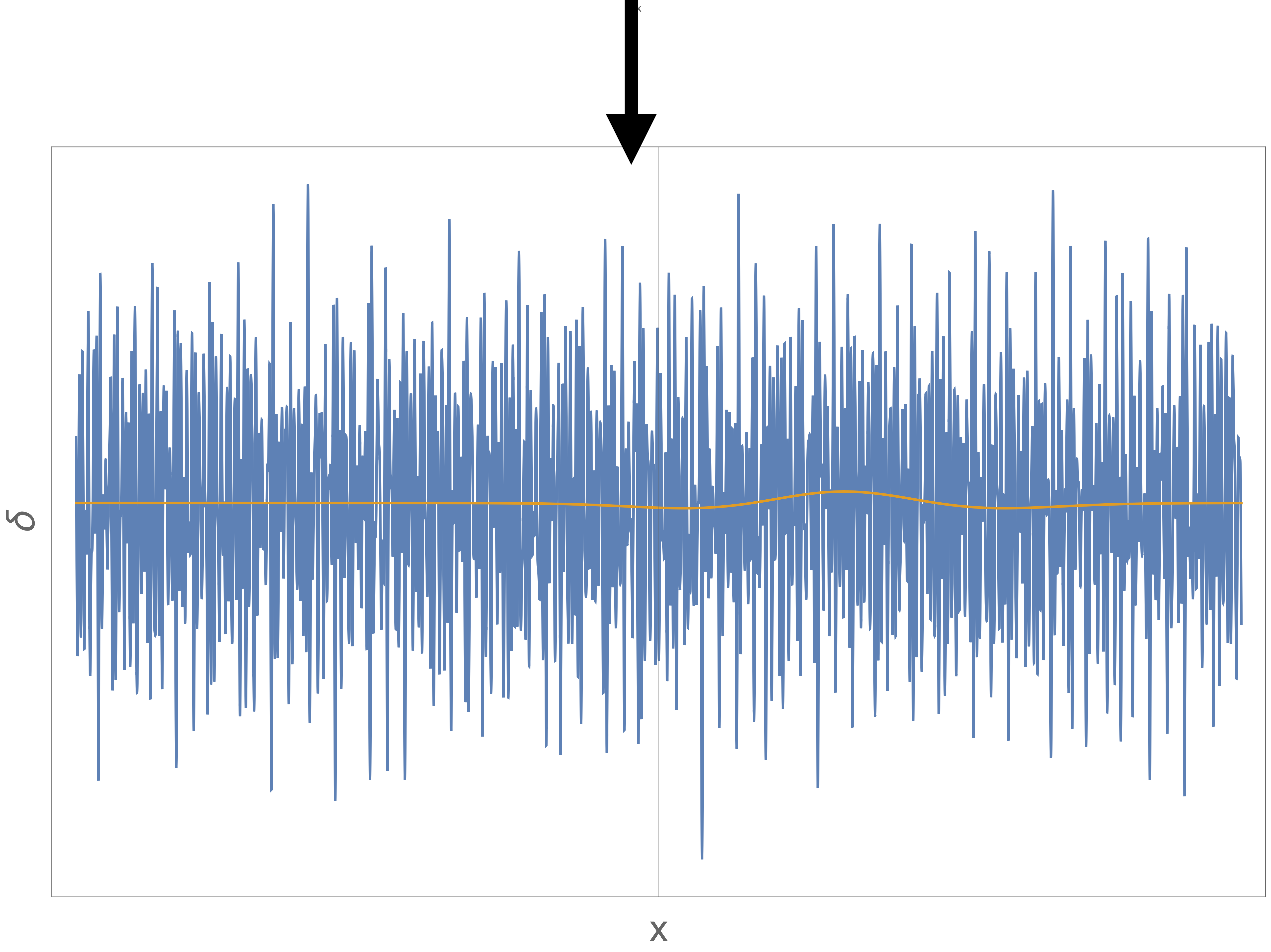
Linear order

$$\delta = -\frac{2(1+\omega)}{5+3\omega}\frac{1}{(aH)^2}\nabla^2\zeta$$

- Dominated by small-scale modes
- Time-dependent

  $\rightarrow$Typically determined at horizon entry with linear transfer function
- Again, not very useful except for narrow power spectra





NB. The equations I'm showing are valid in the super-horizon regime

The density contrast δ
Variations of the parameter

- The density can be split into time-dependant and time-independent components:

$$\delta(t,x) = \epsilon^2(t)\delta_{TI}(x)$$

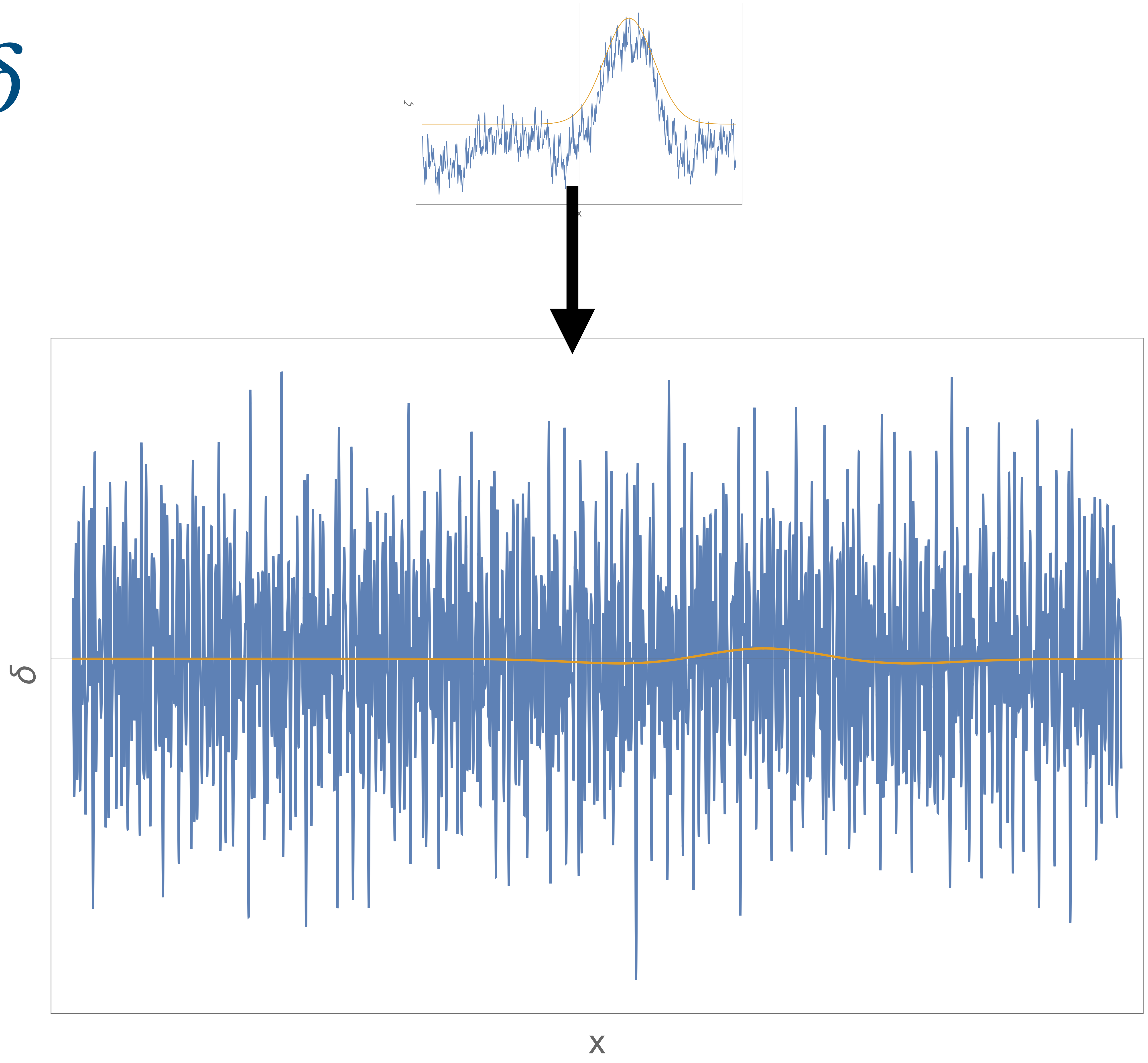
Where $\epsilon = 1/(aH)$, $\delta = -f(\omega)\,\nabla^2\zeta$, and

$$f(\omega) = \frac{2(1+\omega)}{5+3\omega}$$

- We can also consider the second derivative of ζ, which is equivalent (up to multiplicative constants)

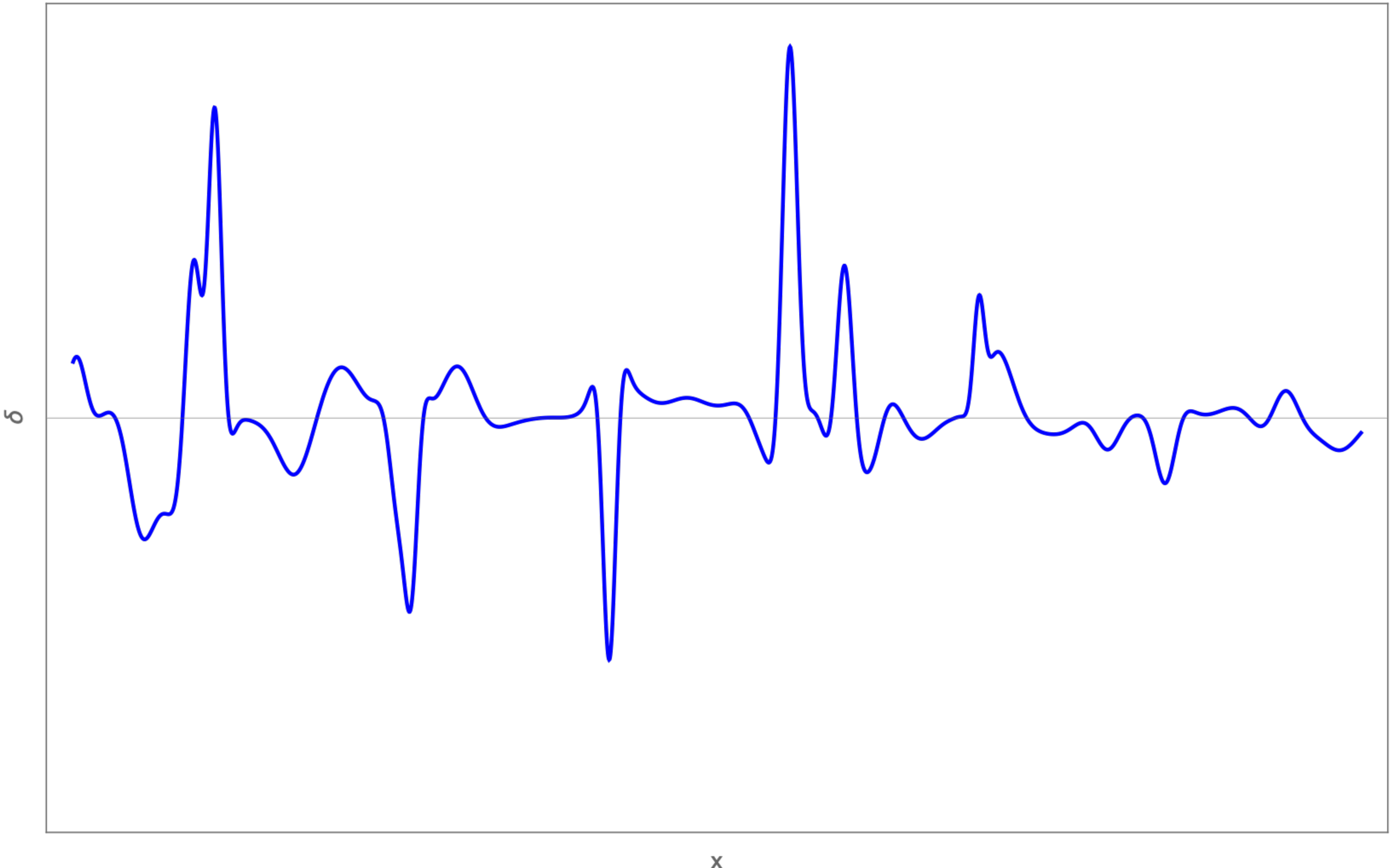
$$\mu = \nabla^2\zeta$$

- But what about non-linearities?



Density contrast and the problem with small scales

Can you spot which perturbation will form a PBH?



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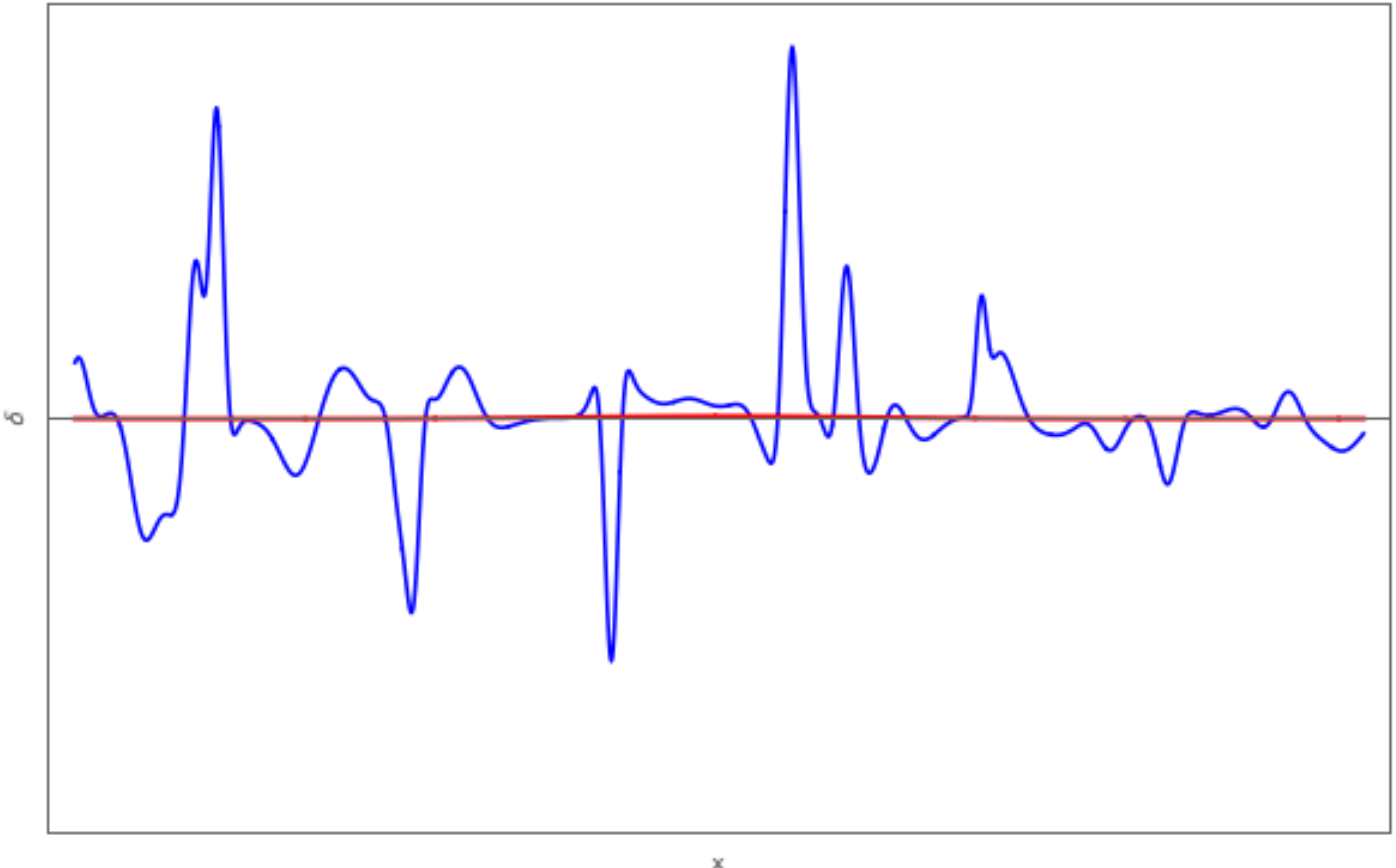
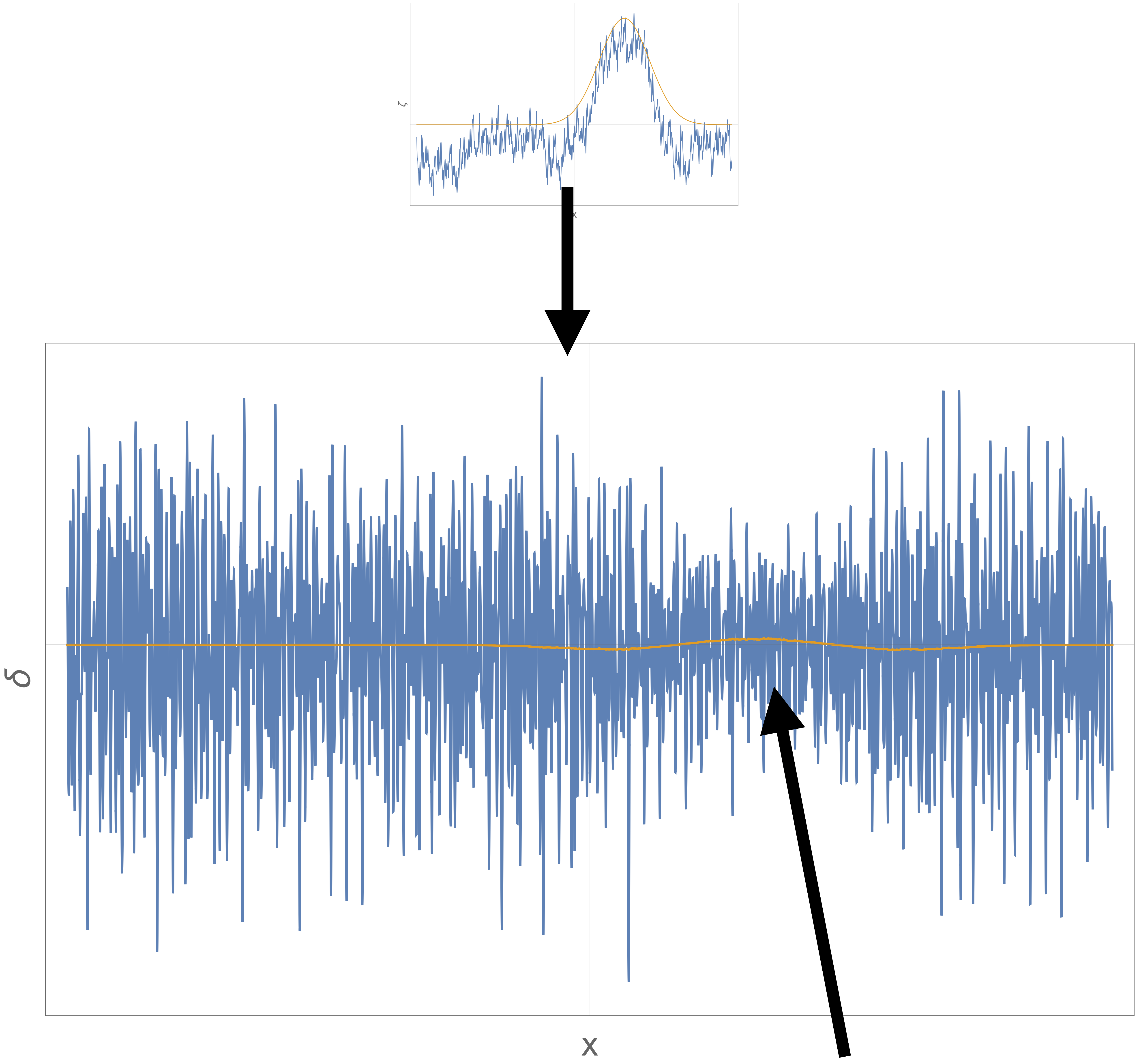


The density contrast

Non-linear

$$\delta = -\frac{2(1+\omega)}{5+3\omega}\frac{1}{(aH)^2}e^{-2\zeta}\left(\nabla^2\zeta + \frac{1}{2}\left(\bar{\nabla}\zeta\right)^2\right)$$

- Still dominated by small-scale modes
- Contamination by background perturbations
- Doesn't increase monotonically with ζ
- Complicated statistics (which are often misleading anyway)



Whats happening here?

The compaction C

The time-dependance of these cancels out

$$C = 2\frac{\delta M}{R} = R^2 H^2 \int \mathrm{d}^3\mathbf{y}\,\delta(\mathbf{x}-\mathbf{y})W_{\mathrm{TH}}(\mathbf{y})$$

- Assuming spherical symmetry, and in the comoving gauge:

$$C = -\frac{2}{a e^{\zeta(r)} r}\frac{3H^2}{8\pi}\int_0^r \mathrm{d}\left(a e^{\zeta(r)} r\right)\left[4\pi\left(a e^{\zeta(r)} r\right)^2\right] \times \frac{4}{9}\left(\frac{1}{aH}\right)^2 e^{-2\zeta(r)}\left(\zeta''(r) + \frac{2}{r}\zeta'(r) + \frac{1}{2}(\zeta'(\mathbf{r}))^2\right)$$

- This has a simple solution

$$C = -\frac{2}{3} r\zeta'(r)\big(2 + r\zeta'(r)\big) = C_1 - \frac{3}{8}C_1^2$$

- NB. $C_1 = -\frac{2}{3} r\zeta'(r)$ is the expression we would obtain if we just used the linear expressions
    - Therefore, we can typically use the same tools as for linear calculations with some modifications
- Not very efficient at removing small-scale modes (see later)
- Spherical symmetry may not be justified

WARNING

- Its tempting to look at $C_1 = -\frac{2}{3}r\zeta'(r)$ and think C_1 is proportional to the first derivative of ζ, and therefore $C(k) \propto k\zeta(k)$
- It is the surface term at radius r, arising from integrating ζ'' over a spherical volume (and assuming spherical symmetry)

$$C(k) \propto k^2 W(k, r)\zeta(k)$$

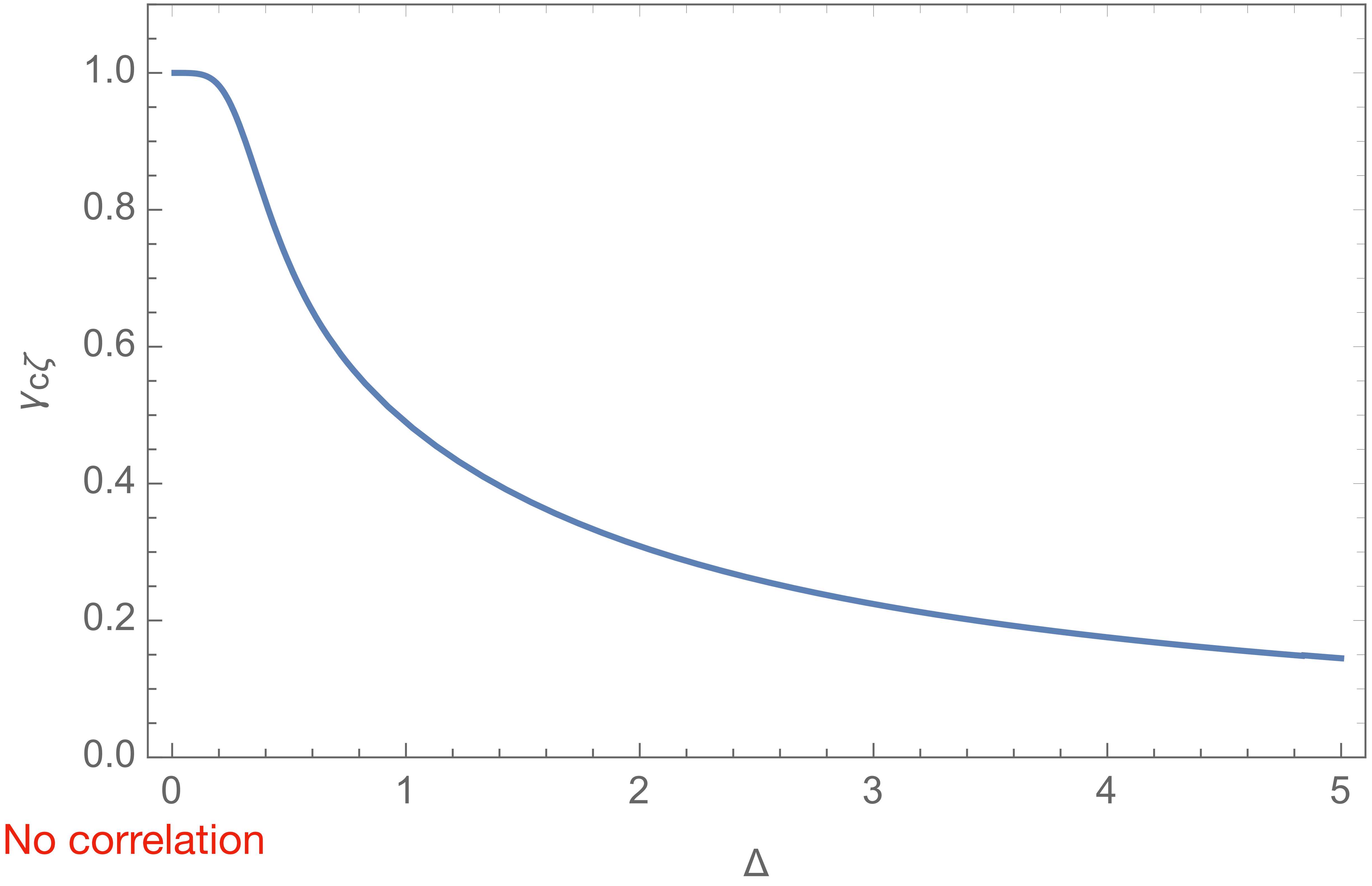
The compaction

Is spherical symmetry justified?

- PBHs are rare in the early universe
  - Large PBH forming perturbations are also rare
  - Large peaks in C are rare
  - The larger (and rarer) a peak is, the more spherical symmetry it has
- Do rare peaks in C also correspond to rare perturbations in ζ?
  - Only for narrow power spectra
  - We can't assume spherical symmetry (for broad power spectra)
  - The quadratic relation on the previous page may not be valid
- What do perturbations look like for broad/narrow power spectra?

$$\mathscr{P}_\zeta \sim \exp\left(-\frac{ln(k)^2}{2\Delta^2}\right)$$

Perfect correlation



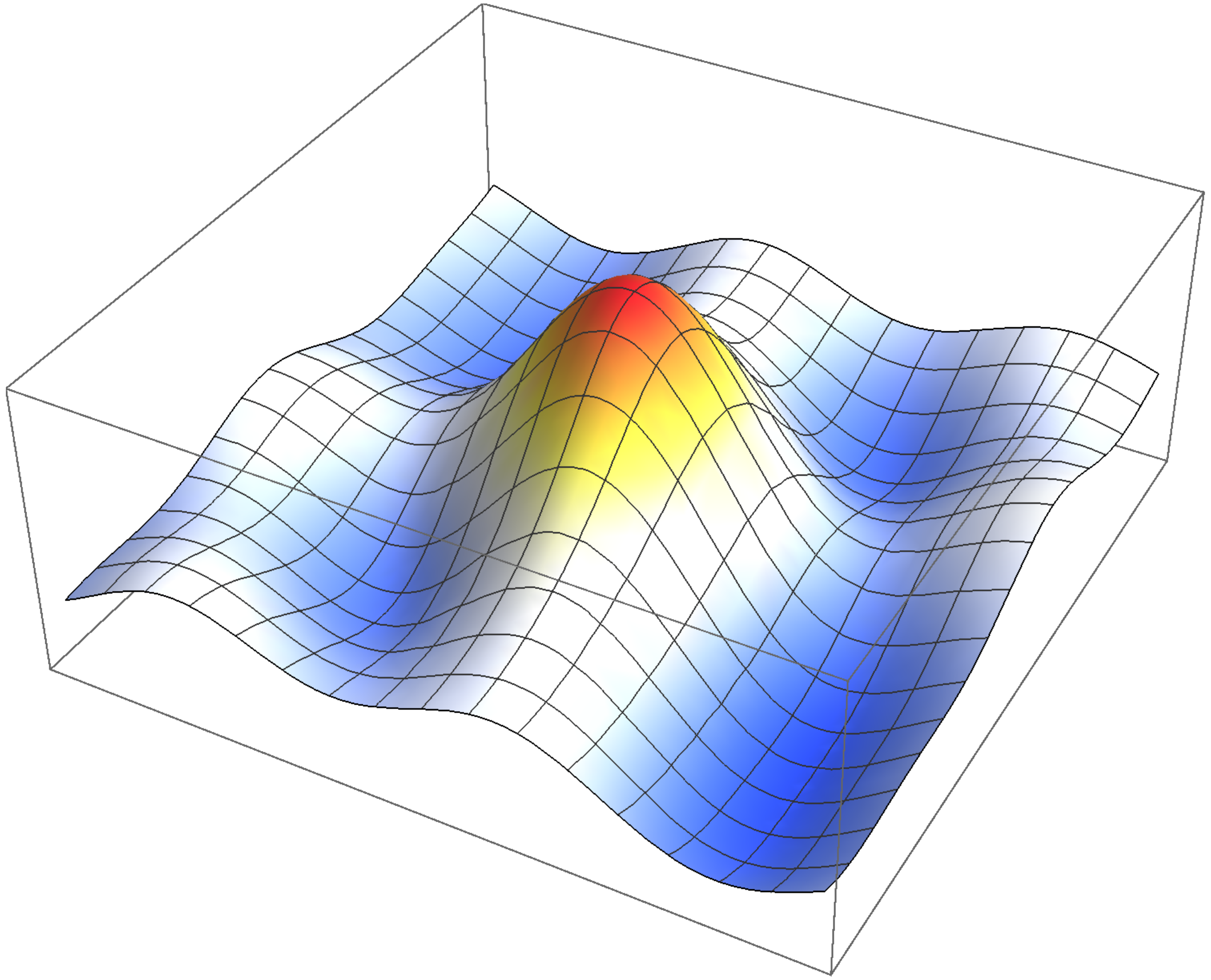
No correlation

SY 2022

Rare perturbations and spherical symmetry

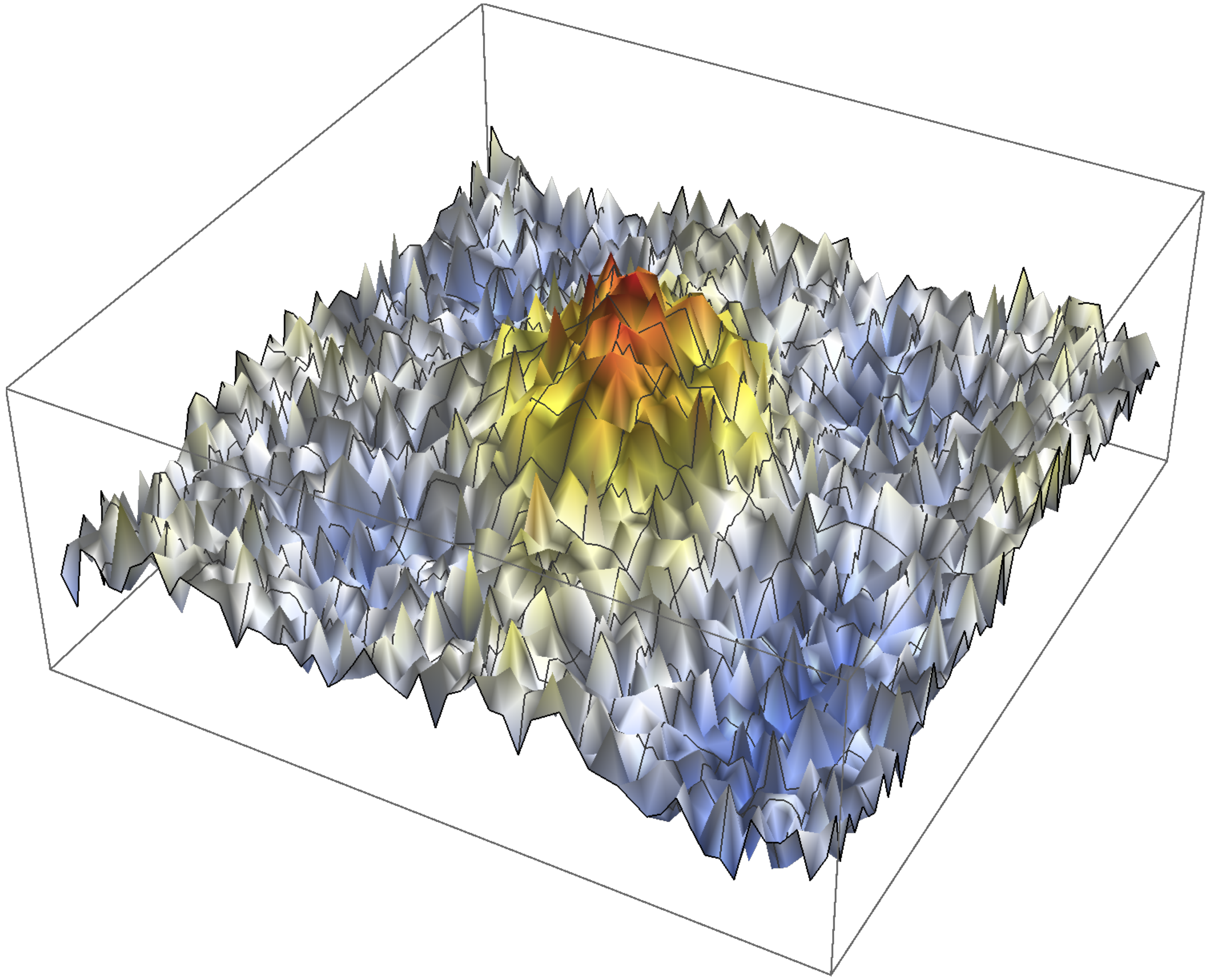
An example of a perturbation in ζ which will create a rare perturbation in C

Narrow power spectrum



Approximately spherically symmetric

Broad power spectrum

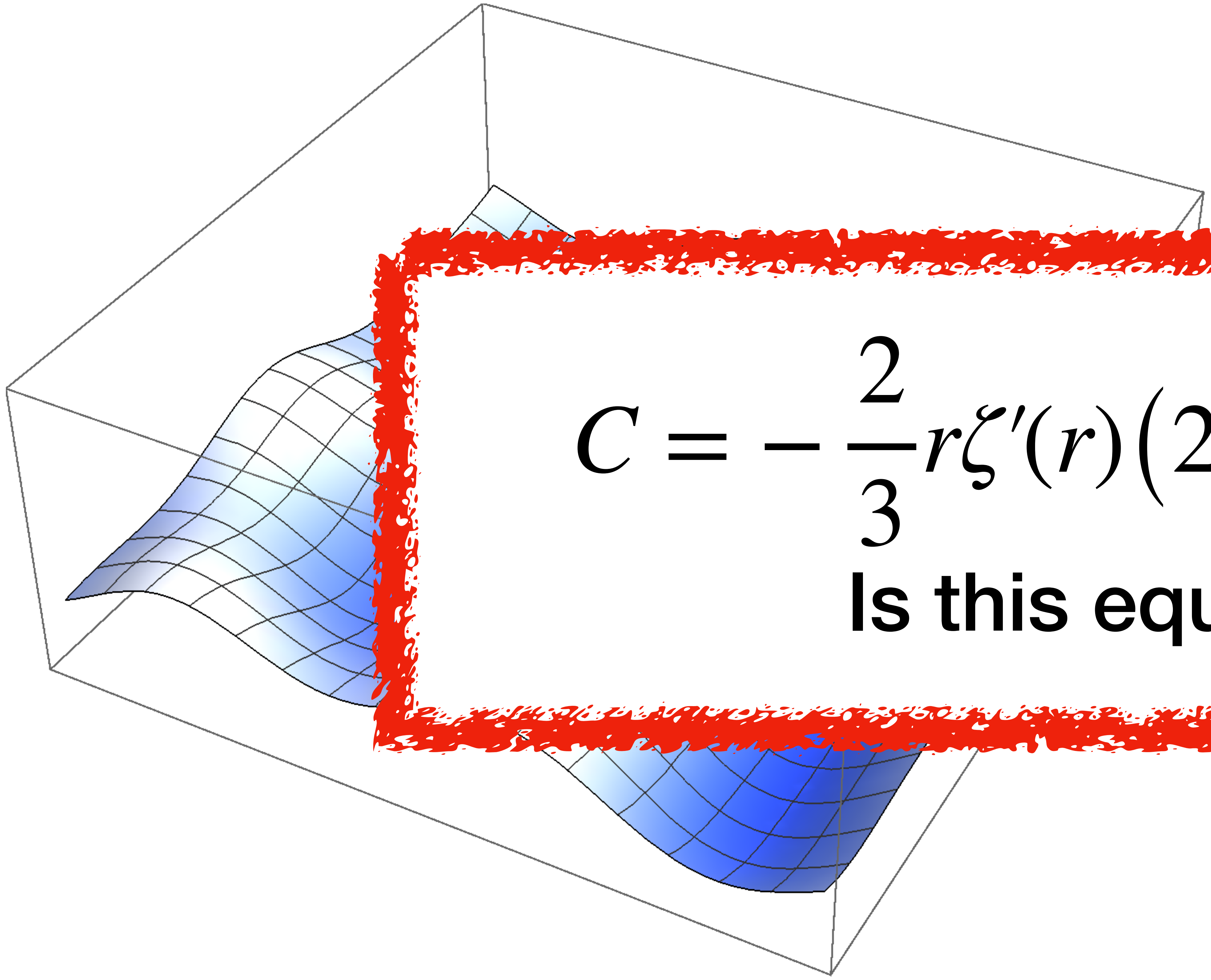


Little spherical symmetry

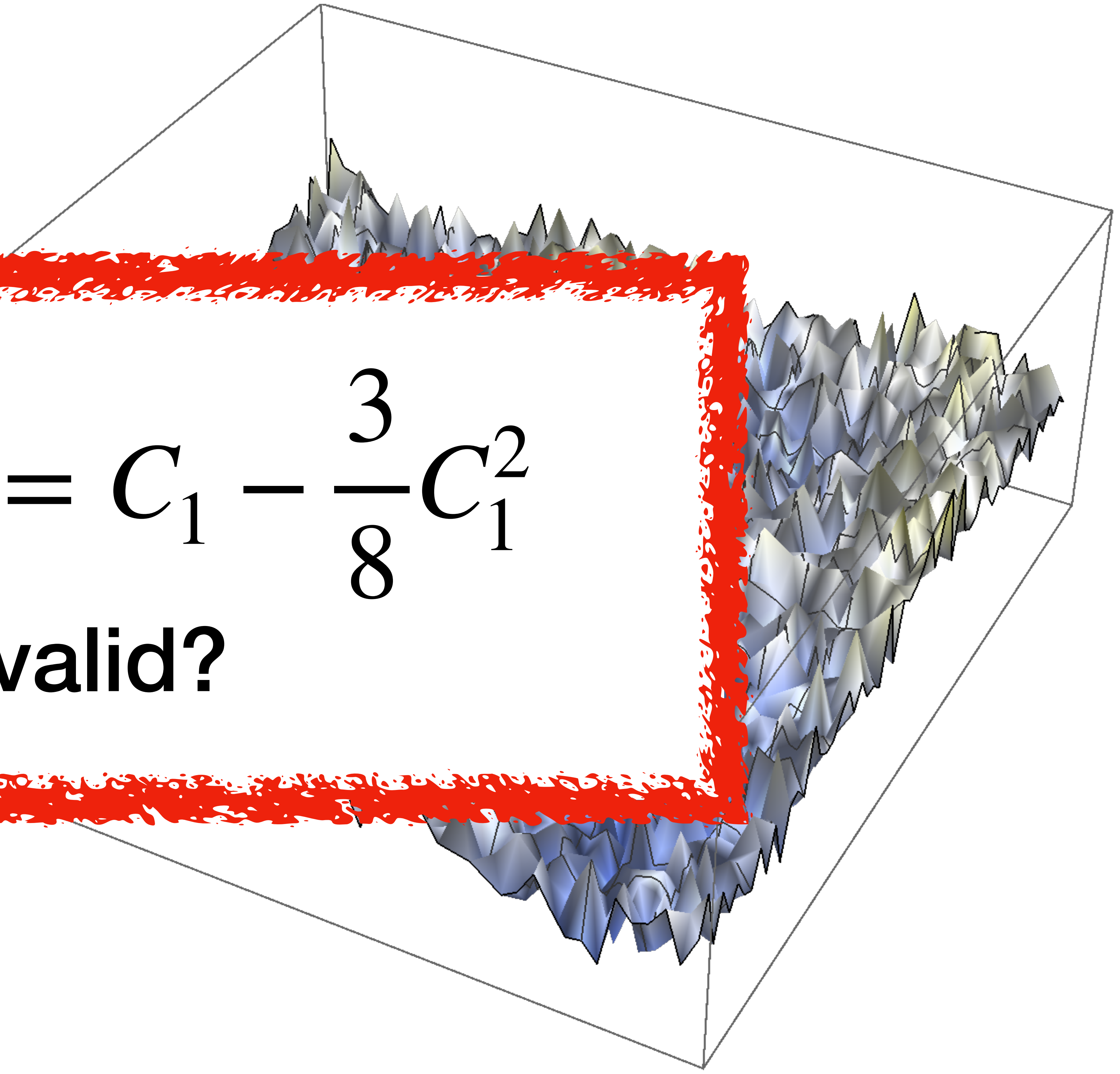
Rare perturbations and spherical symmetry

An example of a perturbation in ζ which will create a rare perturbation in C

Narrow power spectrum



Broad power spectrum



$$C = -\frac{2}{3} r\zeta'(r)\left(2 + r\zeta'(r)\right) = C_1 - \frac{3}{8}C_1^2$$

Is this equation still valid?

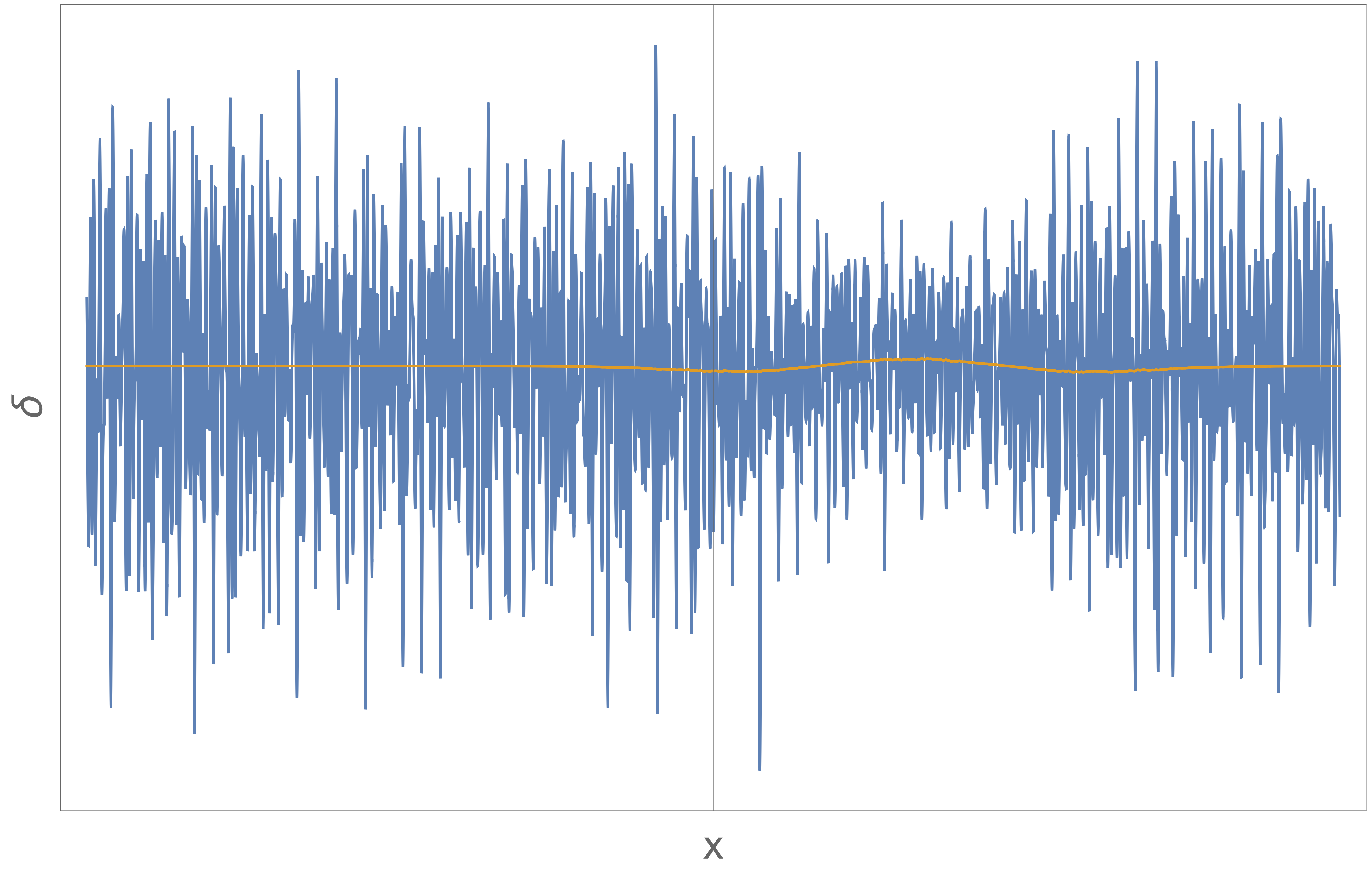
Approximately spherically symmetric

Little spherical symmetry

Smoothing

Smoothing/window functions

How can we get rid of the small-scale modes to see different scale perturbations?



NB. For simplicity, I'll just consider the linear equations from here

Recall that δ depends on the second derivative of ζ

Smoothing functions

Isolating modes of interest

- To isolate specific scale perturbations, smoothing functions are often used to remove small-scale modes

$$\delta_R(\mathbf{x}) = \int \mathrm{d}^3\mathbf{y}\,\delta(\mathbf{y})\,W(\mathbf{x}-\mathbf{y},R)$$

- The idea is average the density within a given volume, and the effect of small-scale perturbations should average out to zero

- Often used functions include top-hat or Gaussian:

$$W_{TH}(\mathbf{x},R) = \frac{3}{4\pi R^3}\Theta_H(R-x),\; W_G(\mathbf{x},R) = \frac{1}{(2\pi R^2)^{3/2}}\exp\left(-\frac{x^2}{2R^2}\right)$$

NB. We can also consider the volume-averaged compaction - the “universal threshold” Escriva et al 2019

- There is *a priori* no reason to think one is better than the other (ongoing work with IM)*
  - Both have advantages
  - Both have disadvantages

* although the top-hat function arises naturally when considering the volume averaged density

Which parameter is best?

δ_R **vs** C **vs** $\mu = \nabla^2 \zeta$

- After smoothing, the equations for each look very similar
- (Here in Fourier space)

$$\delta_R(k) = f(\omega)\epsilon^2(t)W(k,R)k^2\zeta$$
$$C(k) = f(\omega)W(k,R)k^2\zeta$$
$$\mu = W(k,R)k^2\zeta$$

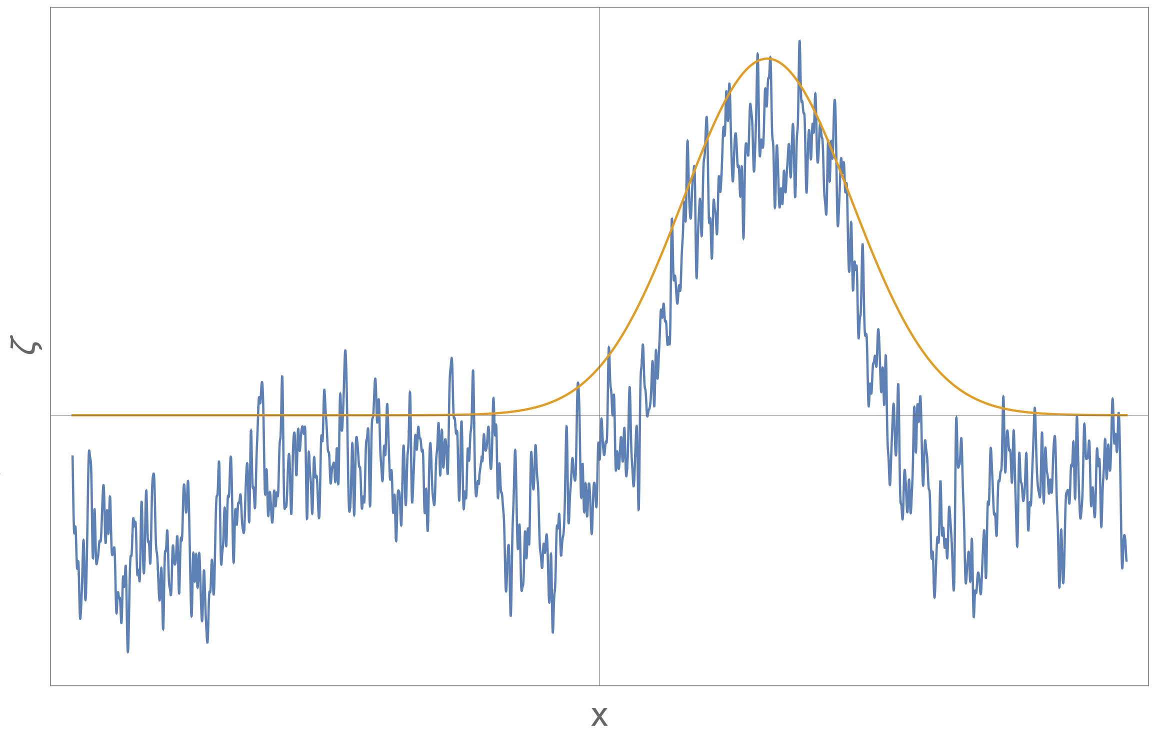
NB. The standard definition of the compaction corresponds to a top-hat smoothing function, but we can choose a different function if desired

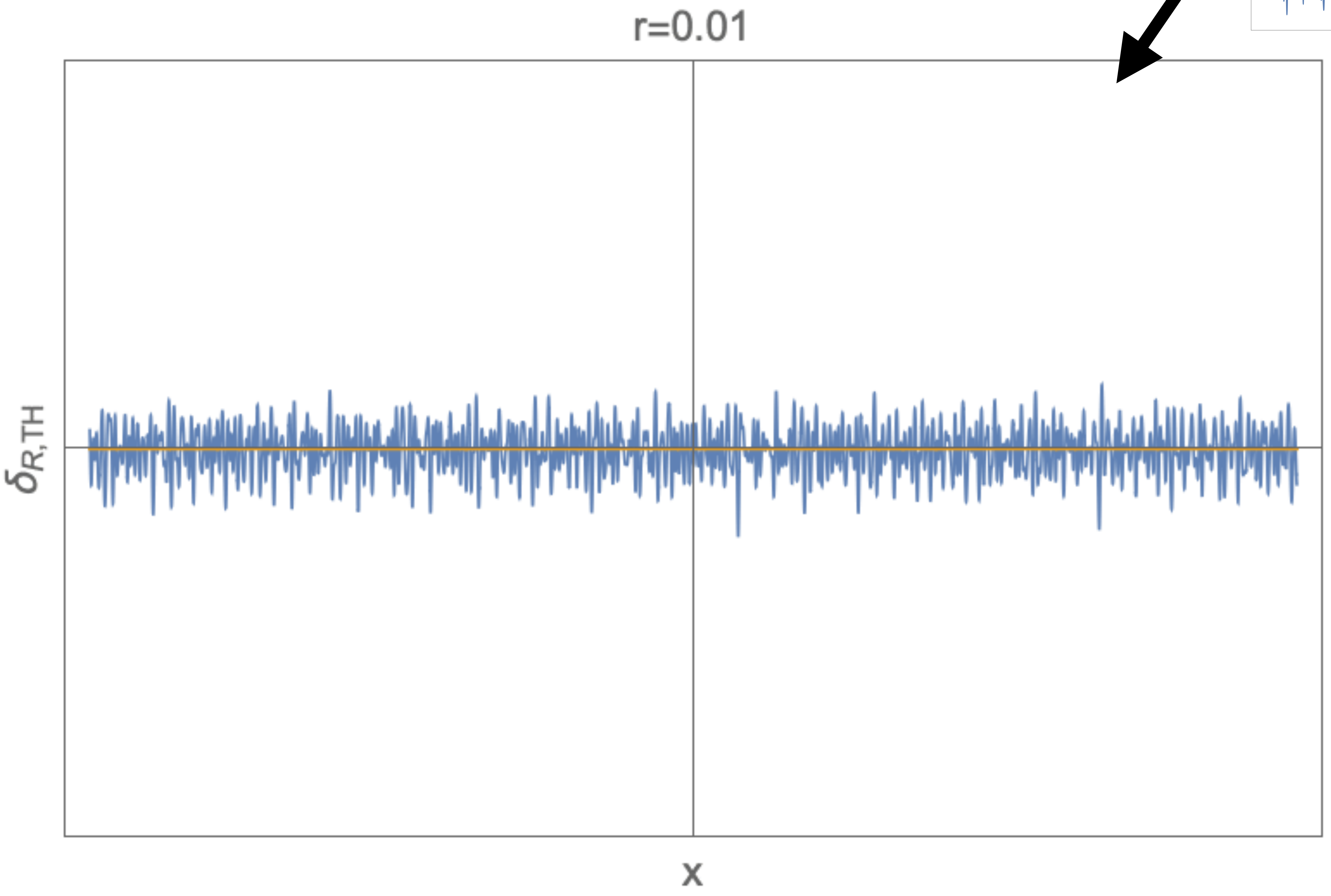
- Large-scale modes are all removed by the k^2 term, whilst small-scale modes are removed by $W(k,R)$
- We could equivalently also absorb k^2 into the definition of our window function and work "directly" with ζ:

$$\zeta_R = \tilde{W}(k,R)\zeta(k)$$

- Generally, then, our choice of parameter boils down to a choice of window function (plus added complications arising from non-linearities)

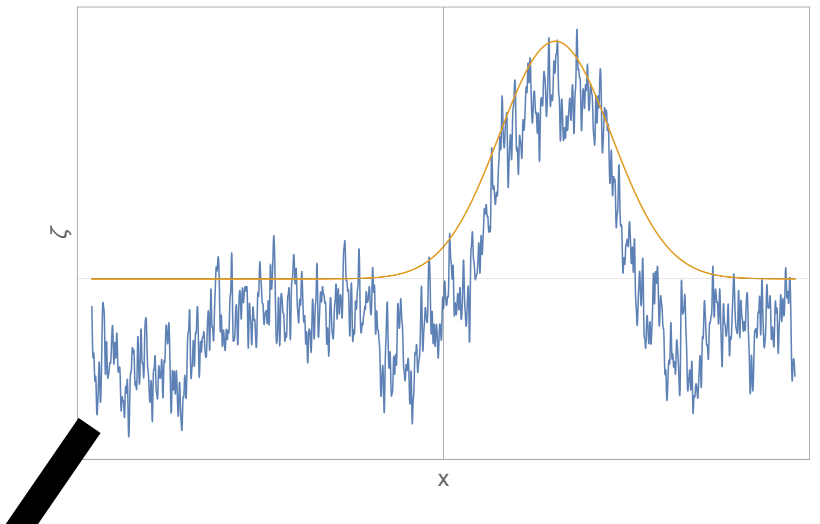
Top-hat smoothing

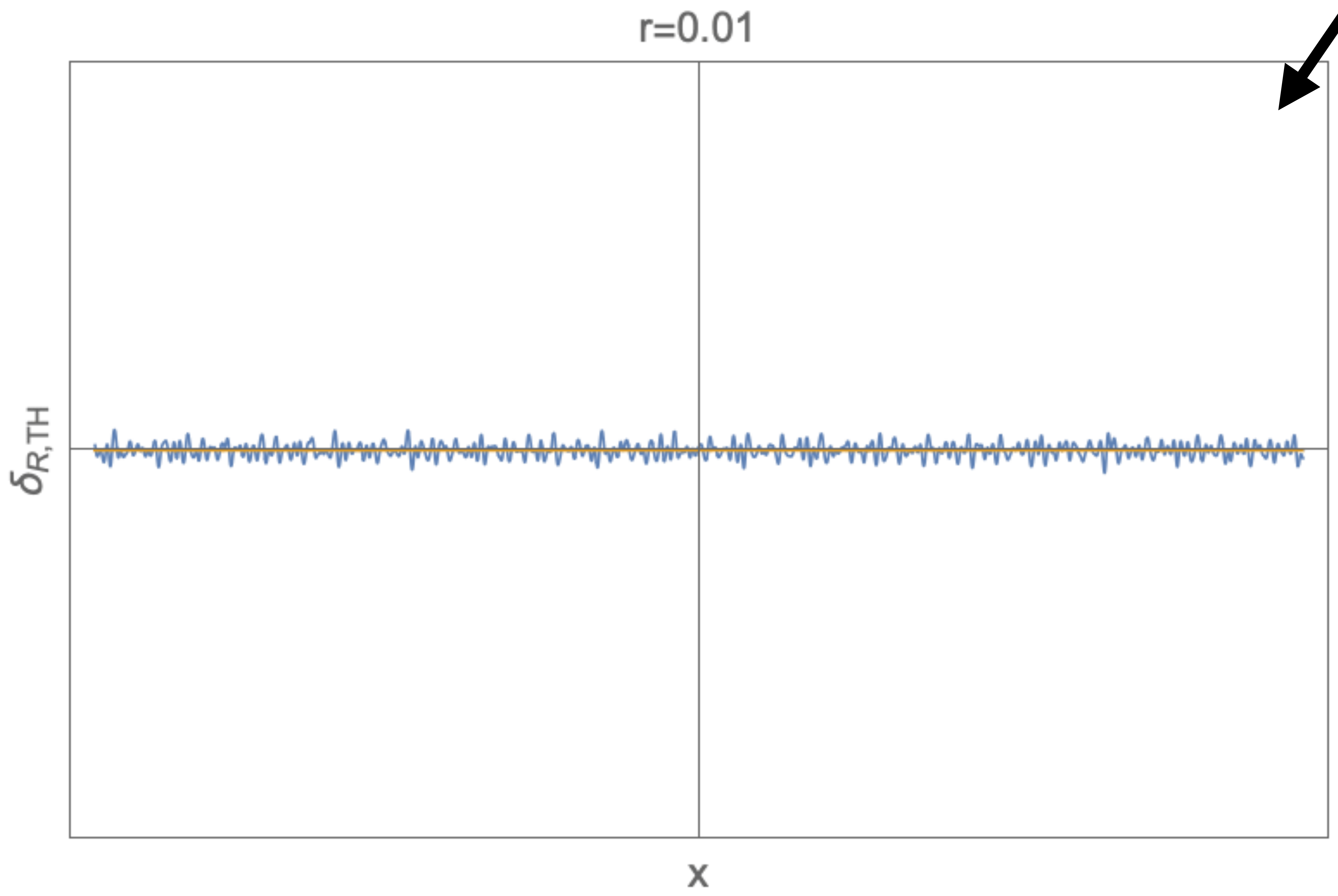




Top-hat smoothing gives us nice relations when we consider non-linearities, but isn't very efficient

Gaussian smoothing



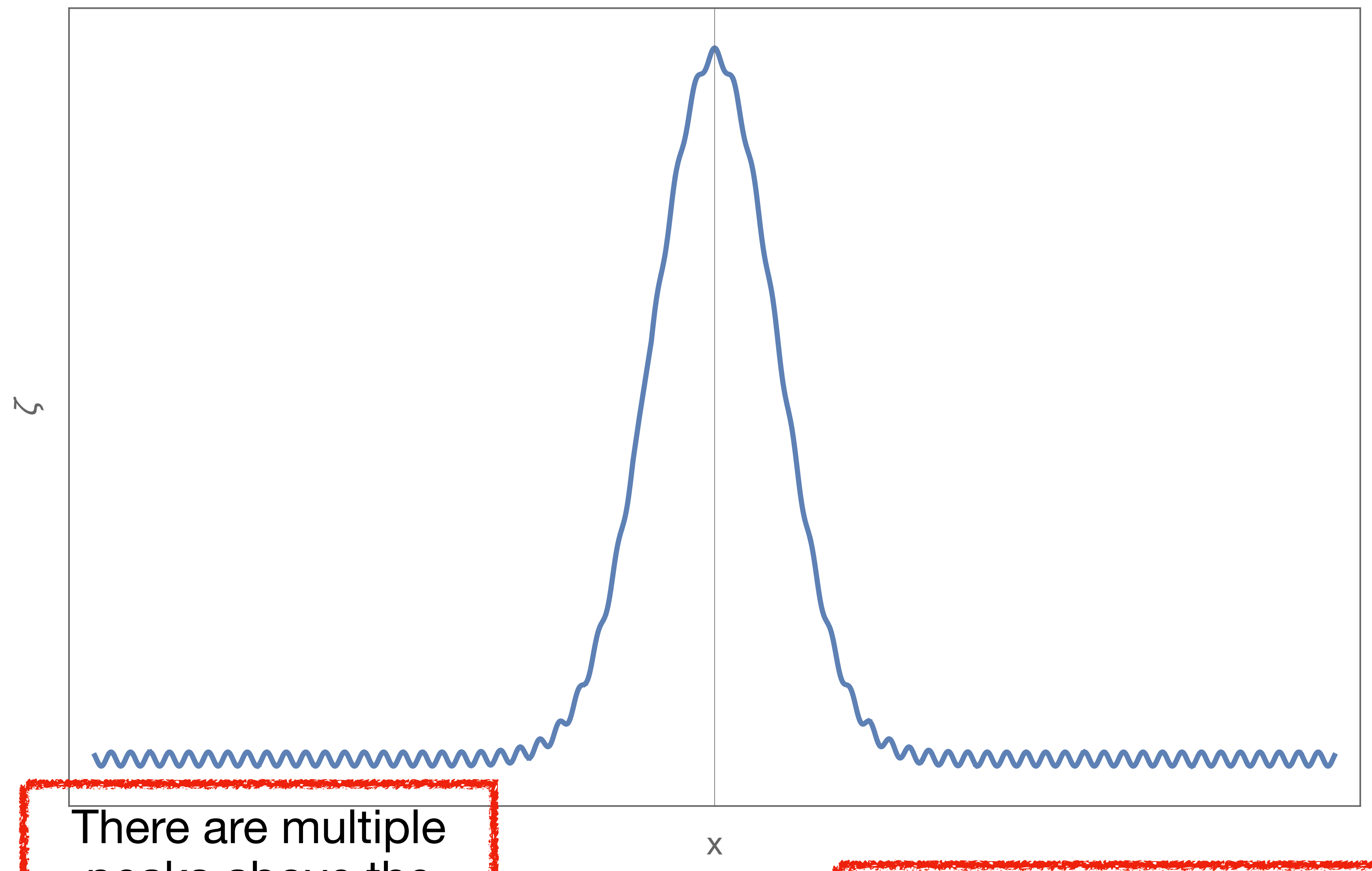


Gassing smoothing is more efficient but non-linearities are a problem

NB. I’ve assumed spherical symmetry here, but there is no reason to assume this for the small-scale perturbation

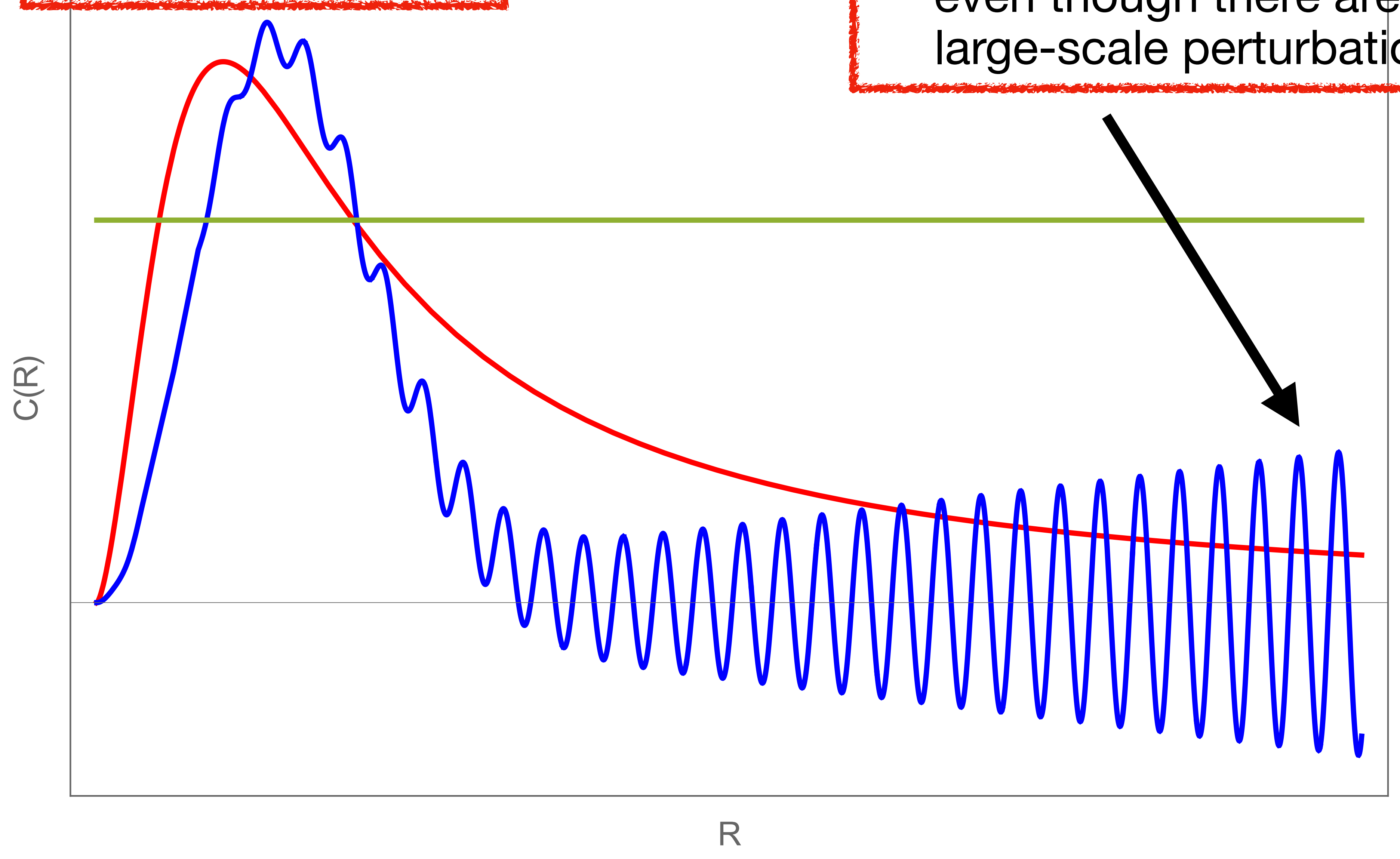
Gaussian vs top-hat

- Generally, the Gaussian window is preferable for calculations
- The top-hat window gives simple analytic relations when non-linearities are considered
- But, just because a window is more convenient, doesn't mean that it is correct
- Which window function is **correct**? (If any)
- How does smoothing affect the abundance calculation?



There are multiple peaks above the threshold for the top-hat

The compaction blows up at large scales for the top-hat, even though there are no large-scale perturbations



Calculating the variance

- When calculating the abundance, we mostly care about the variance (of whichever variable we're using)

$$\sigma^2 = \langle X^2 \rangle = \int_0^\infty \frac{\mathrm{d}k}{k} \mathscr{P}_X(t,k) = \int_0^\infty \frac{\mathrm{d}k}{k} \left(\frac{k}{aH}\right)^2 \mathscr{P}_\zeta(k) W^2(k,R) T^2(k,R_H)$$

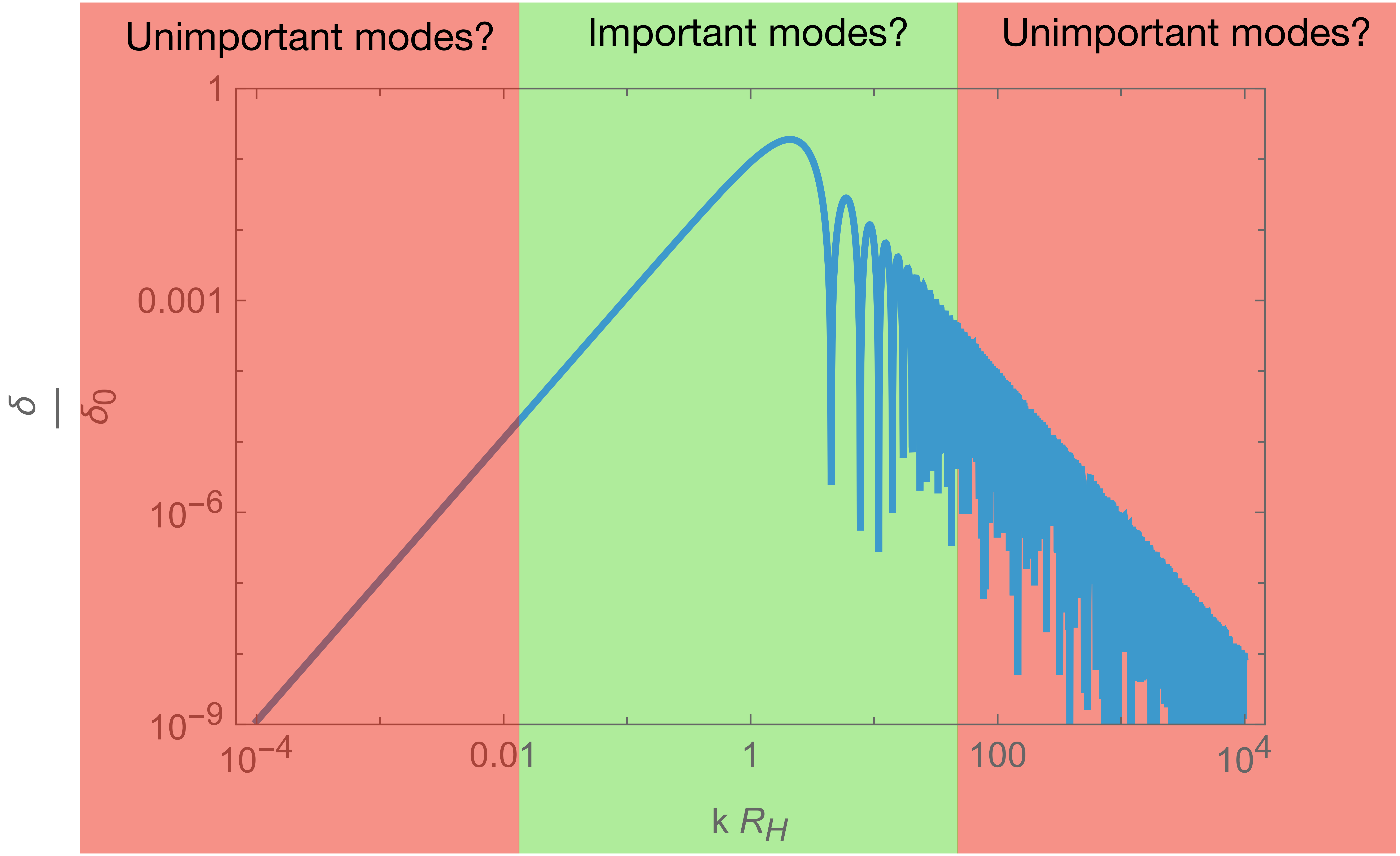
- The transfer function $T(t, R_H)$ describes the time-evolution of the power spectrum, and is given at linear order by

$$T(k,R_H) = 3\frac{\sin\left(3^{-1/2}kR_H\right) + \left(3^{-1/2}kR_H\right)\cos\left(3^{-1/2}kR_H\right)}{\left(3^{-1/2}kR_H\right)^3}$$

  - For super-horizon modes, $T(k, R_H) = 1$
  - But PBH formation happens at horizon entry - so what happens to sub-horizon modes will affect PBH formation to some extent
- What we really want is one function which will tell us how important each different mode is, accounting for non-linearities in the density and the complex evolution at horizon crossing
- How should we choose the right window function? → we need to run some simulations

Choosing an appropriate window

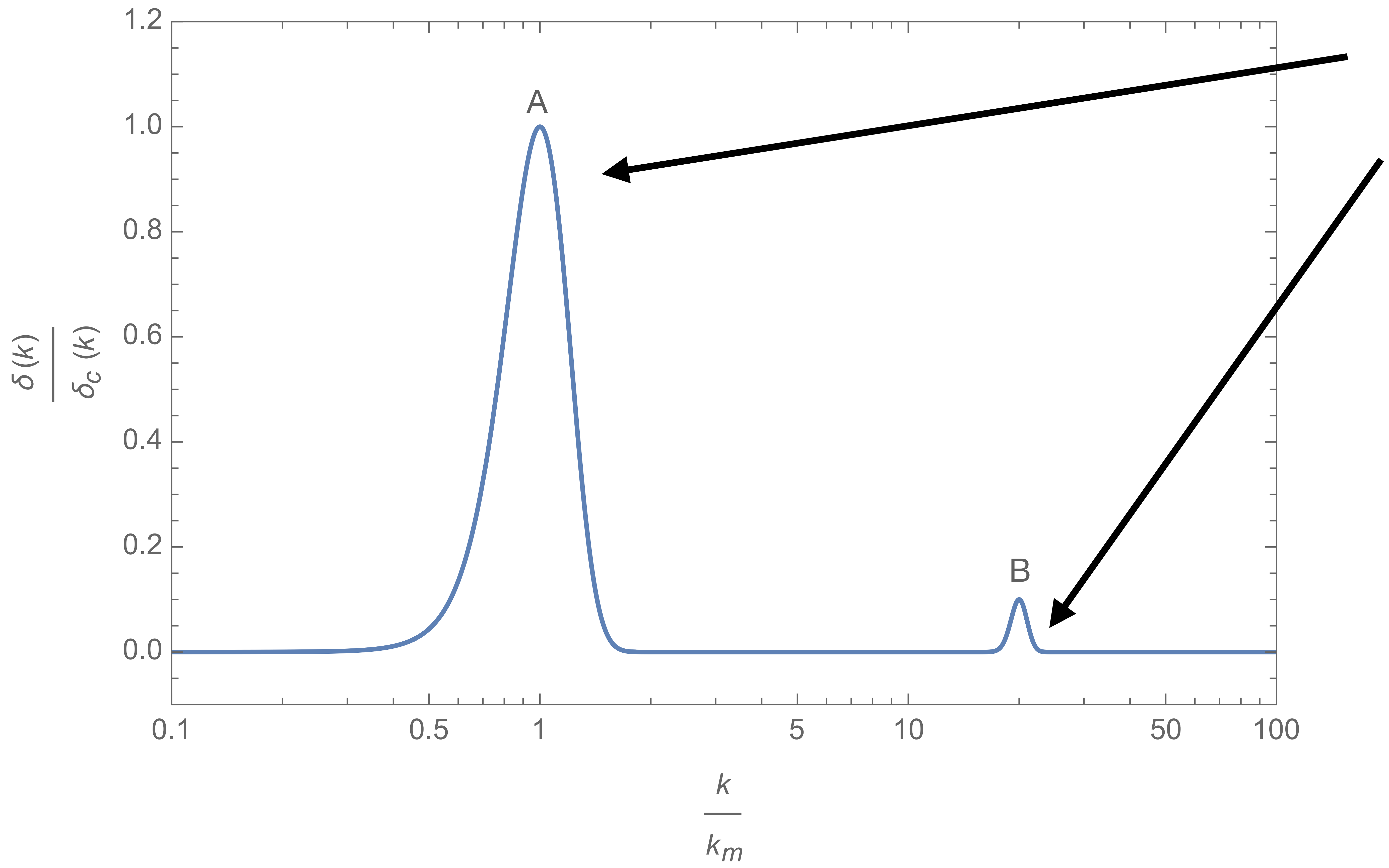
What’s the “correct” window?



An appropriate window functions will quantity how important different modes are to PBH formation

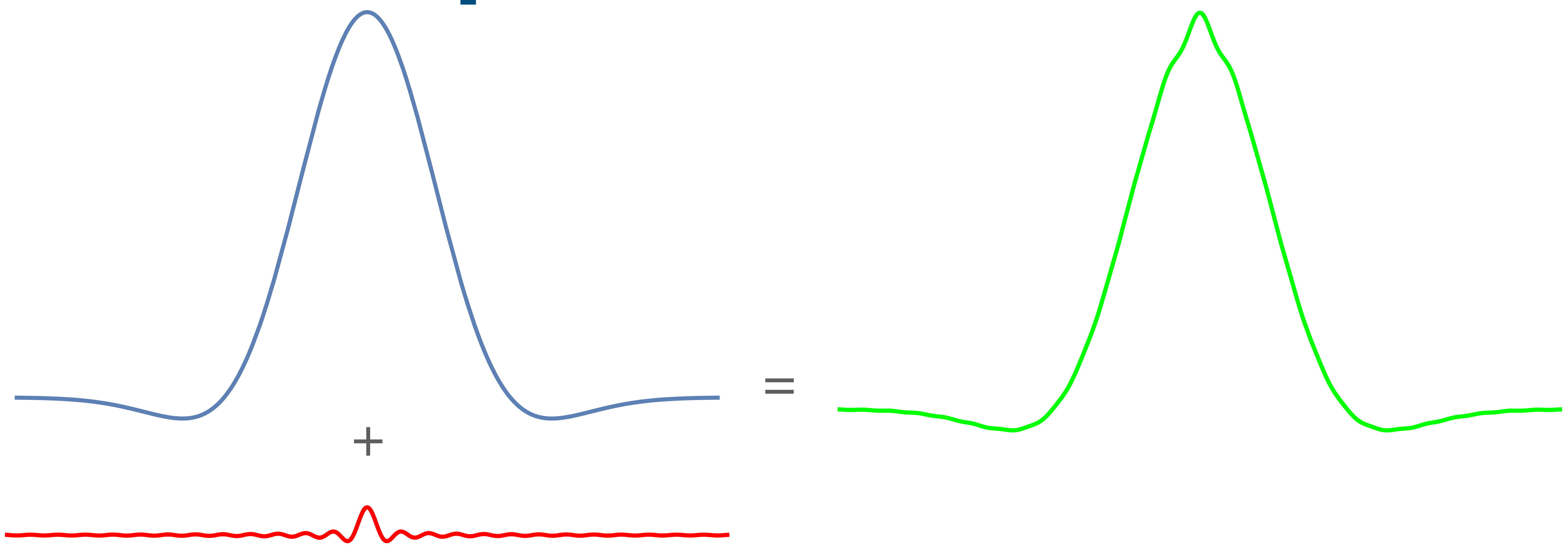
What does a window tell us?

Lots of problems but no solutions (so far)



- Lets consider a PBH forming at this scale
- How important is e.g. this scale?
- How important is e.g the height of A compared to the height of B?
- e.g. If $B = 0.1$, can A be 0.1 smaller and still form a PBH? Or can A only be 0.01 smaller?
- The window assigns a weight to each Fourier mode
- To calculate the correct window, we'll need some simulations

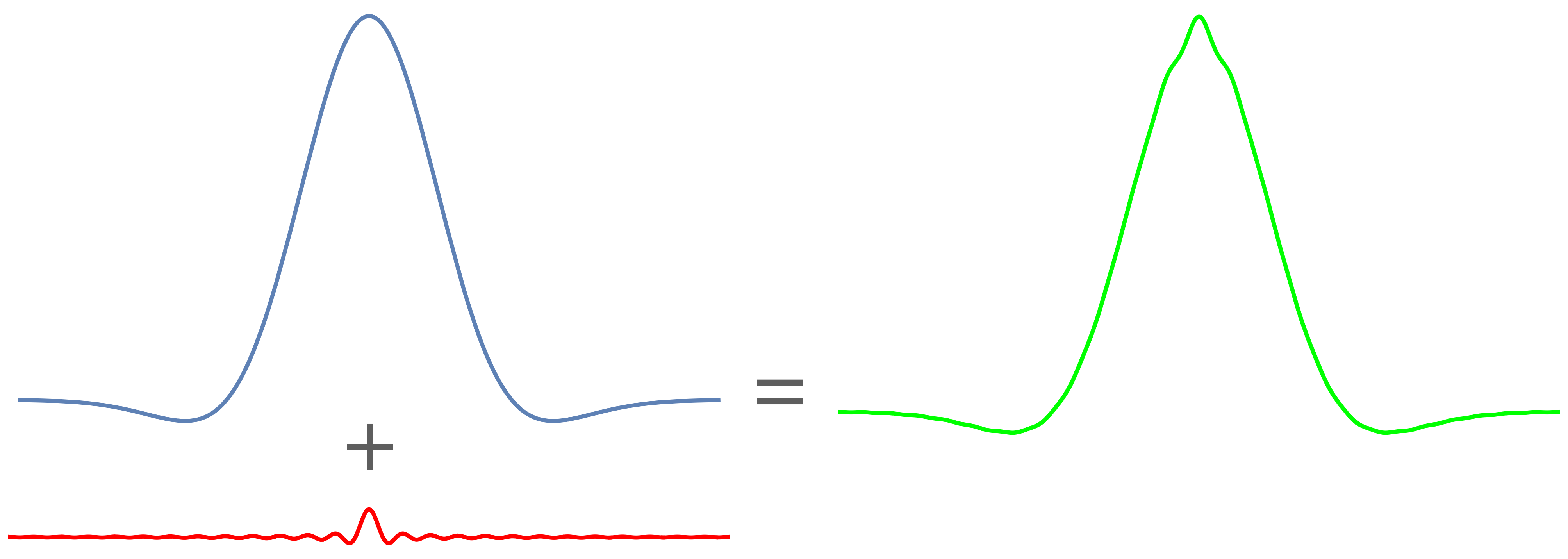
Simulation set-up



- Basic idea:
  1. Simulate PBH formation with a given density profile*
  2. Perturb the profile with a perturbation of varying scale, and see how this affects PBH formation
  3. Quantify the effect of different scales
- We have started by looking at density profiles, but will consider perturbations starting from ζ

*I'll call this the "background" profile, though in reality it is, itself, a perturbation to the uniform background

Simulation set-up



- Background profile: $\delta_{BG}(r) = A\left(1 - \frac{2}{3}\frac{r^2}{r_m^2}\right)\exp\left(-\frac{r^2}{r_m^2}\right)$
- Perturbation to the profile: $\delta_p(r) = Bk^2\frac{\sin(kr)}{kr}$
- Run simulations to find the critical value A_c for given $\{B, k\}$
  - Giving a fiducial value $\bar{A}_c$ when $B = 0$
- Assume that there is a "correct" window $\tilde{W}(k, R)$ which makes the critical value constant (i.e. it is "universal"):

$$\overset{\propto \bar{A}}{\bar{C}_{w,c}} = \overset{\propto A}{C_{BG,w}} + \overset{\propto Bk^2}{C_{p,w}} = \text{constant}$$

$$\downarrow$$

$$c_1\bar{A}_c = c_1 A_c + c_2 Bk^2\tilde{W}(k, R)$$

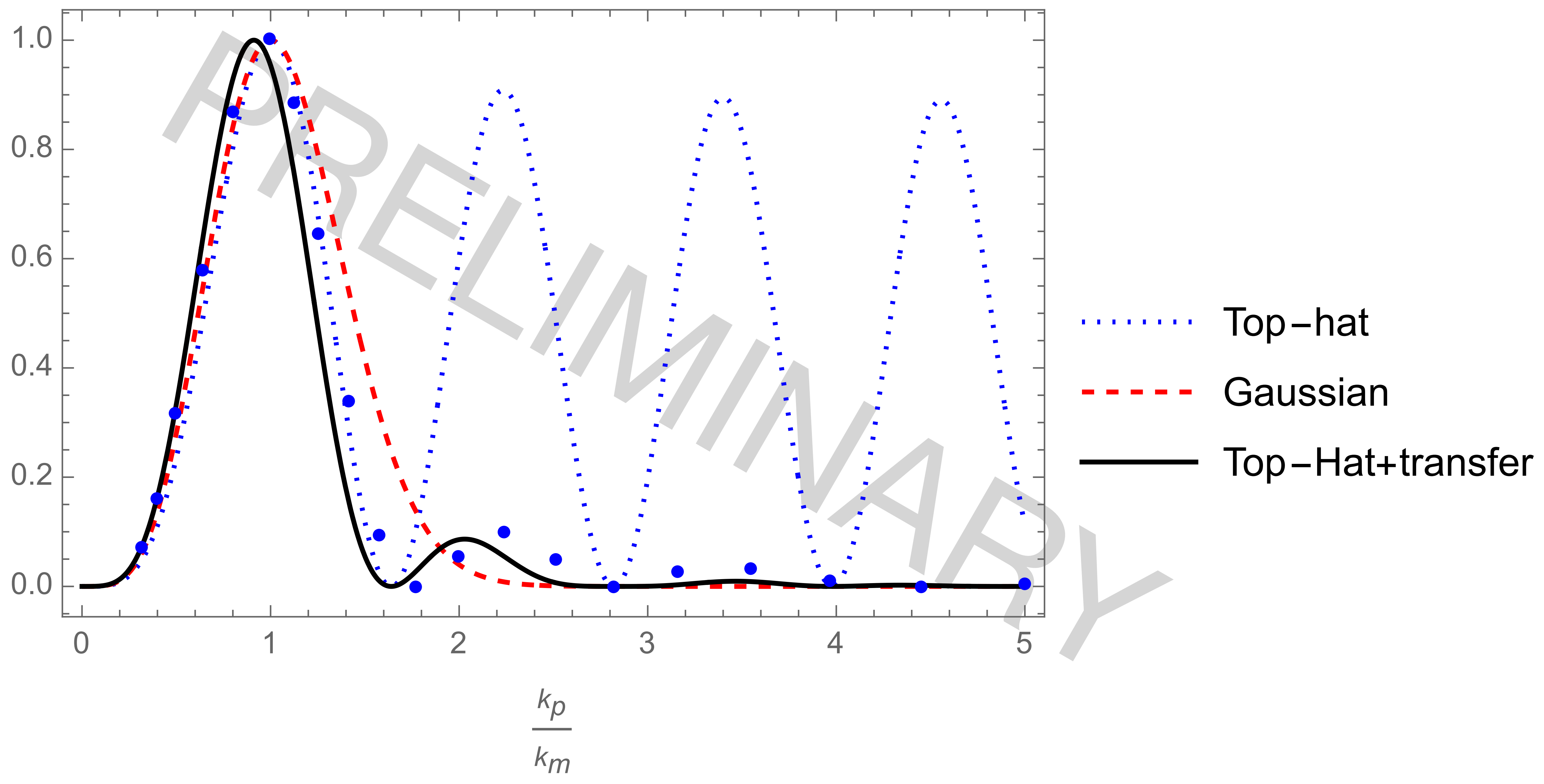
(Preliminary) results from simulations

Using Albert Escriva’s code: [1907.13065]

- Expected: damped oscillatory behaviour
- **Need more data and checks, but results coming soon**
- Moderately well fit by

$$k^2\tilde{W}(k,R) \propto k^2\tilde{W}_{TH}(k,R)T(k,R)$$

- Linear transfer function seems to (surprisingly?) be a good fit



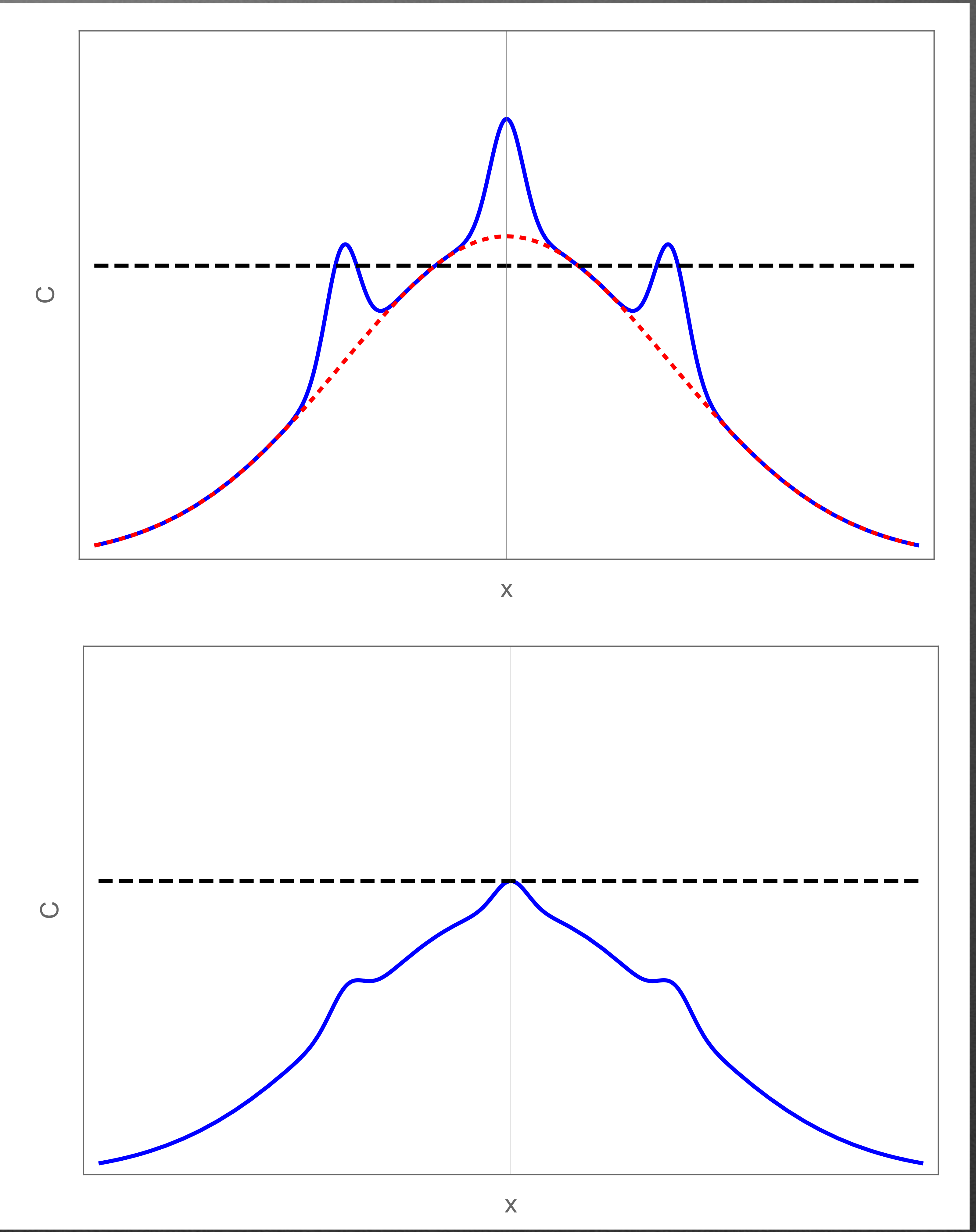
Conclusions

1. There are several parameters which can be used to calculate the abundance
    - They are (mostly) equivalent.
    - (Don’t use ζ without extreme caution)
2. Ongoing work to determine the appropriate window function
    - More work needed still
    - It looks like top-hat+transfer function works well
    - Also need to determine how this will affect the calculation of the abundance, which is non-trivial
        - (e.g. The standard critical value used doesn’t include the effect of the transfer function)

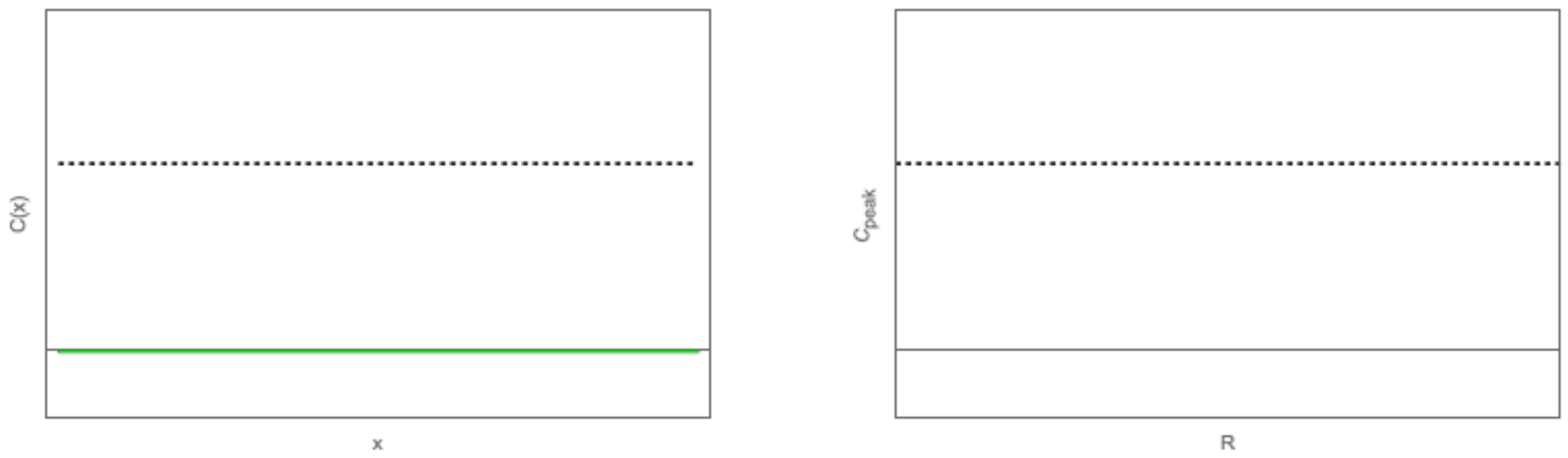
- Don’t forget to email me about the monthly meeting! sam.young@sussex.ac.uk

Appendix: The cloud-in-cloud problem and the excursion set

- A common problem faced in LSS is that small haloes may be located inside larger haloes, resulting in double counting
  - The small haloes are 'swallowed' by the larger haloes
- We need to make sure we are looking at the correct scale
- This is generally solved using the excursion set - which finds the largest smoothing scale at which a perturbation has the threshold value
- Top plot: perturbations smoothed on a small scale. It looks like 3 small compact objects will form
- Bottom plot: smoothed on a larger scale, we instead correctly find 1 large compact object

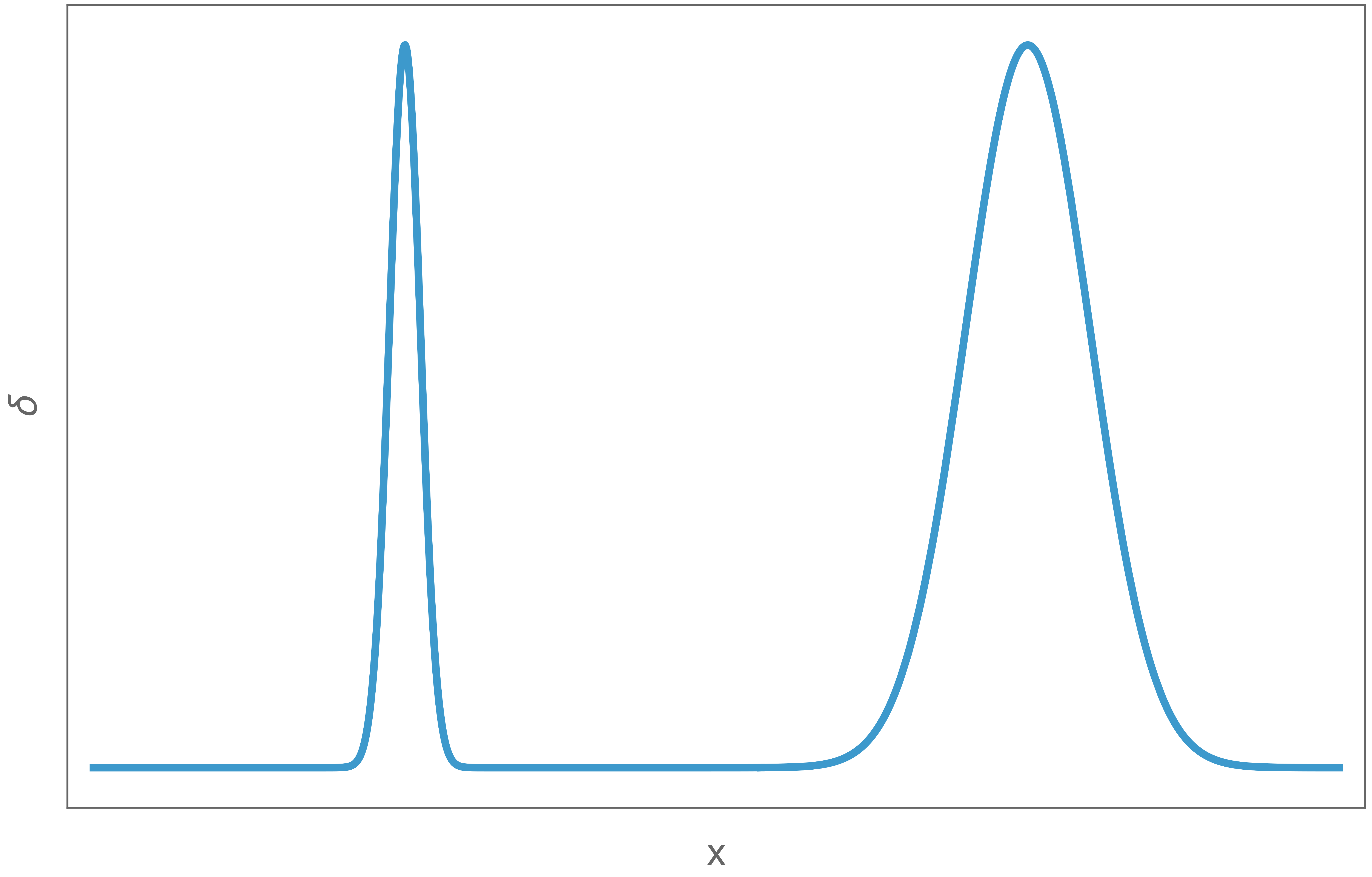


Appendix: The excursion set and PBHs II



- The excursion set approach doesn't work
- All 3 perturbations have $C = C_c$ at the same R, but each forms a different mass PBH
- Simulations define the scale of a perturbation as the smoothing scale at which the compaction peaks
  - We need to include this in the calculation of the abundance

Appendix: defining perturbation scale



Primordial black holes mass

The critical scaling relationship

- As the amplitude of the initial perturbation is increased, the PBH mass is found to increase

$$M_{\mathrm{PBH}} \propto (\delta - \delta_c)^\gamma,$$

with $\gamma \sim 0.36$

- As the scale is increased, PBH mass also increases

$$M_{\mathrm{PBH}} \propto R^2 \propto M_H$$

- Combining these gives the critical-scaling law:

$$M_{\mathrm{PBH}} = \mathscr{K} M_H (\delta - \delta_c)^\gamma$$

- This equation will hold regardless of how the scale or amplitude is defined, but will need a different $\mathscr{K}$