Non-Smooth Horizons in Kerr Black Hole Mergers

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Stationary Black Holes

By now, stationary black holes are rather well-understood. We have

- explicit solutions for a number of black holes (Schwarzschild, Kerr, etc.).
- uniqueness theorems (under certain assumptions) suggesting that stationary vacuum black holes can be described with only four parameters: M, a, Q, P.
  e.g. Carter 1971; Robinson 1975; Mazur 1982; reviewed in Chruściel *et al.* 2012
- some mathematically-rigorous understanding of their stability, especially in the case of Schwarzschild, and numerical evidence of stability for Kerr.
  e.g. Dafermos *et al.* 2019, 2021; Klainerman & Szeftel 2023; Press & Teukolsky 1973
- constraints on the topologies that stationary black holes may have on a spacelike hypersurface.
  Hawking 1972, Galloway & Schoen 2005

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We can get useful insights by studying the non-smooth structures that may develop at the horizon. It turns out that this analysis can be done mathematically!

Introduction

1. Generic Non-Smooth Horizons: how to mathematically study non-smooth horizons
    - Assumptions
    - Classification of non-smooth structures
    - Evolution of such structures ('perestroikas')

2. Kerr Merger in the Infinite Mass Ratio Limit: a specific example
    - Methodology
    - Results
    - Comparison with the mathematical predictions

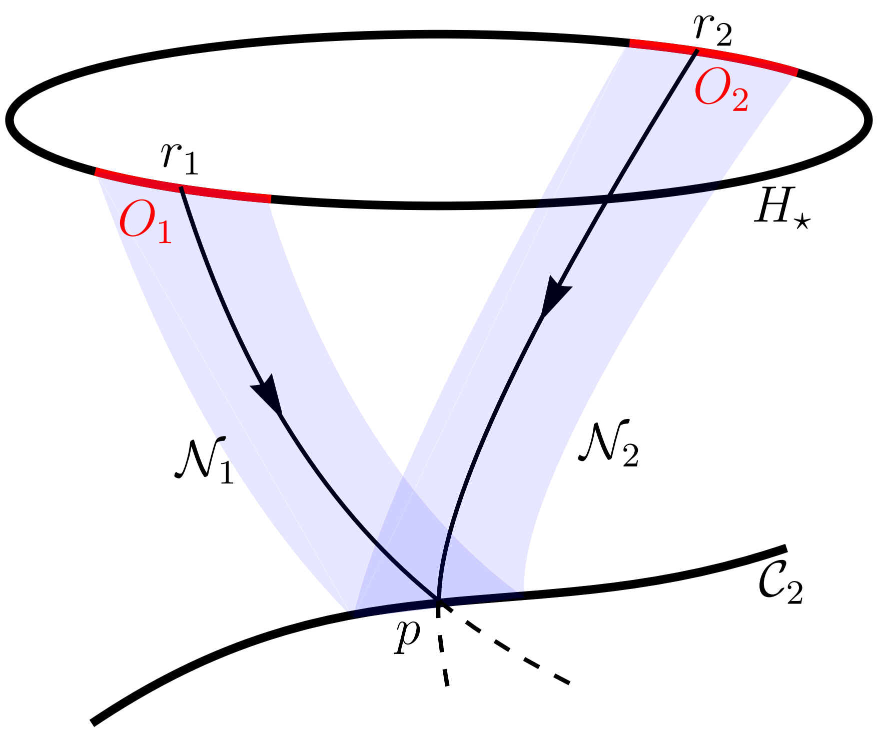
Part I: Generic Non-Smooth Horizons

Definitions and Assumptions

- The event horizon $\mathcal{H}$ is a null hypersurface that is achronal, i.e. no timelike curve can connect any two points of the horizon.
- There exists a future-inextendible null geodesic (generator) through each point $p \in \mathcal{H}$. This implies that once a null geodesic joins onto the horizon, it will stay on it. (For a white hole horizon, we would assume the generators are past-inextendible instead.)
- Spacetime is globally hyperbolic.
- There exists a time $t_\star$ at which the horizon is smooth. This encapsulates the expectation that black holes settle down to a stationary black hole after a sufficiently long time.

Finding Past Endpoints

- Start from our smooth horizon cross-section $H_\star$ at $t_\star$. Smoothness implies that there is exactly one generator through each point on $H_\star$. Beem and Królak 1998



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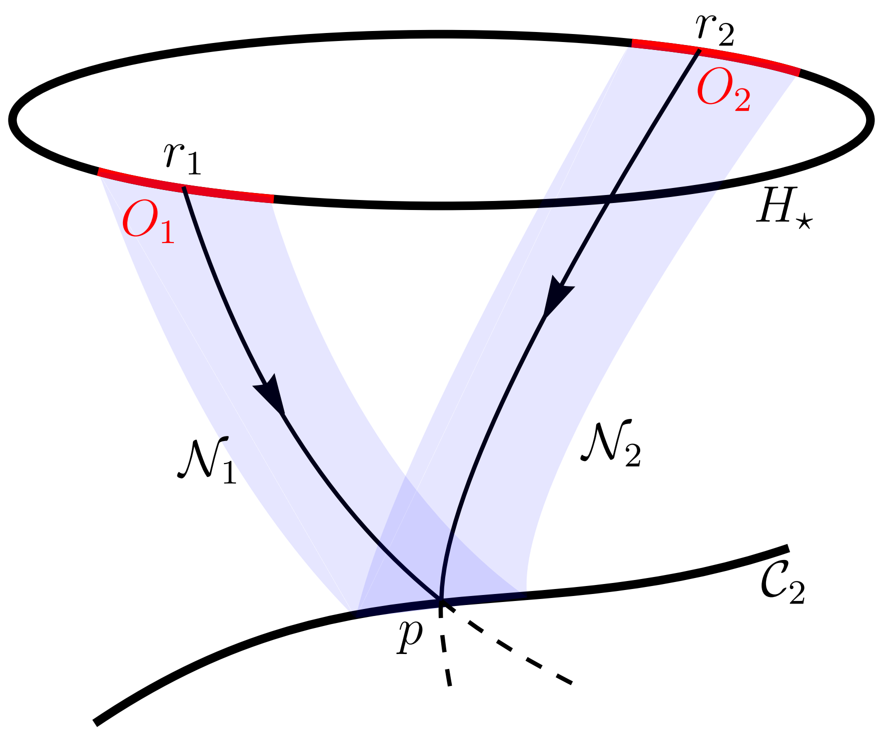

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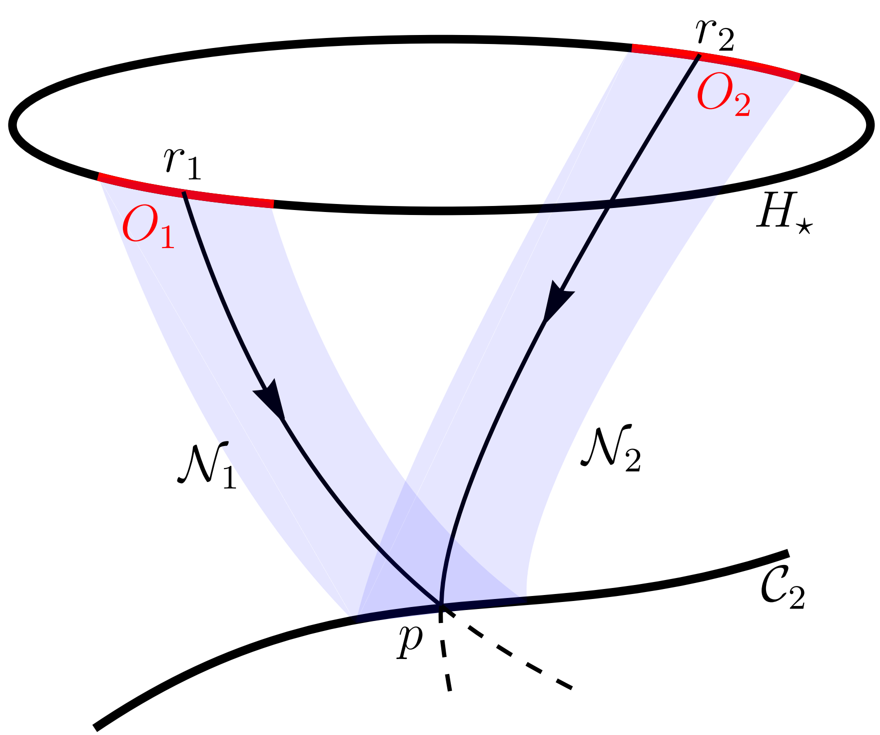

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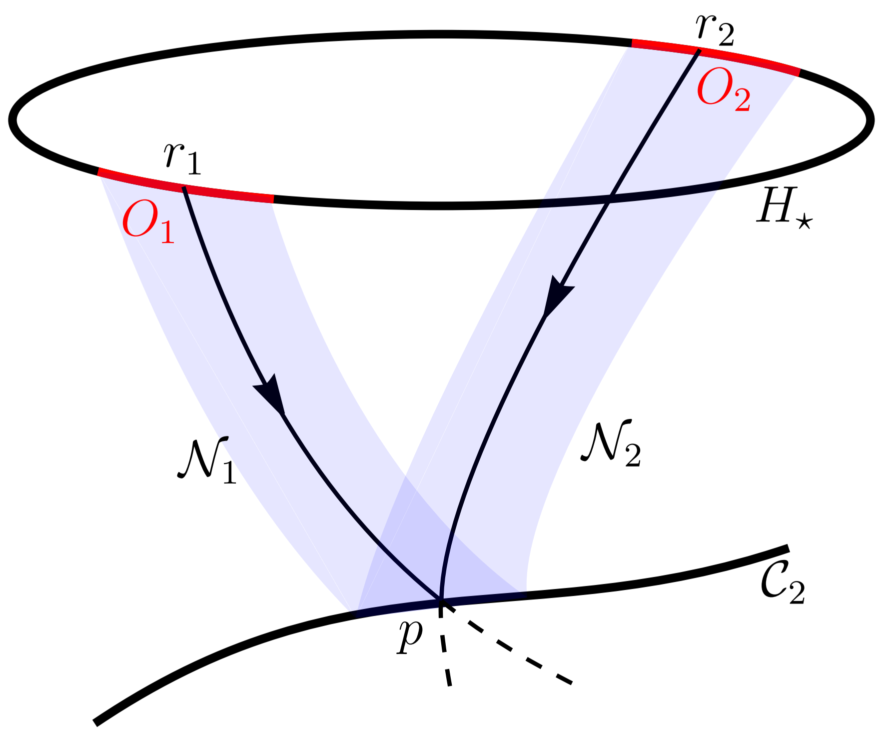

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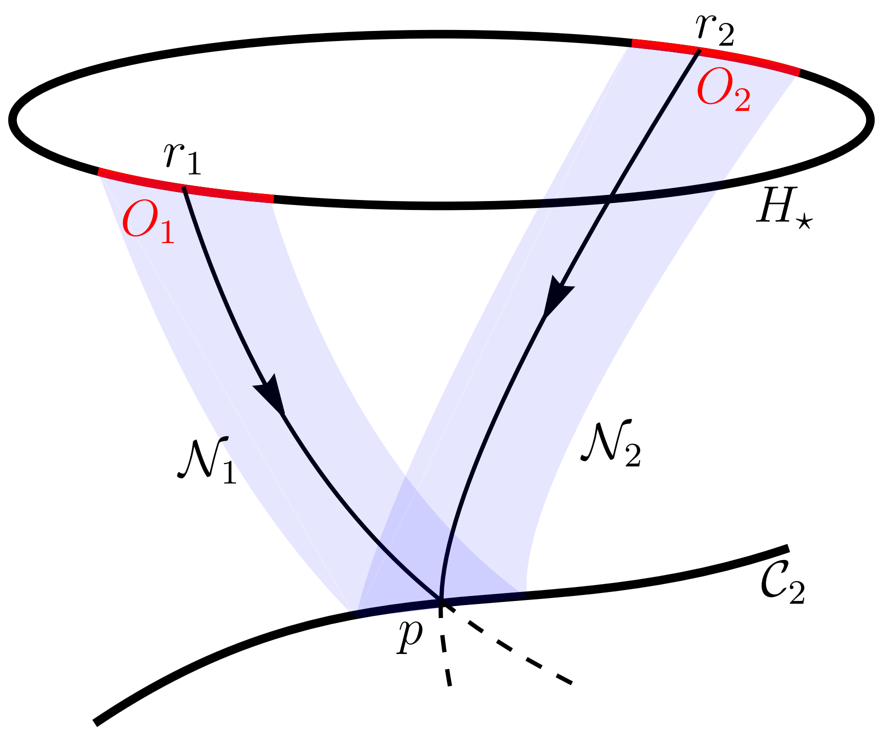
- Past endpoints are precisely where non-smoothness arises.
- There are two main types of past endpoints: caustic points (points conjugate to $H_\star$) and non-caustic points.



Classification in Four Dimensions

We are interested in *generic* non-smooth features, i.e. features that are stable under perturbations of the metric. We assume that this is equivalent to stability under perturbations of the null hypersurface.

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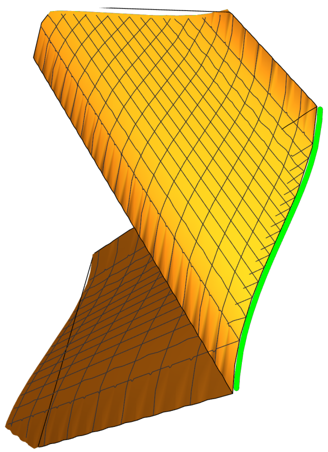

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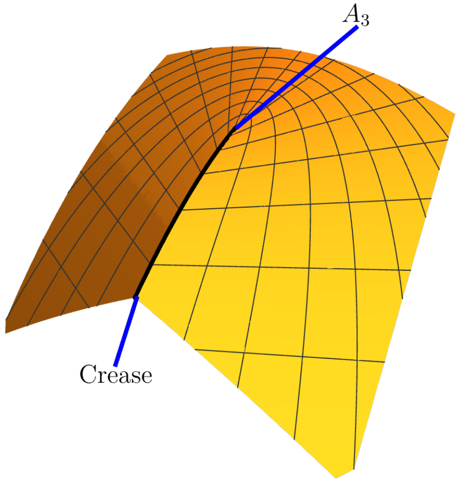
In four dimensions, we obtain the following list of possibilities:

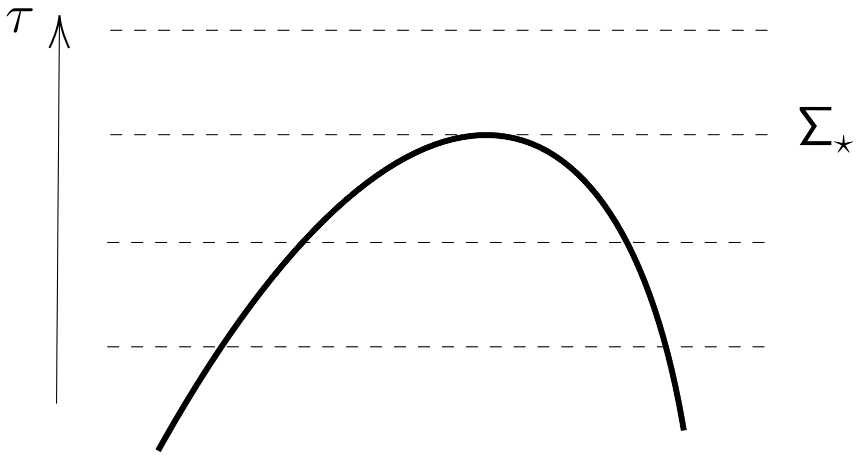
Type			Dimension	# of generators
Non-caustic	A_1	Regular point	3	1
Non-caustic	(A_1, A_1)	Normal crease point	2	2
Non-caustic	(A_1, A_1, A_1)	Normal corner point	1	3
Non-caustic	(A_1, A_1, A_1, A_1)		0	4
Caustic	A_3		1	1
Caustic	(A_3, A_1)		0	2

Siino & Koike 2011



Perestroikas

- Let us now focus on normal crease points, which are non-caustic points at which exactly two generators enter the horizon. Such points form a smooth, 2-dimensional spacelike submanifold in spacetime.



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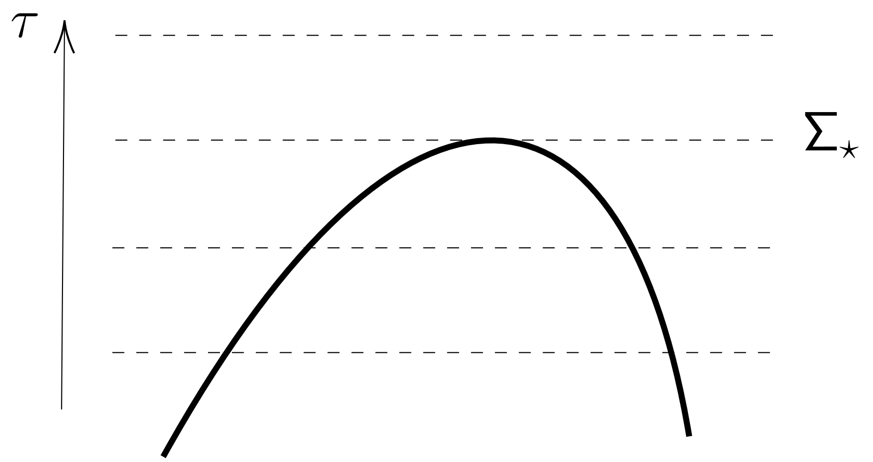

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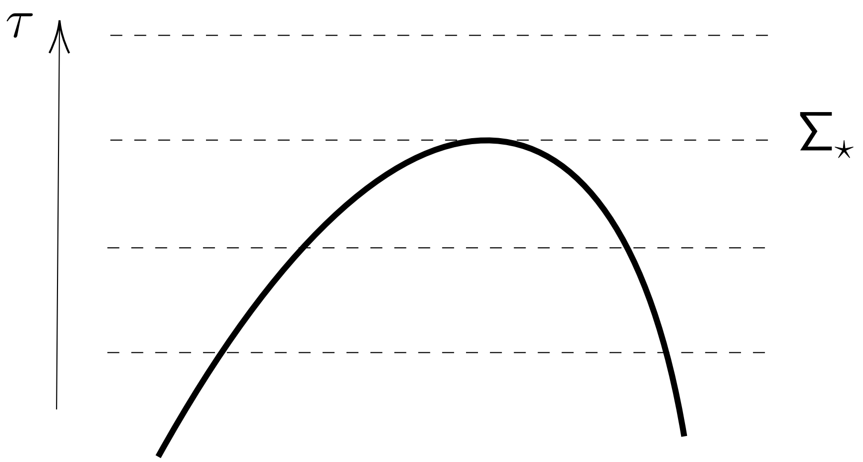
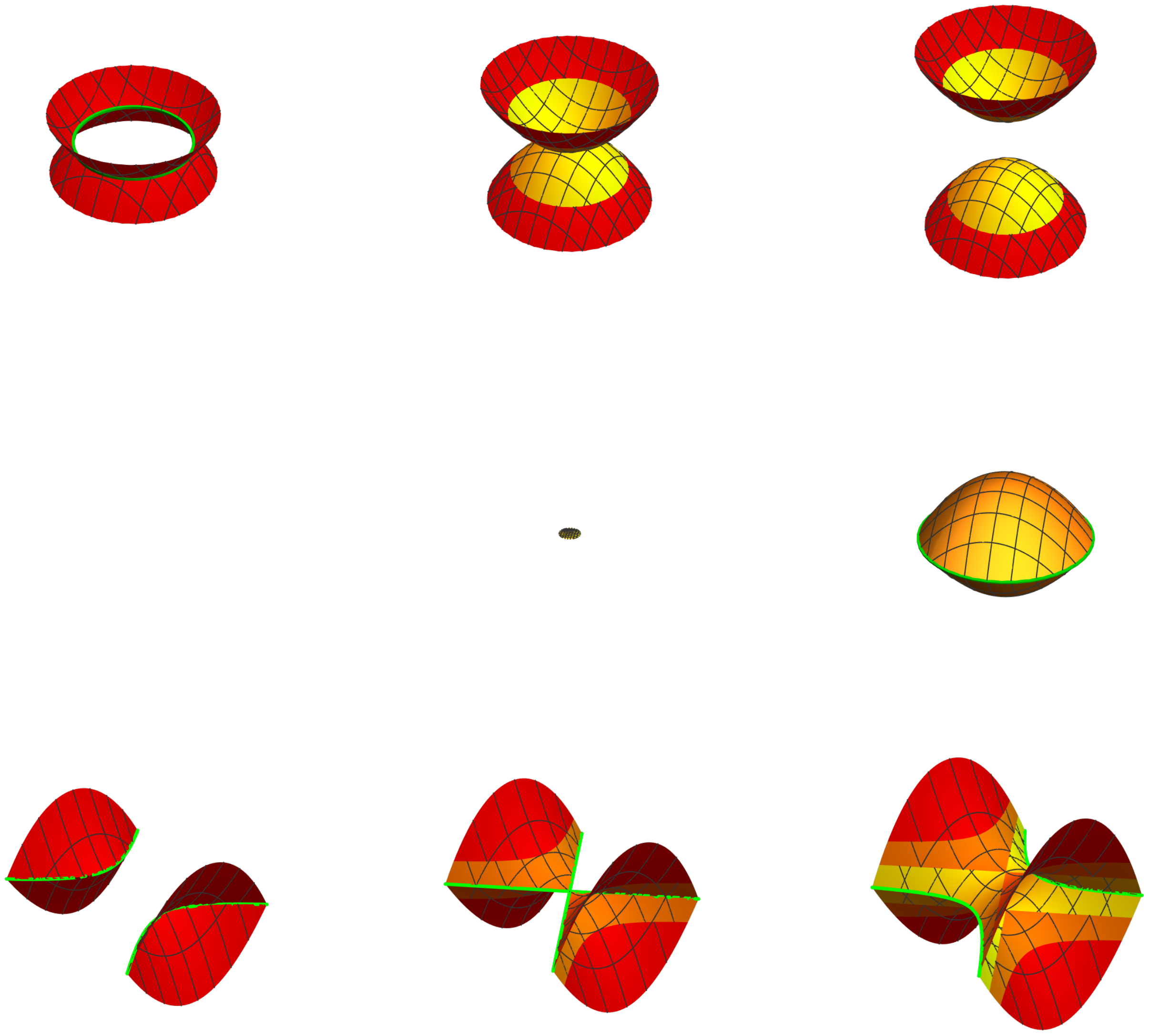
- Choose a time function τ. Then we can foliate our spacetime with Cauchy surfaces Σ_τ of constant τ.
- Generically, these Σ_τ will intersect the crease submanifold transversely.
- However, at special moments of time the intersection will be tangential, resulting in a qualitative change in the non-smooth structure. Such changes are known as perestroikas.



Crease Perestroikas



$\tau < 0$ $\tau = 0$ $\tau > 0$

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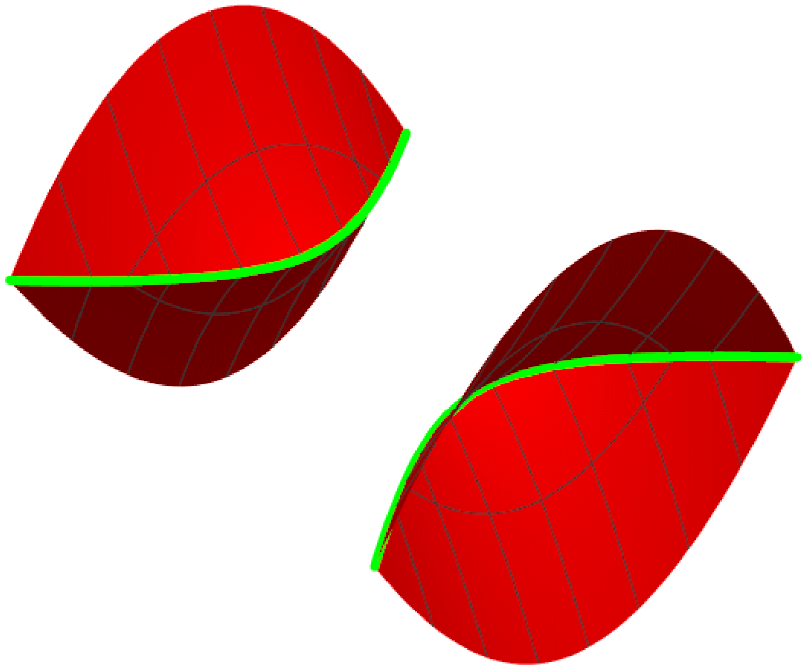
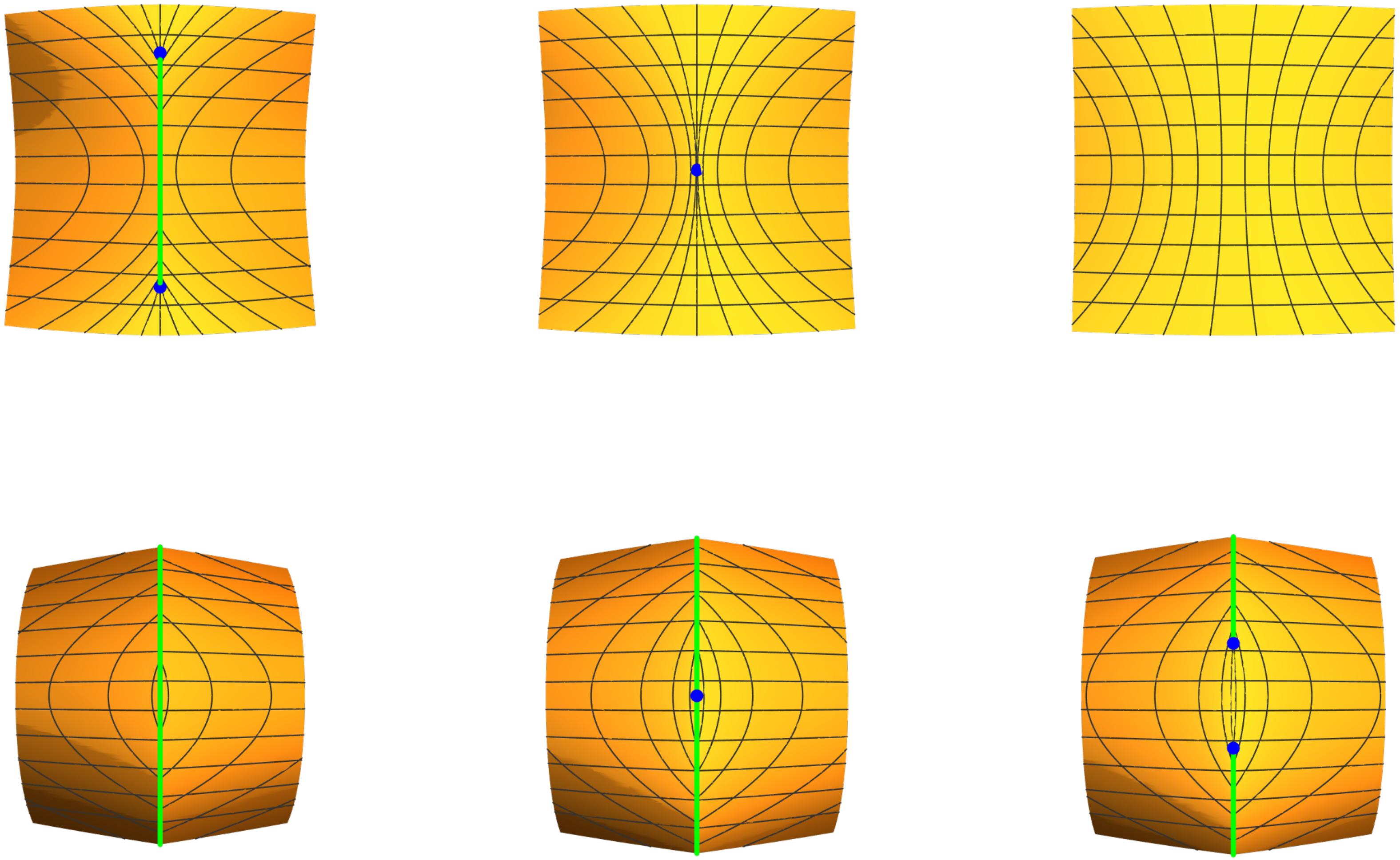


Caustic Perestroikas



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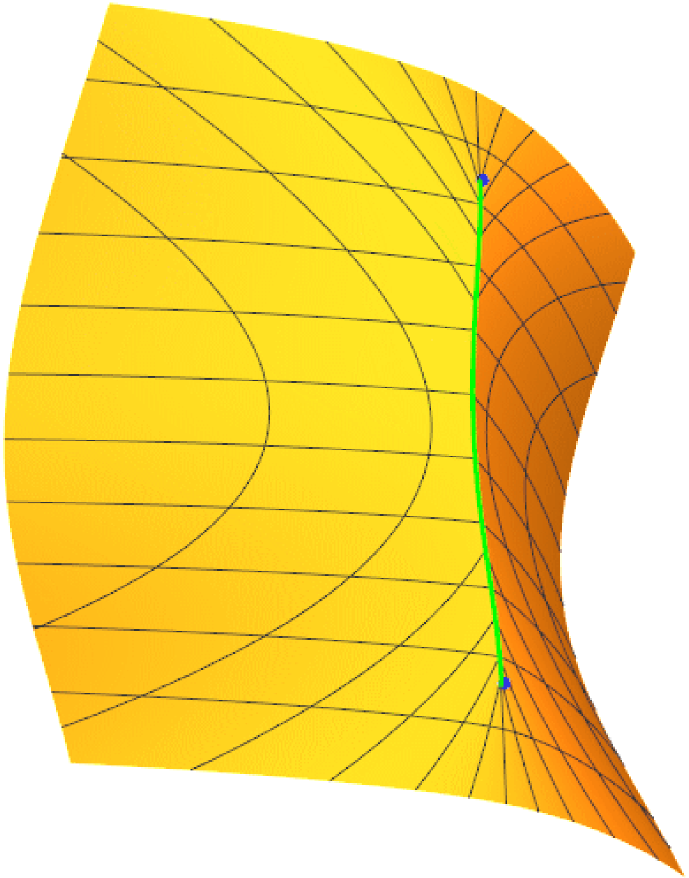
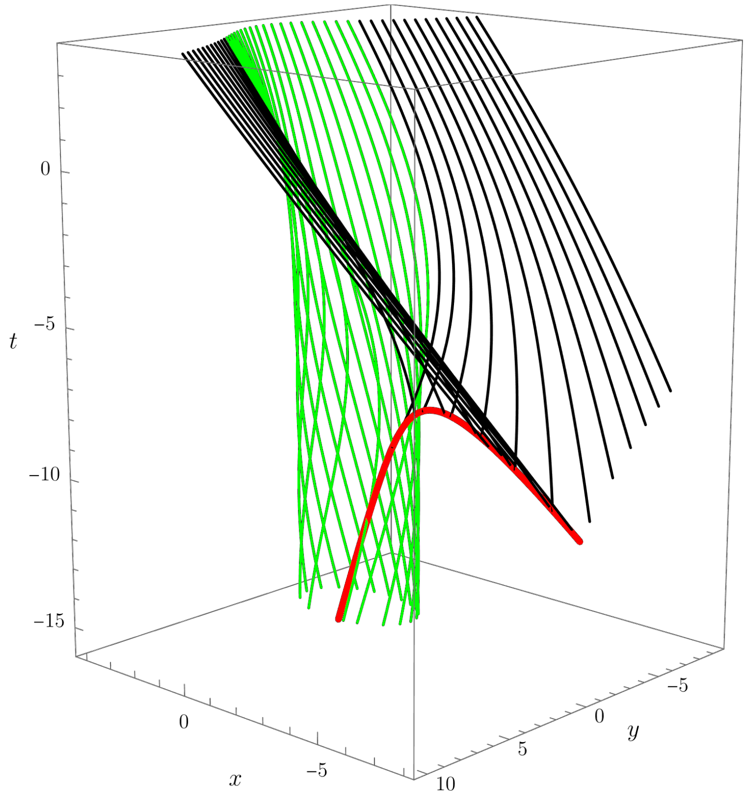


Part II: Kerr Merger in the Infinite Mass Ratio Limit

Why this limit?

- The curvature of a Kerr black hole of mass M_L is relevant on scales $\sim M_L$.
- Consider a merger of a small black hole, of mass $M = 1$, with a much larger black hole with $M_L \gg 1$. Then near the small black hole (i.e. on scales much smaller than M_L), the metric is Kerr with some small corrections from the curvature of the large black hole.
- In the limit $M_L \to \infty$, these corrections vanish, so the metric is exactly Kerr! Emparan *et al.* 2018



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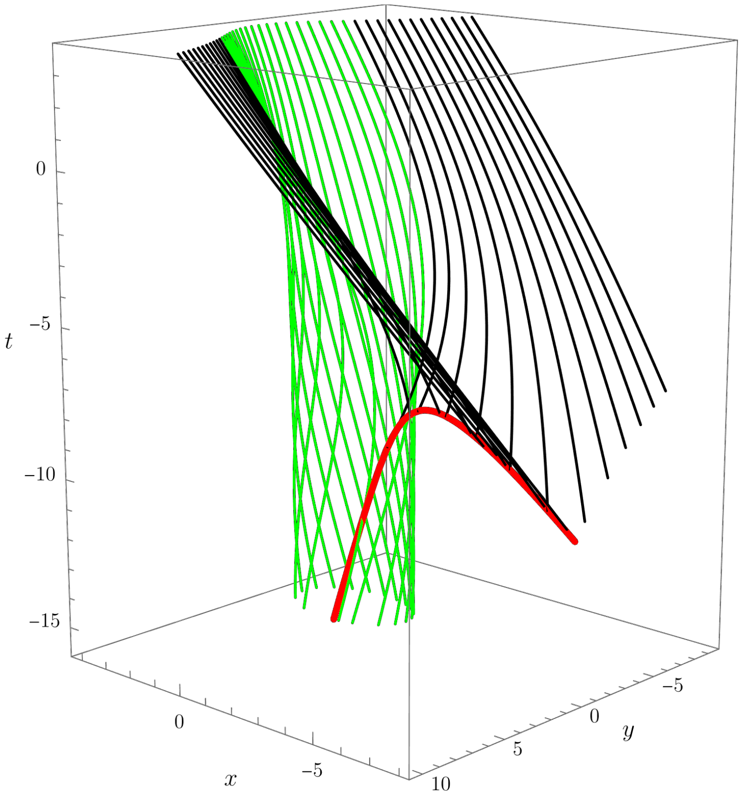
- However, the horizon of the combined system is now asymptotic to a Rindler or acceleration horizon, rather than the usual Kerr horizon.
- The correct null hypersurface is the one that asymptotes to a null plane at late times, representing the infinitely-large black hole that has settled down after merger.



Equations of Motion

The equations of motion are simply the null geodesic equations in Kerr, with appropriate final conditions. With an affine parameter τ, these read

$$\frac{\Sigma}{E}\frac{dt}{d\tau} = \frac{r^2+a^2}{\Delta}(r^2+a^2-a\lambda) + a(\lambda - a\sin^2\theta)$$
$$\frac{\Sigma}{E}\frac{dr}{d\tau} = \pm_r \left[r^4 + \left(a^2-\lambda^2-\eta\right) r^2 + 2M\left((\lambda-a)^2+\eta\right) r - a^2\eta\right]^{1/2}$$
$$\frac{\Sigma}{E}\frac{d\theta}{d\tau} = \pm_\theta \left[\left(a^2-\lambda^2\csc^2\theta\right)\cos^2\theta + \eta\right]^{1/2}$$
$$\frac{\Sigma}{E}\frac{d\phi}{d\tau} = \frac{a}{\Delta}(r^2+a^2-a\lambda) + \frac{\lambda}{\sin^2\theta} - a$$

where E, λ and η are constants of motion, and where

$$\Delta = r^2 - 2Mr + a^2\,, \quad \Sigma = r^2 + a^2\cos^2\theta.$$

For null geodesics we can always rescale the affine parameter to set $E = 1$.

We also introduce two impact parameters α, β which span the null plane at infinity (which is at an angle θ_o),

$$\alpha = -\frac{\lambda}{\sin\theta_o} \qquad \beta = \pm_o\sqrt{\left(a^2-\lambda^2\csc^2\theta_o\right)\cos^2\theta_o + \eta}.$$

Perturbative vs Numerical Methods

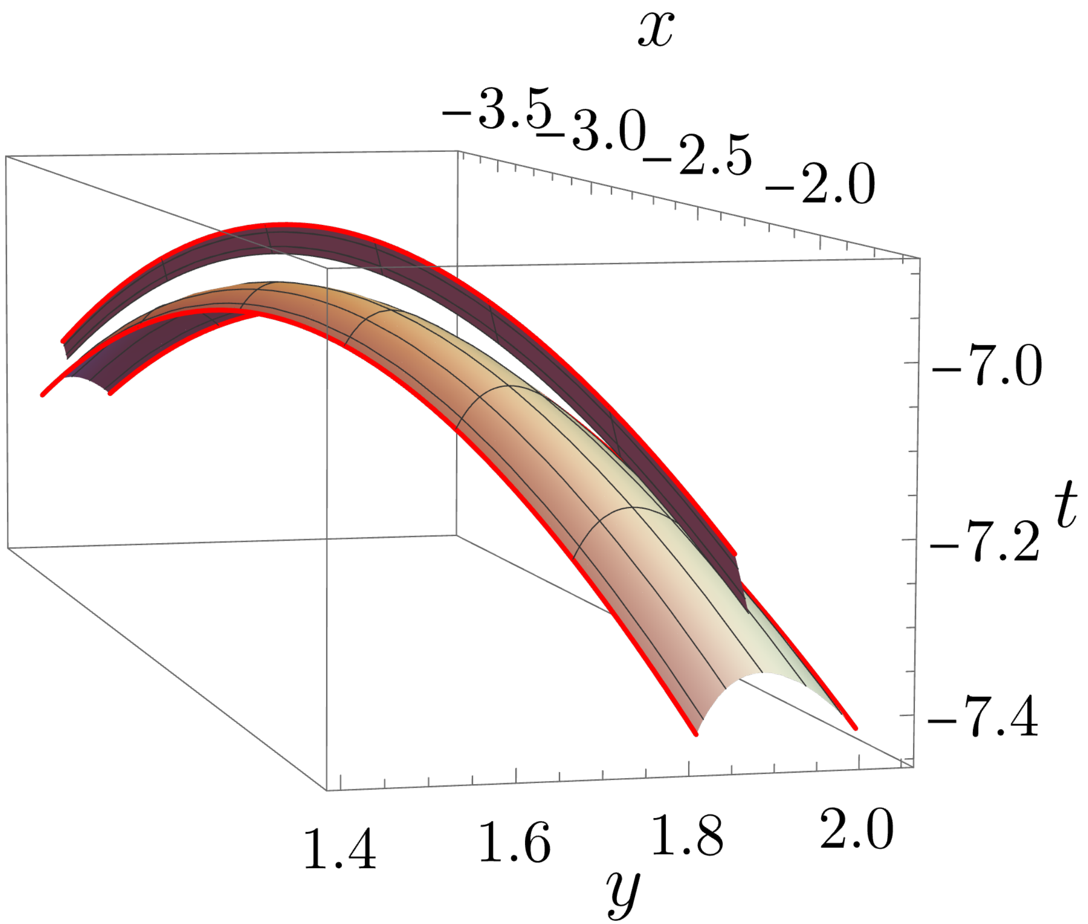
Perturbative Method

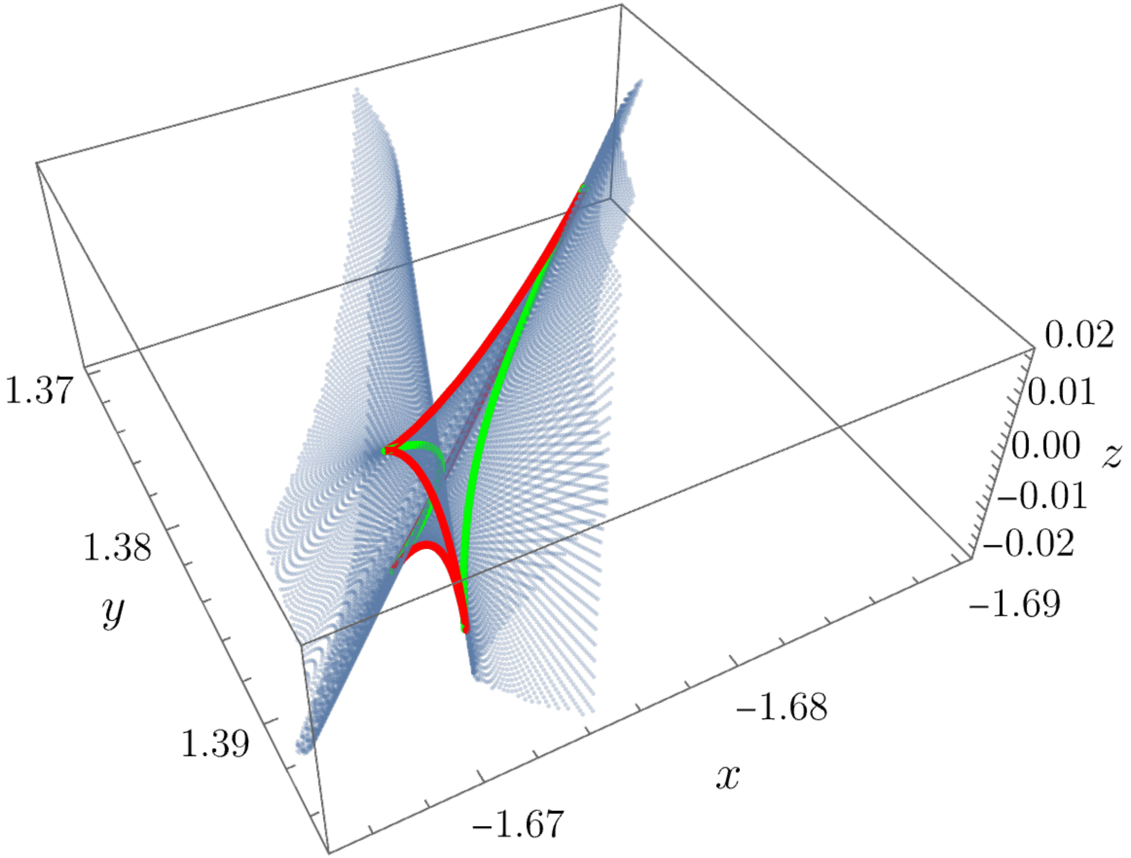
- Expand the equations (in integral form) in inverse powers of λ, $b \equiv \sqrt{\lambda^2 + \eta}$ and r_s, where r_s is the initial r coordinate of the geodesic.
- Can then determine the coordinates of caustic and crease points perturbatively.
- Holds for large impact parameter and large radius, i.e. it is best suited to study the non-smoothness of the large black hole.
- Nevertheless gives the correct qualitative behaviour near the point of merger, when the series expansion is expected to break down.

Numerical Method

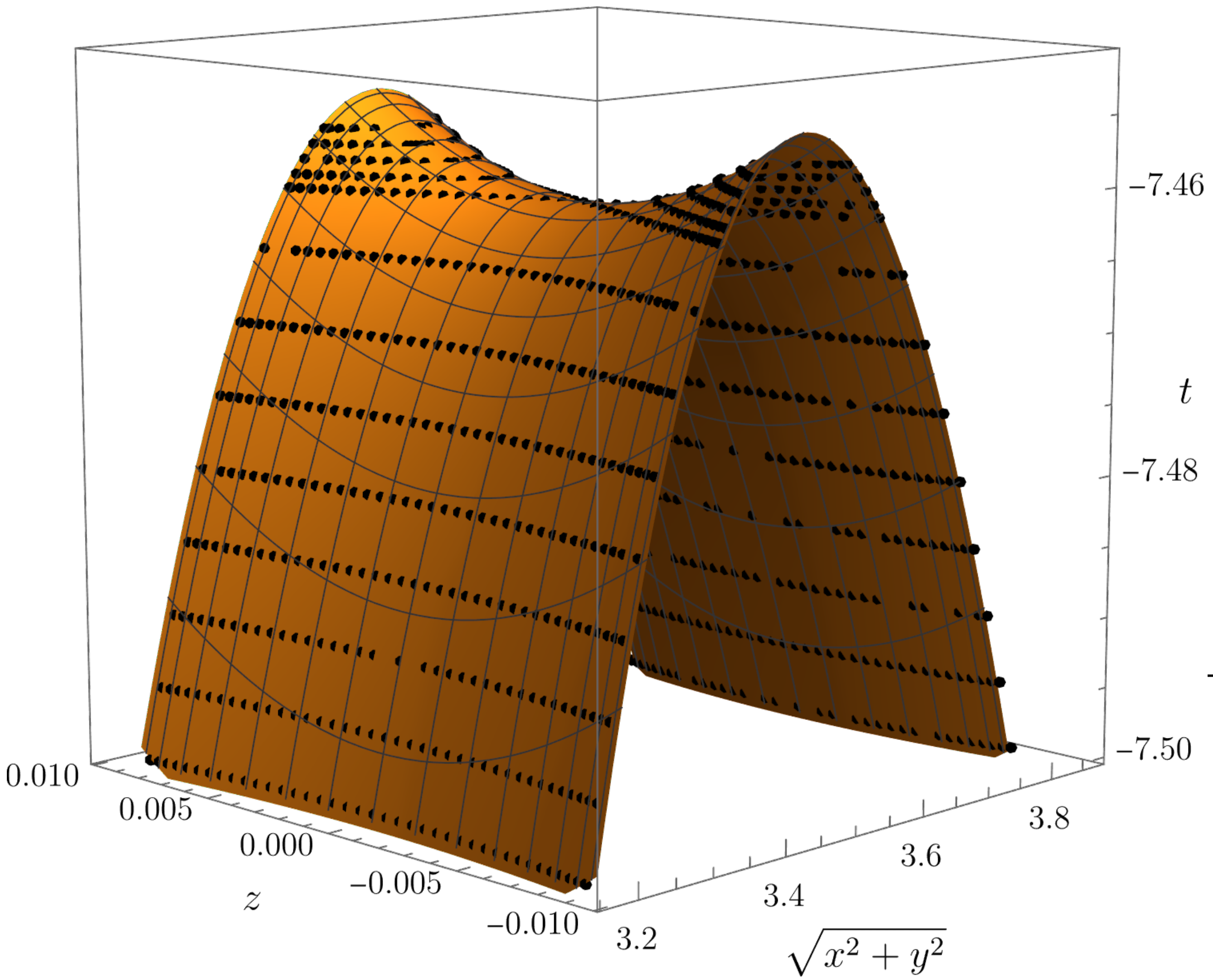
- Solve the ODEs numerically, starting from late time and evolving backwards in time.
- Determine when two geodesics intersect to find crease points.
- Can quantitatively probe up to and beyond the time of merger. However, at early times, the geodesics forming creases intersect with an angle that is almost π, causing difficulties in identifying crease points.
- Does not give us formulae that can illuminate the physics.

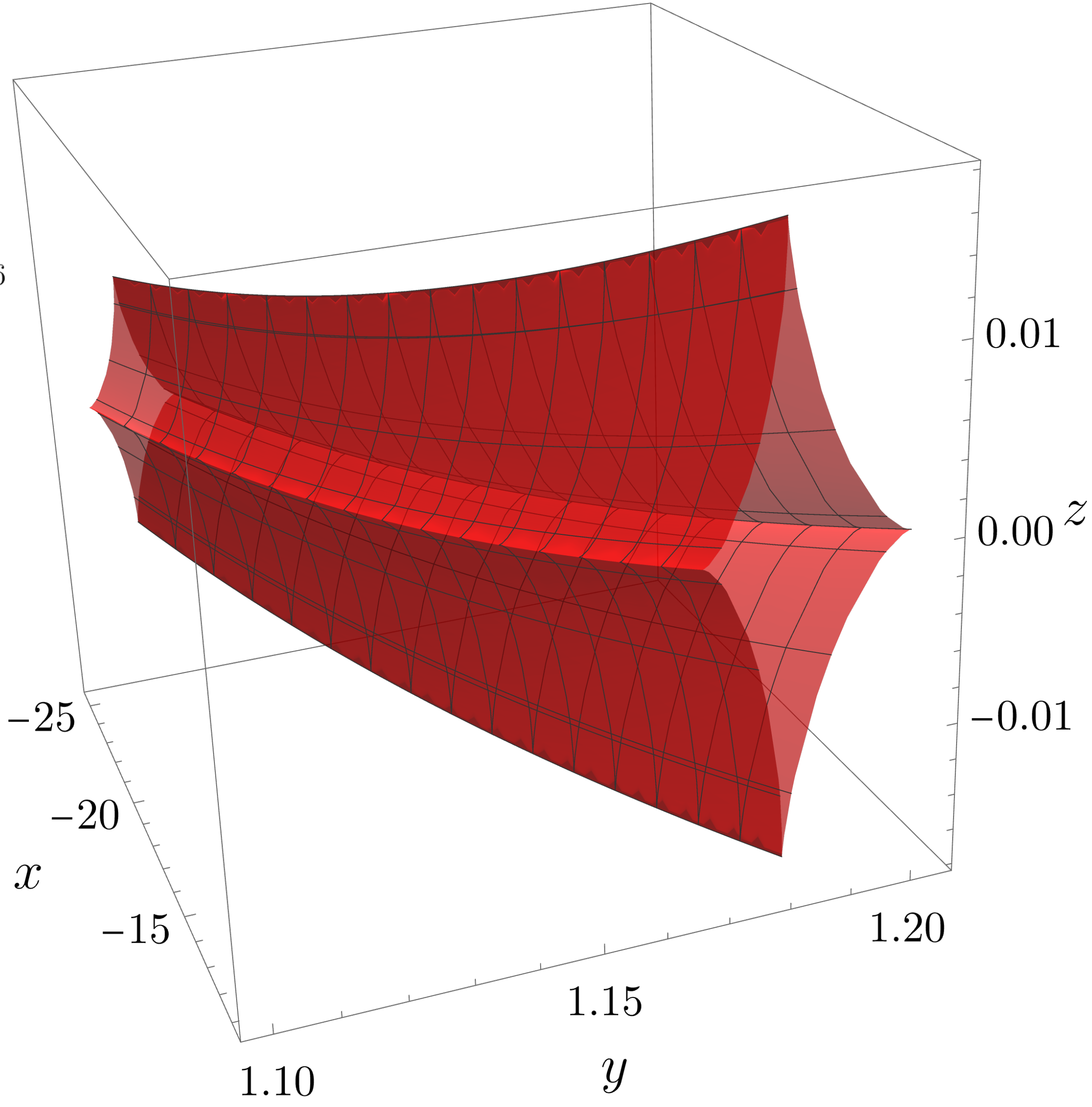
True and False Creases



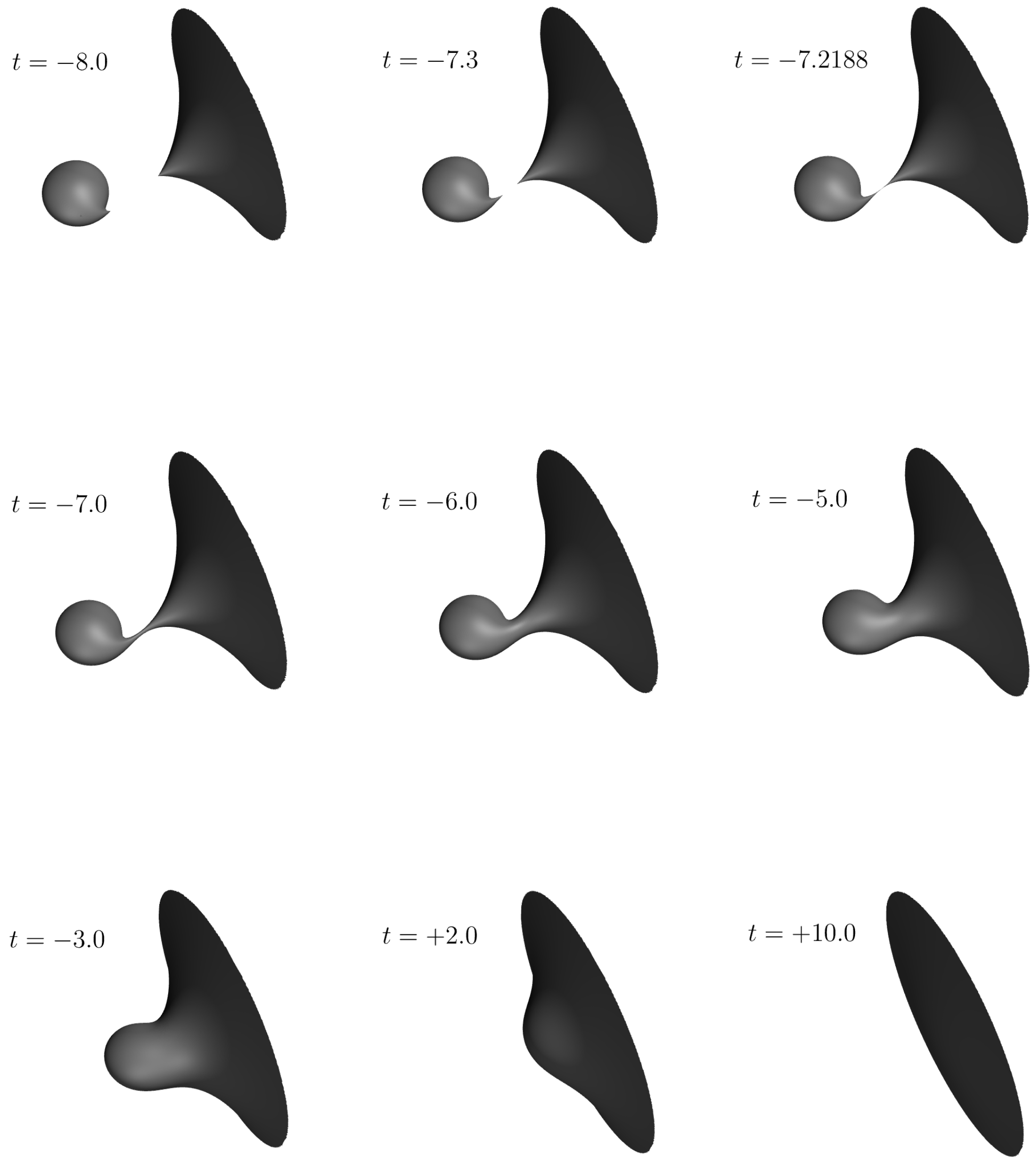


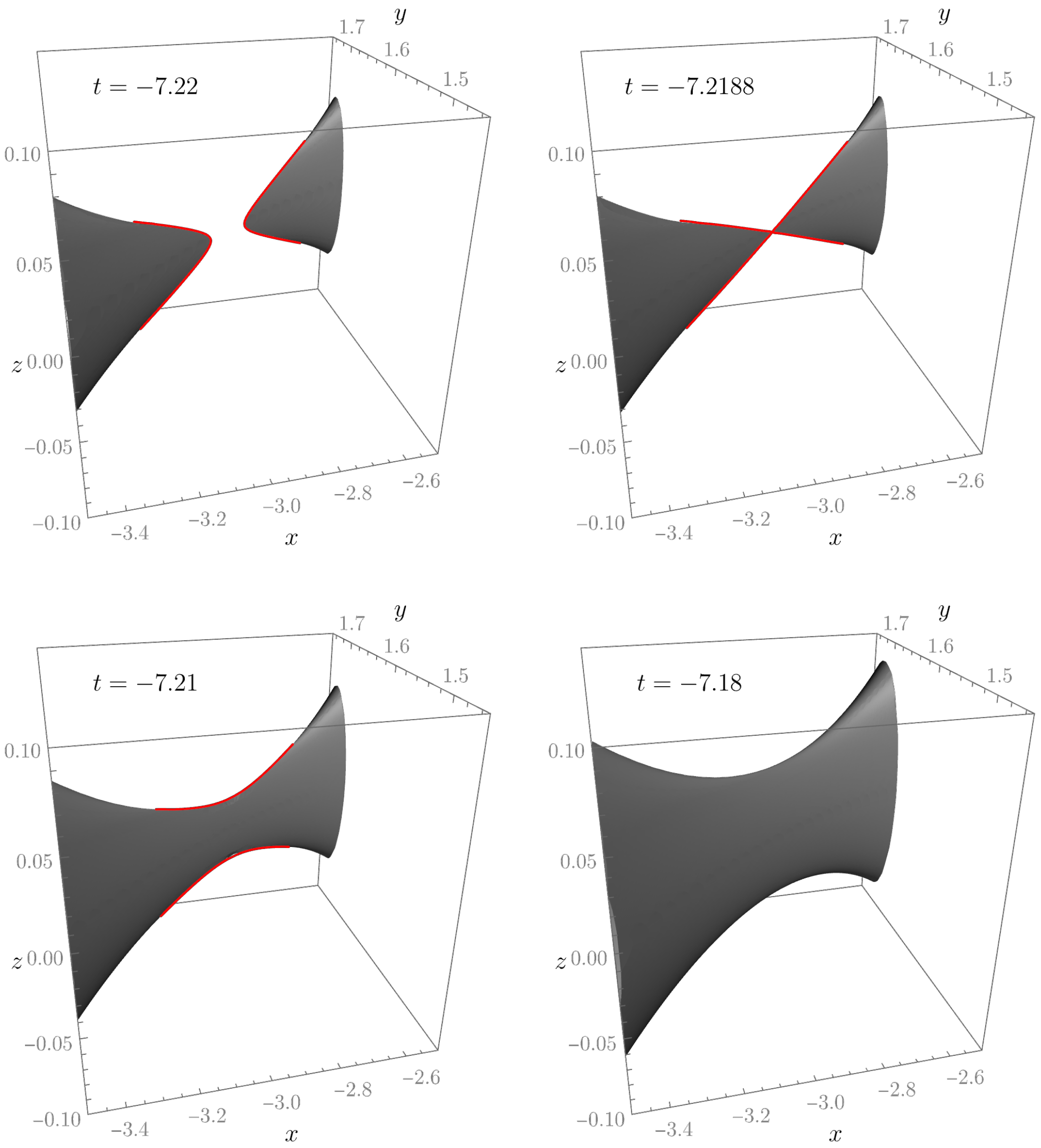
Crease and Caustic Sets



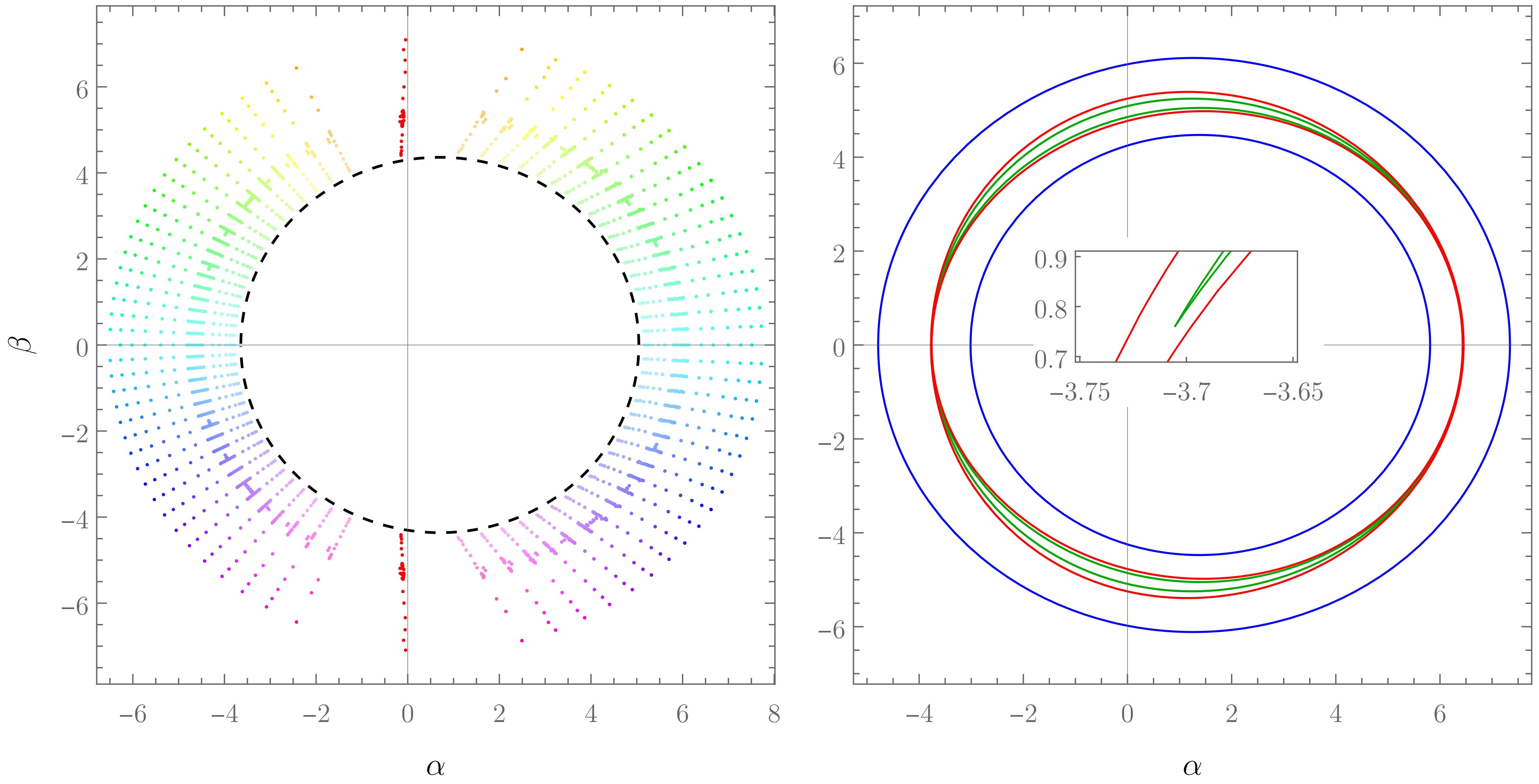


Horizon Cross-Sections





Structure in Parameter Space



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- Nevertheless, the crease submanifold has finite area. We can compute the area perturbatively for the large black hole's section of the crease submanifold for some cutoff radius r_{min},

$$A_{\mathrm{Crease}} = \frac{15\pi a^2(1-\cos^2\theta_o)}{64}\sqrt{\frac{M}{4r_{\mathrm{min}}}}\left[1+\frac{15\pi}{64}\sqrt{\frac{M}{4r_{\mathrm{min}}}}+\cdots\right].$$

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- In a constant t cross-section, the length of the crease increases near the time of merger, as the crease is stretched towards the other black hole. In the infinite past, the large black hole's crease has zero length, but the small black hole's crease has finite length. Perturbatively, for the large black hole,

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- Near the caustic points, the opening angle of the crease tends to π, as is expected.

Crease Angle in the Local Model

In a neighbourhood of a normal crease point, the horizon can be locally described as the union of two null hypersurfaces. We can use Riemannian normal coordinates (T, X^1, X^2, Z) at the point of merger to give a local description of how the opening angle should behave.

The two null hypersurfaces can be written as

$$k\tau = Z + \left(b_{AB} - \frac{1}{2}K_{AB}\right) X^A X^B + \dots \qquad k\tau = -Z + \left(\hat{b}_{AB} - \frac{1}{2}K_{AB}\right) X^A X^B + \dots$$

where $\tau \equiv t - t_{\mathrm{merger}}$ is our time function, b_{AB} and $\hat{b}_{AB}$ describe the shear and expansion of the null hypersurfaces, and K_{AB} describes the extrinsic curvature of the $\tau = 0$ slice at the origin.

From these equations we can find the crease submanifold

$$2k\tau = (b_{AB} + \hat{b}_{AB} - K_{AB})X^A X^B + \mathcal{O}(|X^A|^3) \qquad 2Z = (\hat{b}_{AB} - b_{AB})X^A X^B + \mathcal{O}(|X^A|^3)$$

and find the normals there:

$$n_1 = dZ + [(2b_{AB} - K_{AB})X^B + \mathcal{O}(|X^B|^2)]dX^A \qquad n_2 = -dZ + [(2\hat{b}_{AB} - K_{AB})X^B + \mathcal{O}(|X^B|^2)]dX^A.$$

Crease Angle in the Local Model

The crease's opening angle Ω can then be obtained from

$$\cos(\pi - \Omega) = \frac{n_1 \cdot n_2}{||n_1||\,||n_2||}.$$

We obtain

$$\Omega^2 = 4\mu_{AB}X^B\mu_{AC}X^C + \mathcal{O}(|X^A|^3), \qquad \mu_{AB} = b_{AB} + \hat{b}_{AB} - K_{AB}.$$

μ_{AB} must have indefinite signature for a merger perestroika, so it has eigenvalues $-\lambda_-$, λ_+ for $\lambda_\pm > 0$. Hence, by suitably rotating X^A,

$$\Omega^2 = 4\left[\lambda_-^2(X^1)^2 + \lambda_+^2(X^2)^2\right] + \mathcal{O}(|X^A|^3).$$

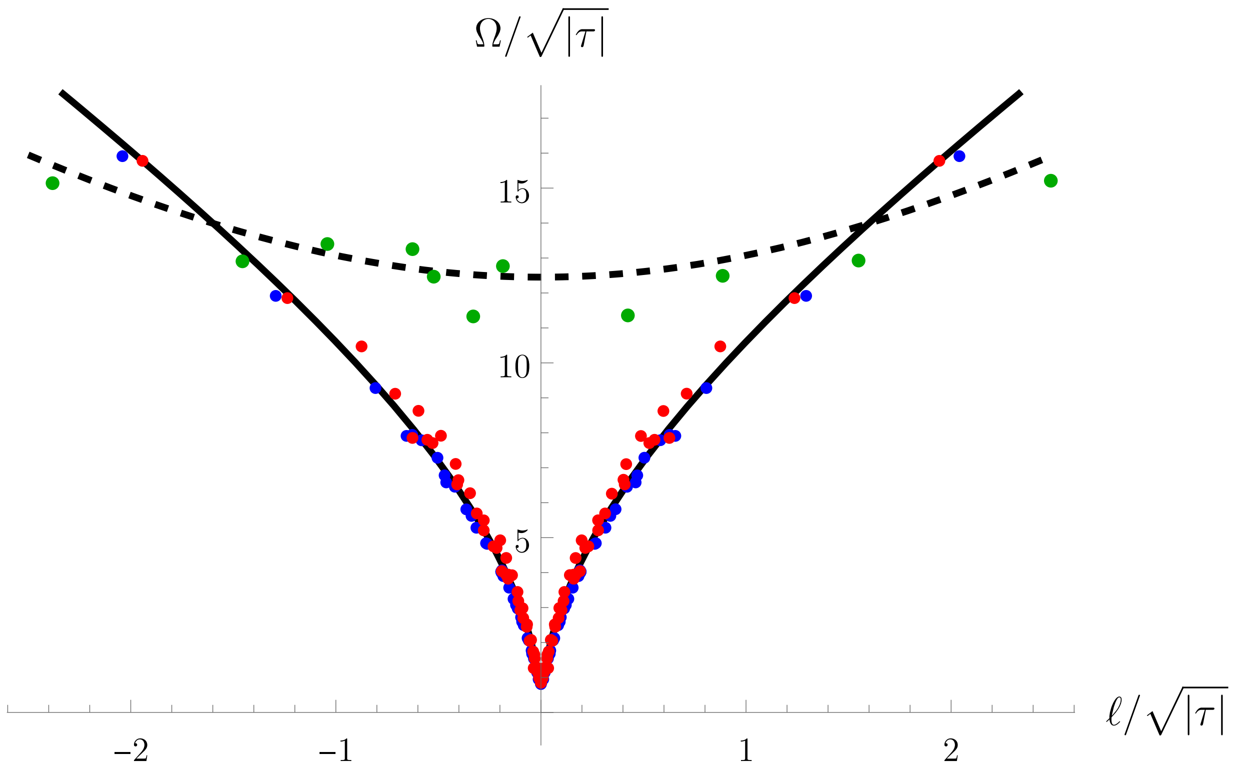
After some further manipulation we can write the angle as a function of proper length along the crease, with parametric solution

$$\Omega = \sqrt{8k\lambda_\pm|\tau|}f_\pm(\xi) + \dots \qquad \ell = \sqrt{\frac{2k|\tau|}{\lambda_\mp}}\int_0^\xi f_\pm(x)dx + \dots$$

where

$$f_\pm(x) = \left(\cosh^2 x + \frac{\lambda_\mp}{\lambda_\pm}\sinh^2 x\right)^{1/2}$$

Crease Angle in the Extreme Kerr Merger



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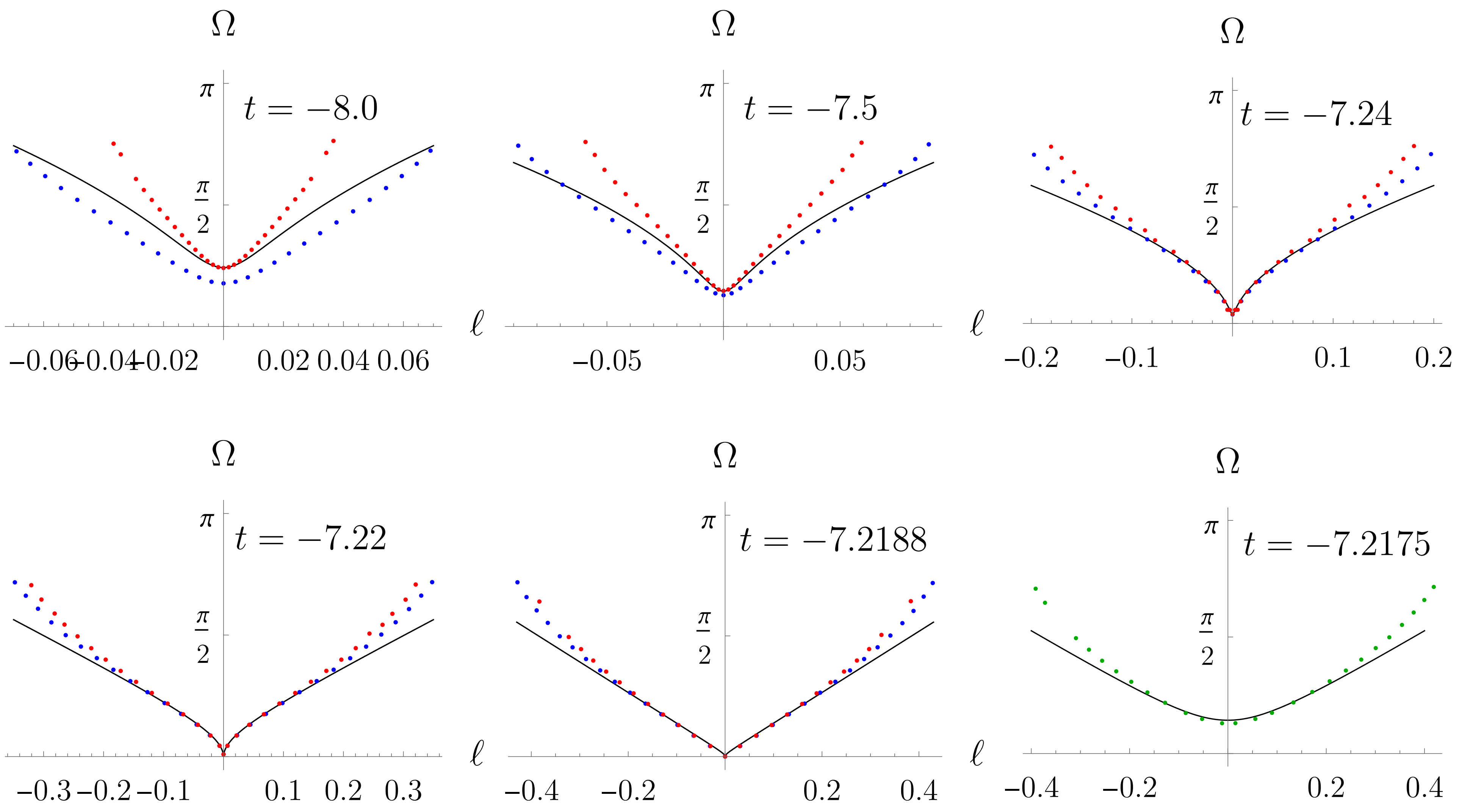
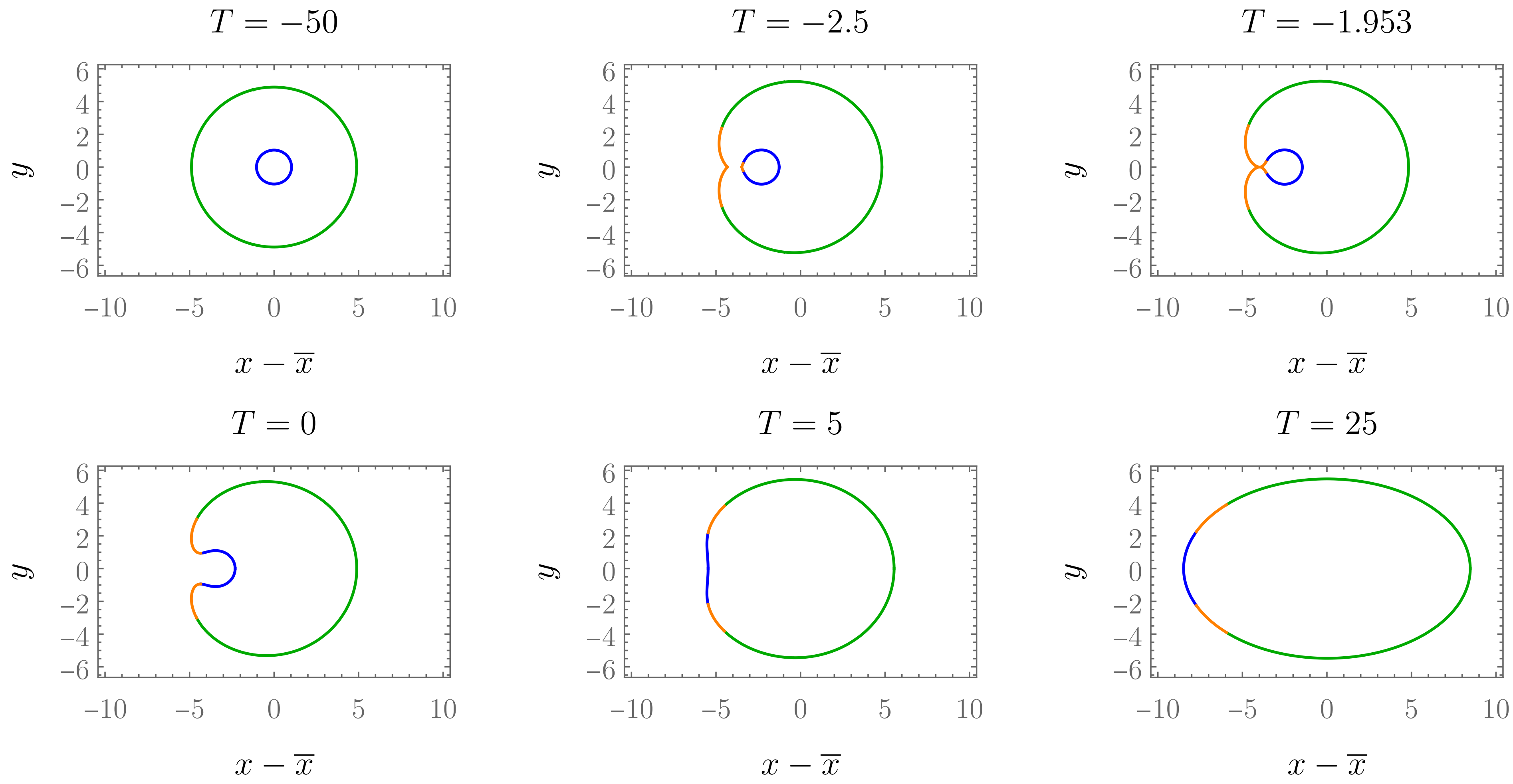


Merger of a Black Hole and a Cosmological Horizon

Our results on creases and caustics apply to any future horizon. Suppose a black hole disappears behind the cosmological horizon of some observer. The combined horizon can be obtained in a similar way to that of the black hole merger. MG & H. Wang, arXiv:2412.04551



Summary and Future Directions

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- These structures provide useful insights into the otherwise very complicated dynamics that black holes may undergo, e.g. in a merger.
- We have provided an exact local description of the event horizon when black holes merge.
- The extreme mass ratio merger allows us to employ the well-known properties of the Kerr spacetime to obtain a precise view of how the merger occurs.
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Future Directions

- The area of the crease submanifold is a geometric invariant of the spacetime. Is there a physical interpretation of it?
- Implications of non-smooth structures on black hole entropy?
- Non-smooth horizons in higher dimensions or higher-derivative theories of gravity?