Regularised Higher Derivative Effective Field Theories

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The Dynamics of Black Hole Mergers and Gravitational Wave Generation
IFPU, SISSA
15th of January 2026

Overview

- Era of gravitational wave astronomy:

➡ New tests of the strong field regime of gravity

➡ Understanding of fundamental nature of gravity

- Relativistic viscous hydrodynamics (and pre-hydrodynamics)

- Mathematical understanding of classical EFTs

Overview

- Some issues in carrying out this program in gravity:
  - Predictions from (the strong field regime of) alternative theories of gravity are needed
  - No preferred alternative theory
  - How can we extract the strong field predictions of these alternative theories?

    ➡ So what should we be looking for?

    ➡ Do they have a well-posed initial value problem? Predictive?

Classical EFTs

- EFT provides a framework to parametrise the effects of the (unknown) high energy physics at low energies in the strong field regime

- An EFT is defined by:

  1. The low energy degrees of freedom

  2. The low energy symmetries

  3. A power counting scheme (derivatives of the fields)

- Construct the most general effective action 1) and 2), and organise the terms in a series expansion based on 3)

Classical EFTs

- Redundancies in the expansion: one can make field redefinitions to simplify the action

- The coefficients in the expansion can be constrained by demanding that the UV theory satisfies certain properties (unitarity, locality,…) [Adams et al. ’06; de Rham,…; Cheung and Remmen ’17; Alexander et al. ’25]

- In situations with a large separation of scales, accurate predictions can be made by truncating the series to the first few terms

Classical EFTs

Examples

- Scalar-tensor theories of gravity:

$$S = \int d^4x\sqrt{-g}\left[\frac{m_{Pl}^2}{2}R - \frac{1}{2}(\nabla_a\phi)(\nabla^a\phi) - V(\phi) + \ell^2\left(m_{Pl}\,G(\phi)\,\mathscr{L}_{GB} + m_{Pl}\,F(\phi)\,\tilde{R}_{abcd}\,R^{abcd} + \frac{1}{4\,m_{Pl}^2}\,H(\phi)\big((\nabla_a\phi)(\nabla^a\phi)\big)^2\right)\right]$$

$$\mathscr{L}_{GB} = R^2 - 4\,R_{ab}R^{ab} + R_{abcd}R^{abcd} \qquad \tilde{R}_{abcd} \equiv \frac{1}{2}\,\varepsilon_{ab}{}^{ef}R_{efcd}$$

Classical EFTs

Examples

- Pure gravity in *D*=4 spacetime dimensions:

$$S = \int d^4x \sqrt{-g} \Big[R + \ell^4 \big(c_3\, \mathcal{Q} + \tilde{c}_3\, \tilde{\mathcal{Q}} \big) + \ell^6 \big(d_{(4,1)}\, \mathcal{C}^2 + d_{(4,2)}\, \tilde{\mathcal{C}}^2 + \tilde{d}_4\, \mathcal{C}\, \tilde{\mathcal{C}} \big) \Big]$$

$$\mathcal{Q} \equiv R_{ab}{}^{cd}\, R_{cd}{}^{ef}\, R_{ef}{}^{ab}\,, \quad \tilde{\mathcal{Q}} \equiv R_{ab}{}^{cd}\, R_{cd}{}^{ef}\, \tilde{R}_{ef}{}^{ab}\,,$$

$$\mathcal{C} \equiv R_{abcd}\, R^{abcd}\,, \quad \tilde{\mathcal{C}} \equiv R_{abcd} \tilde{R}^{abcd}\,.$$

Classical EFTs

Examples

- Relativistic viscous (conformal) hydrodynamics in *d* dimensions [Baier et al. '08]:

$$T_{\mu\nu} = \frac{\rho}{d-1}(d\, u_\mu u_\nu + \eta_{\mu\nu}) + \Pi_{\mu\nu}$$

$$\Pi_{\mu\nu} = -2\,\eta\,\sigma_{\mu\nu} + 2\,\eta\,\tau_\Pi \left(\langle u^\alpha \partial_\alpha \sigma_{\mu\nu}\rangle + \frac{1}{d-1}\,\sigma_{\mu\nu}\,\partial_\alpha u^\alpha\right) + \langle \lambda_1\,\sigma_{\mu\alpha}\sigma_\nu{}^\alpha + \lambda_2\,\sigma_{\mu\alpha}\omega_\nu{}^\alpha + \lambda_3\,\omega_{\mu\alpha}\omega_\nu{}^\alpha\rangle$$

$$\sigma_{\mu\nu} = \partial_{\langle\mu}u_{\nu\rangle}\,, \quad \omega_{\mu\nu} = \partial_{[\mu}u_{\nu]}$$

Classical EFTs

General issues

- The equations of motion of these truncated EFTs typically have higher than 2nd order equations of motion (Lovelock and Horndeski theories are the exception)
- Problematic:
  - No general classification for higher than 2nd order PDEs
  - Well-posedness? Determined by the highest derivative terms but those should be the least important ones in EFT
  - Need to specify more initial data corresponding to additional dofs not present in the EFT
  - Ghosts [Ostrogradsky 1850]: how can prevent runaway solutions?

Higher order EOMs

- Various methods have been proposed over the years to deal with higher order evolution equations:
  - Perturbation theory
  - Reduction of order
  - Fixing-of-the-equations
  - Regularisation

Outline of the talk

- Perturbation theory and Reduction of order
- Special theories with 2nd order equations of motion
- Fixing-of-the-equations
- Regularisation
- Conclusions

Perturbation theory and Reduction of Order

Perturbative solutions

- Consider the general structure of the higher derivative equations:

$$\mathcal{O}^{(2)}(h) = \sum_{n>2} \ell^{n-2} \mathcal{O}^{(n)}(h)$$

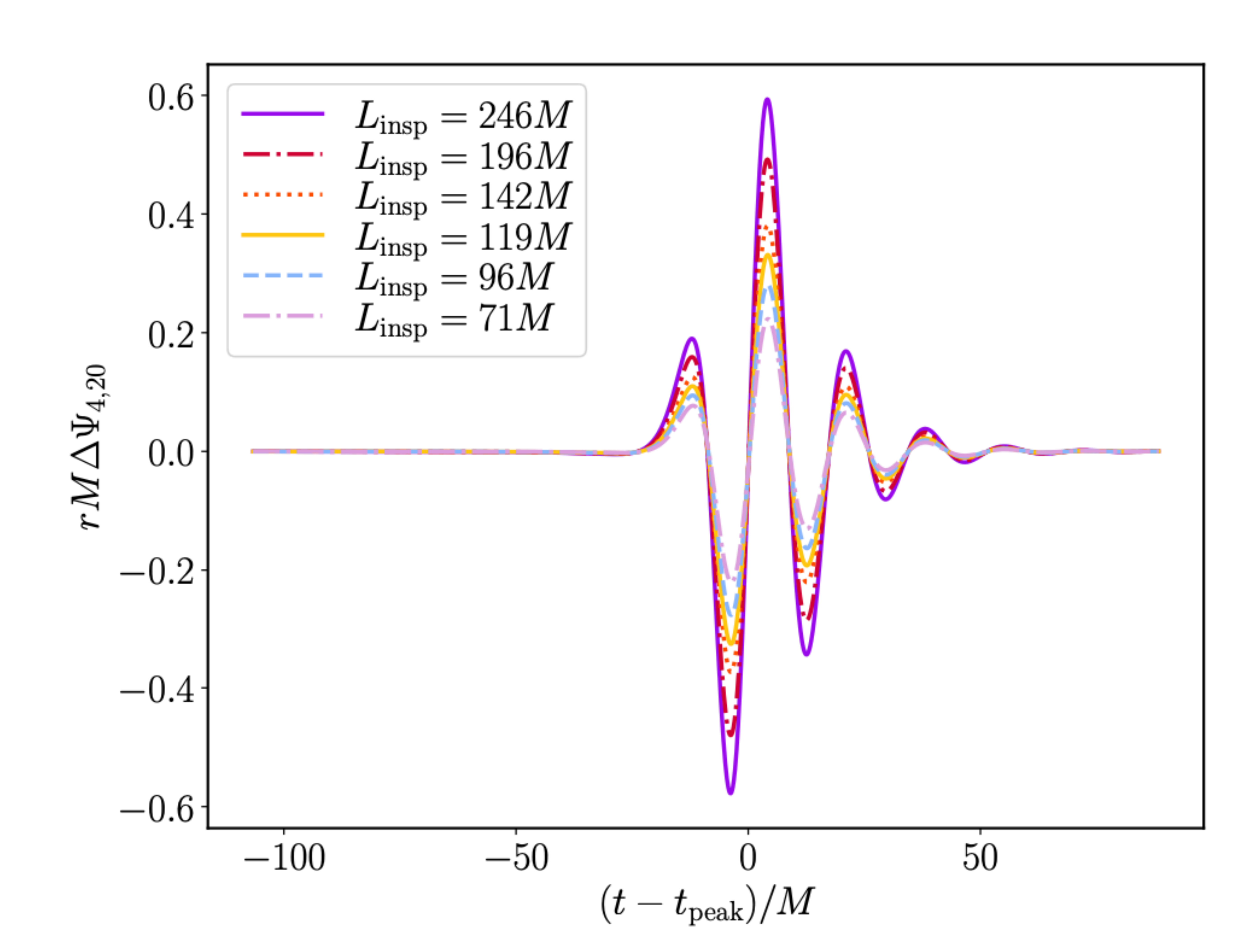
- Construct solutions as a series expansion in the small parameter:

$$h = h^{(0)} + \sum_{n=1} \ell^n h^{(n)}$$

$$\Rightarrow \quad \mathcal{O}^{(2)}(h^{(k)}) = \sum_{n=0}^{k-1} \mathcal{S}^{(n)}(h^{(i)} \ldots)$$

Perturbative solutions

- At every order in perturbation theory, one has to solve the zeroth order (two derivative) equation sourced by the lower order solutions
- Runaway solutions are explicitly left out
- But only solutions to the full theory that are expandable in perturbation theory can be constructed (i.e., analytic)
- Unphysical secular growth in time



[Corman, Lehner, East, Dideron ’24]

Example: radiation reaction problem

- Motion of a non-relativistic, charged particle with charge q on a fixed external electric field:

$$\ddot{\mathbf{x}} = \frac{q}{m}\,\mathbf{E}(\mathbf{x}, t) + \tau\,\dddot{\mathbf{x}}$$

- Generic solutions exhibit runaway behaviour at late times, with $\ddot{\mathbf{x}} \propto e^{t/\tau}$

- They exit the regime of validity of the equation very quickly

- Solutions that remain bounded are non-generic: they require a fine-tune "pre-acceleration"

Example: radiation reaction problem

Reduction of order

- Idea: self-consistently modify the equation to obtain "equally accurate" solutions

- Differentiate (in time) the original equation to compute the higher derivatives time and substitute them into the higher derivative terms and expand in the small parameter:

$$\dddot{\mathbf{x}} = \frac{q}{m}\left[\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x},t) + (\dot{\mathbf{x}}\cdot\nabla)\mathbf{E}(\mathbf{x},t)\right] + \tau\,\ddddot{\mathbf{x}}$$

$$\Rightarrow \quad \ddot{\mathbf{x}} = \frac{q}{m}\left[\mathbf{E}(\mathbf{x},t) + \tau\,\frac{\partial \mathbf{E}}{\partial t}(\mathbf{x},t) + \tau(\dot{\mathbf{x}}\cdot\nabla)\mathbf{E}(\mathbf{x},t)\right] + O(\tau^2)$$

Example: radiation reaction problem

Reduction of order

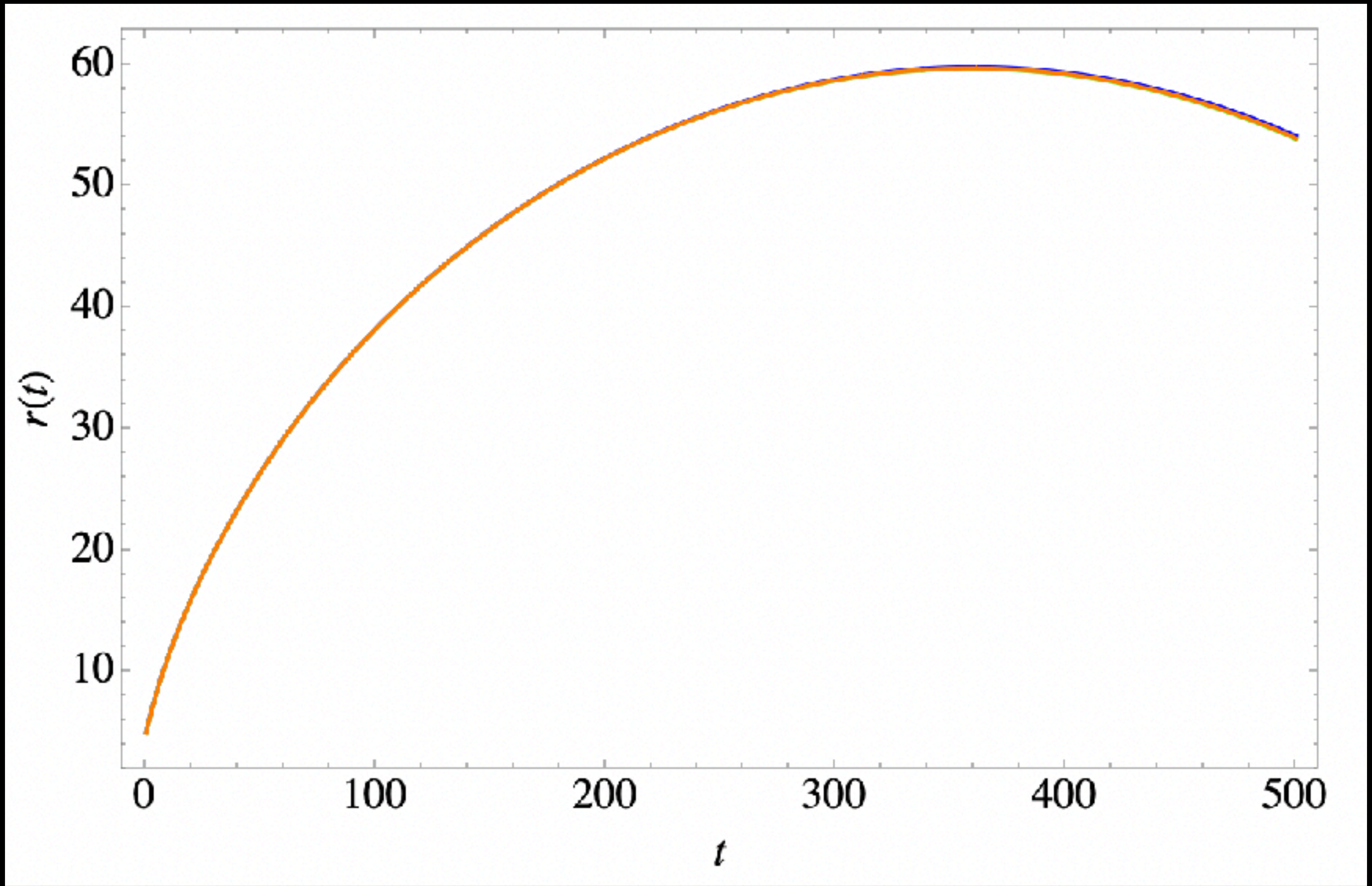
- Consider radial trajectories only (for simplicity) of a charged particle with charge q on an electric field created by a point particle of charge Q at the origin:

$$\ddot{r} = \frac{q\,Q}{m\,r^2} + \tau\,\dddot{r}$$

- Order reduction prescription:

$$\dddot{r} = -\frac{2\,q\,Q}{m\,r^3}\,\dot{r} + O(\tau)$$

$$\Rightarrow \ddot{r} = \frac{q\,Q}{m\,r^2}\left(1 - 2\,\tau\,\frac{\dot{r}}{r}\right) + O(\tau^2)$$



Reduction of order

- Not covariant!
- One generates higher order diffusive-type equations with no control of the signs!

Special theories with 2nd order eoms

4 derivative scalar-tensor theory

- Most general general scalar-tensor theory of gravity up to 4 derivatives [Weinberg '10]:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\left[R + X - V(\phi) + H(\phi)\,X^2 + \lambda(\phi)\mathscr{L}_{GB}\right]$$

$$X = -\tfrac{1}{2}(\nabla_a\phi)(\nabla^a\phi) \qquad \mathscr{L}_{GB} = R^2 - 4\,R_{ab}R^{ab} + R_{abcd}R^{abcd}$$

- EFT of inflation
- Horndeski class
- 2nd order eoms!

Einstein-Gauss-Bonet gravity

- Leading derivative correction to vacuum GR in D>4

$$S = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\,\left[R + \lambda\,\mathscr{L}_{GB}\right]$$

- 2nd order eoms!

mGHC/mCCZ4

- Horndeski and Lovelock theories are not well posed in harmonic gauge due to degeneracies [Papallo and Reall '17]

- Solution: break the degeneracies by introducing auxiliary metrics that control the propagation of different modes [Kovacs and Reall '20]

Evolving EGB with NR

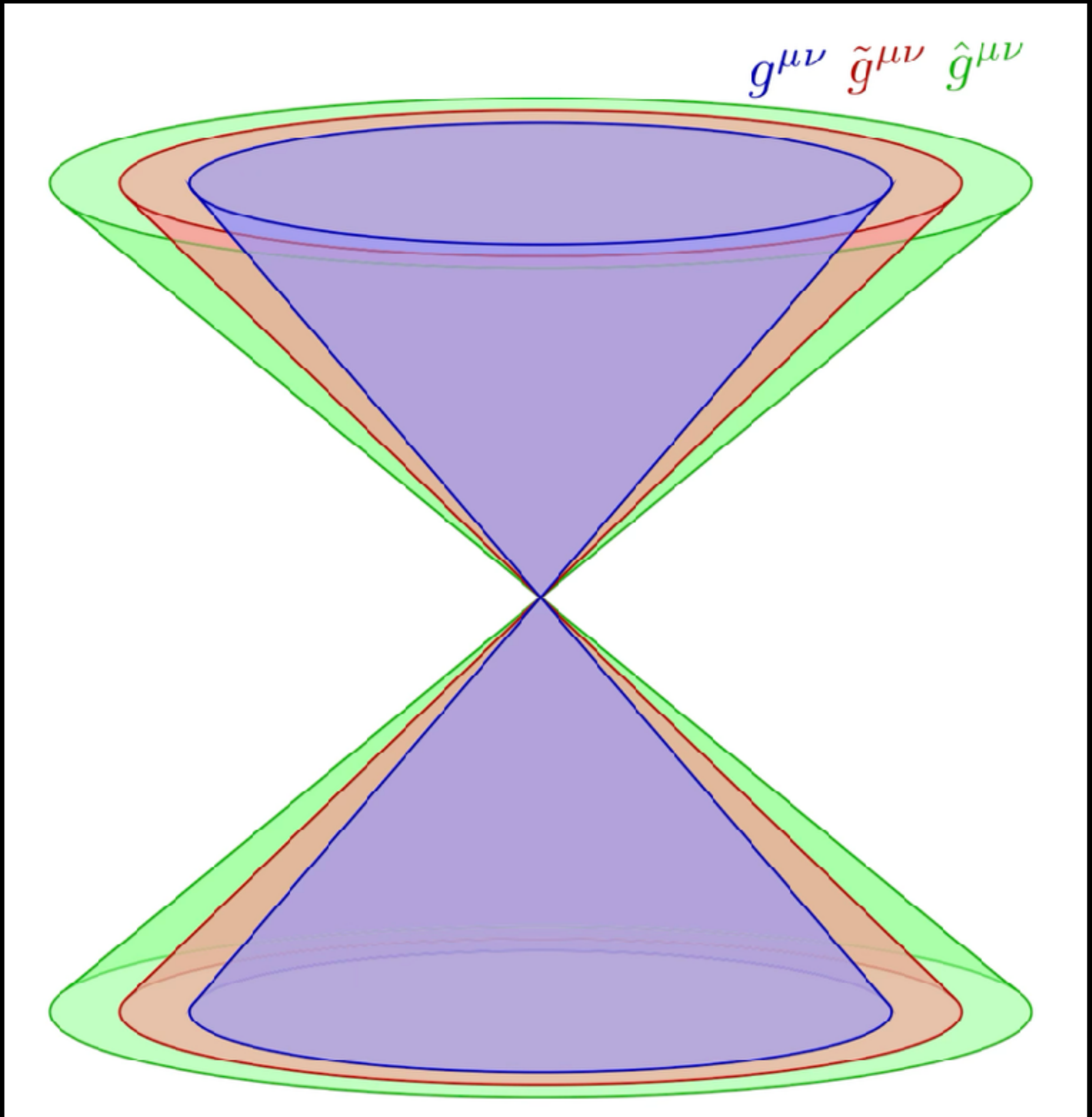
- Solution: break the degeneracies → modify the harmonic gauge conditions so that the unphysical modes propagate on the light cone of some auxiliary metrics [Kovacs and Reall '20]

$$\tilde{g}^{ab} = g^{ab} - a(x)\, n^a\, n^b$$
$$\hat{g}^{ab} = g^{ab} - b(x)\, n^a\, n^b$$

$$0 < a(x) < b(x)$$

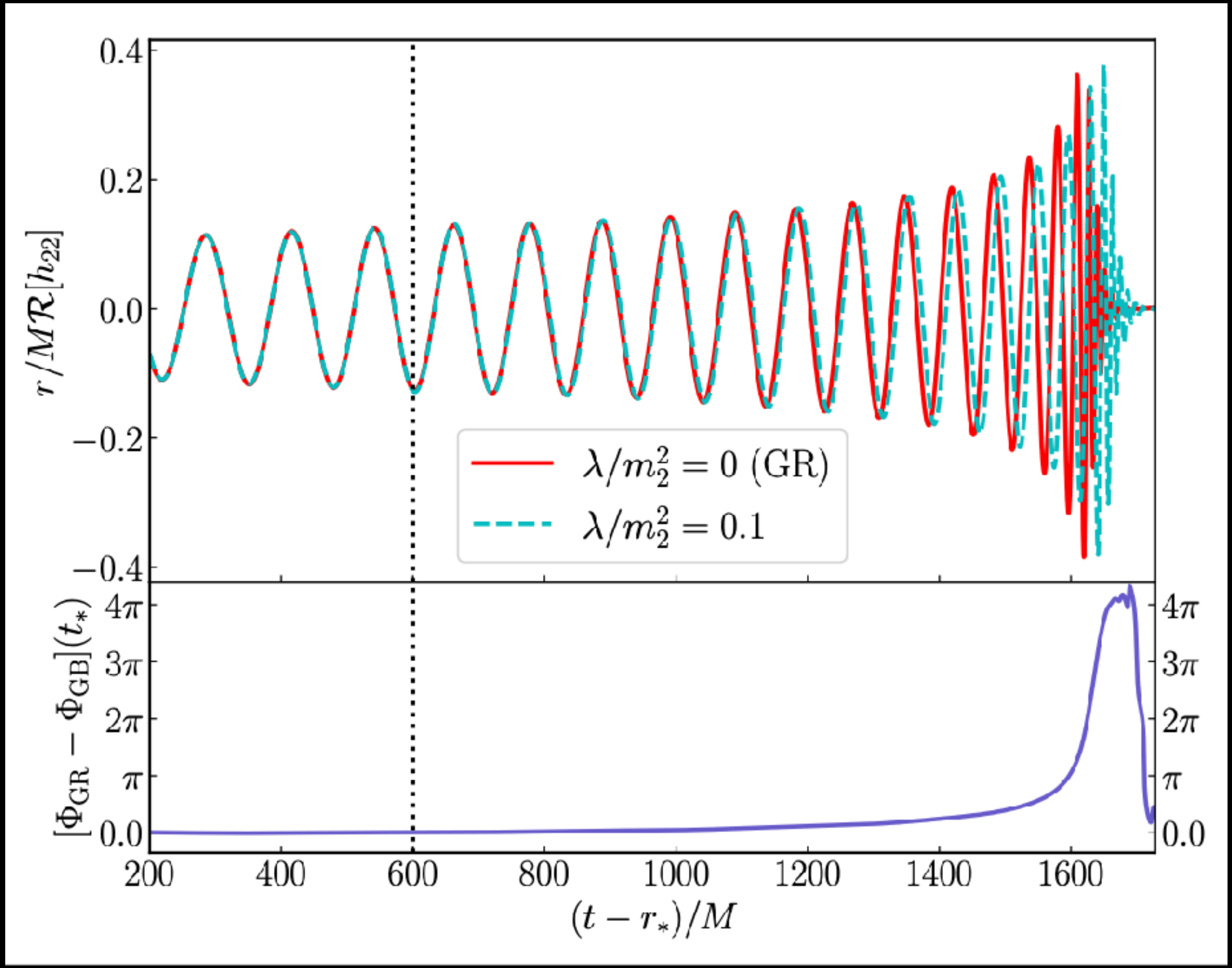
- Can be generalised to singularity avoiding coordinates (puncture gauge) [Aresté Saló, Clough and PF '23]

- Well-posedness only holds in the weakly coupled regime



$\lambda(\phi) = \frac{\lambda^{GB}}{4}\phi$ theory

[Aresté-Saló, Clough and PF ’23]

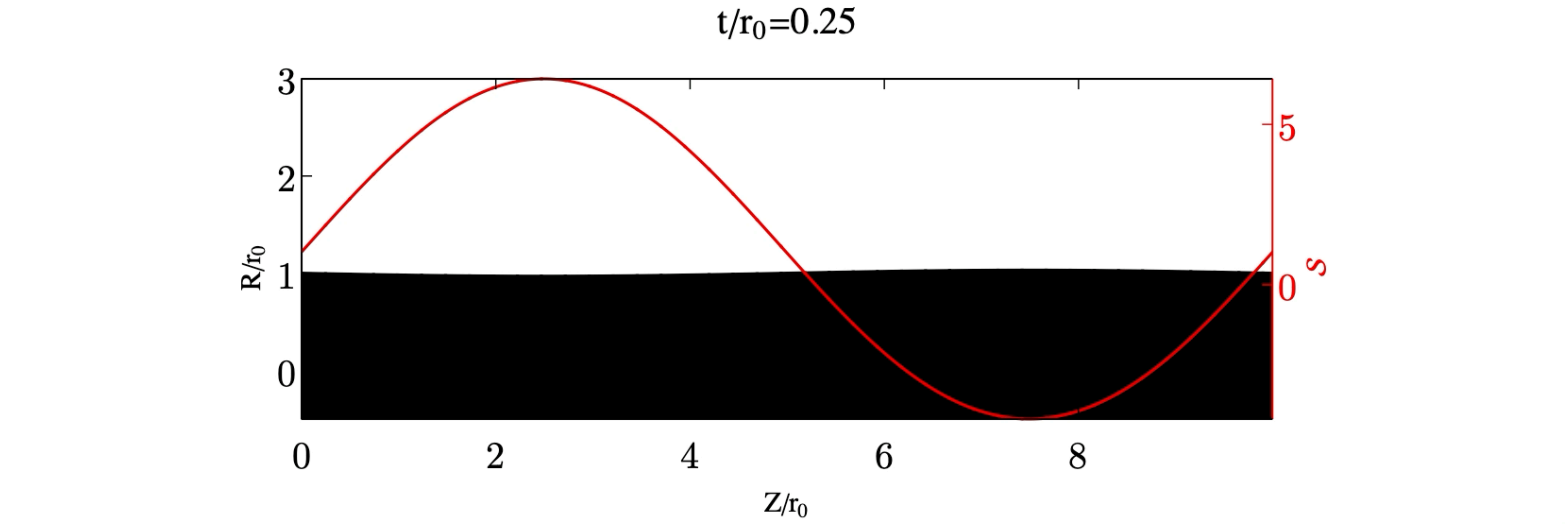


[Aresté-Saló, Clough and Corman ’25]

- Black hole binaries in this theory merge more slowly than in GR

GL instability of black strings in EGB

$$\lambda / r_0^2 = 10^{-5}$$



Fixing of the equations

8 derivative theory of gravity

[Cayuso, PF, França and Lehner ’23]

- Most general higher derivative theory of gravity (in vacuum) up to 8 derivatives:

$$I = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R - \frac{1}{\Lambda^6}\mathscr{C}^2 - \frac{1}{\tilde{\Lambda}^6}\tilde{\mathscr{C}}^2 - \frac{1}{\Lambda_-^6}\mathscr{C}\tilde{\mathscr{C}}\right)$$

$$\mathscr{C} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \quad \tilde{\mathscr{C}} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\,, \quad \tilde{R}^{\mu\nu\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}$$

➡ EOMs with 4th order derivatives ($\epsilon \equiv \Lambda^{-6}$):

$$\begin{aligned} G_{\mu\nu} = 8\epsilon\Big\{&\mathscr{C}\Big[\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu\nabla_\nu R - \frac{1}{16}\mathscr{C}\,g_{\mu\nu} - R_{\mu\lambda}R^\lambda{}_\nu \\ &+ R^{\alpha\beta}R_{\mu\alpha\nu\beta} + \frac{1}{2}R_{\mu\sigma\rho\lambda}R_\nu{}^{\sigma\rho\lambda}\Big] \\ &+ 2(\nabla^\alpha\mathscr{C})\Big[\nabla_\alpha R_{\mu\nu} - \nabla_{(\mu}R_{\nu)\alpha}\Big] + R_\mu{}^\alpha{}_\nu{}^\beta\nabla_\alpha\nabla_\beta\mathscr{C}\Big\} \end{aligned}$$

‘Fixing’ higher derivative theories of gravity

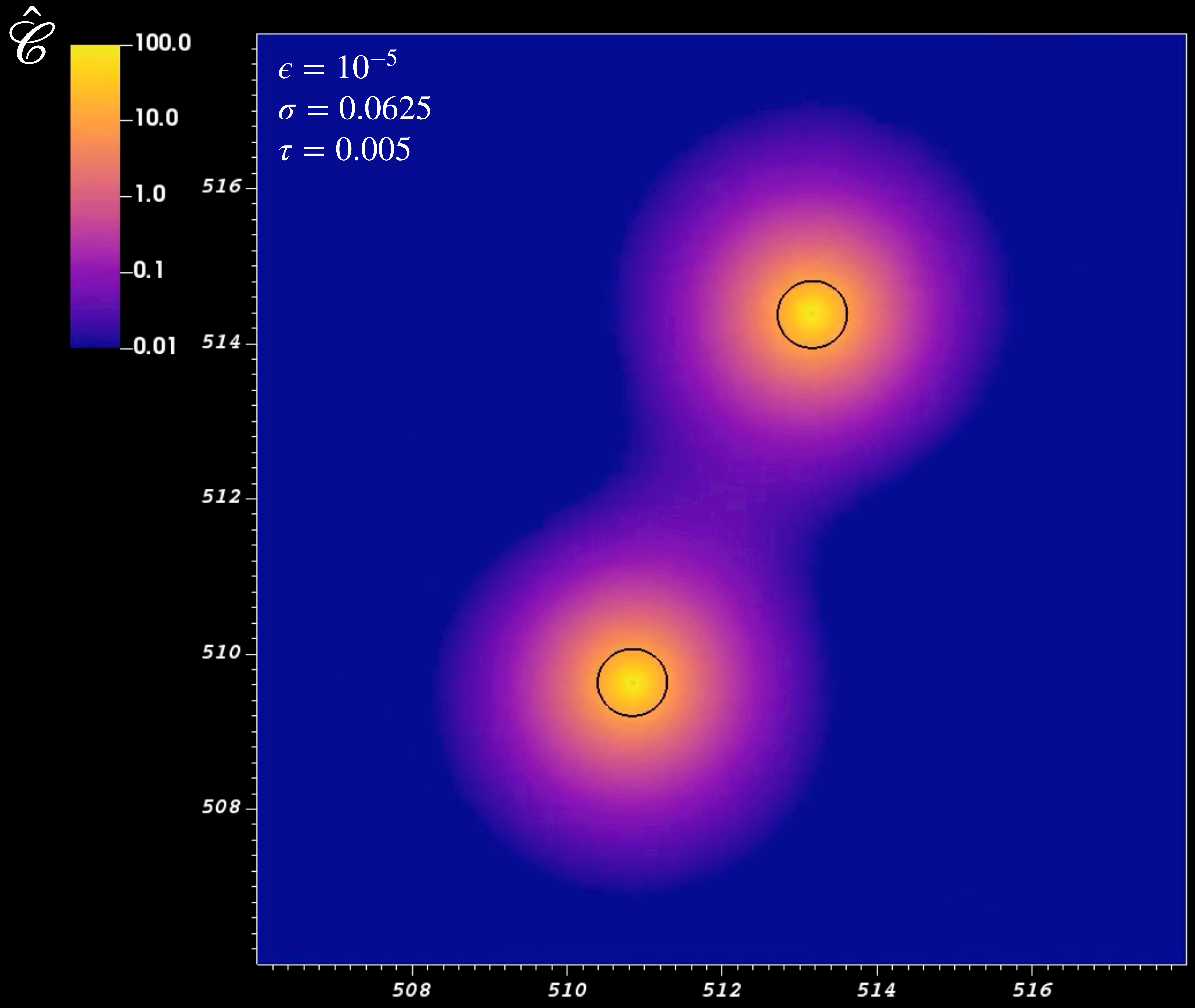
- Order reduction: Ric $\sim O(\epsilon)$ $\Rightarrow$ keep only the $O(\epsilon)$ terms in the EOMs

$$G_{\mu\nu} = \epsilon \left(4\, \mathscr{C}\, C_{\mu}{}^{\alpha\beta\gamma}\, C_{\nu\alpha\beta\gamma} - \frac{1}{2}\, g_{\mu\nu}\, \mathscr{C}^2 + 8\, C_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\, \nabla_{\alpha} \nabla_{\beta} \mathscr{C} \right), \quad \mathscr{C} = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$$

- Want:
    - Well-posed, 2nd order (in time) equations
    - Consistently incorporate the small corrections at long wavelengths whilst controlling to flow of energy to the UV
    - Study strong fields whilst remaining in the regime of validity of EFT

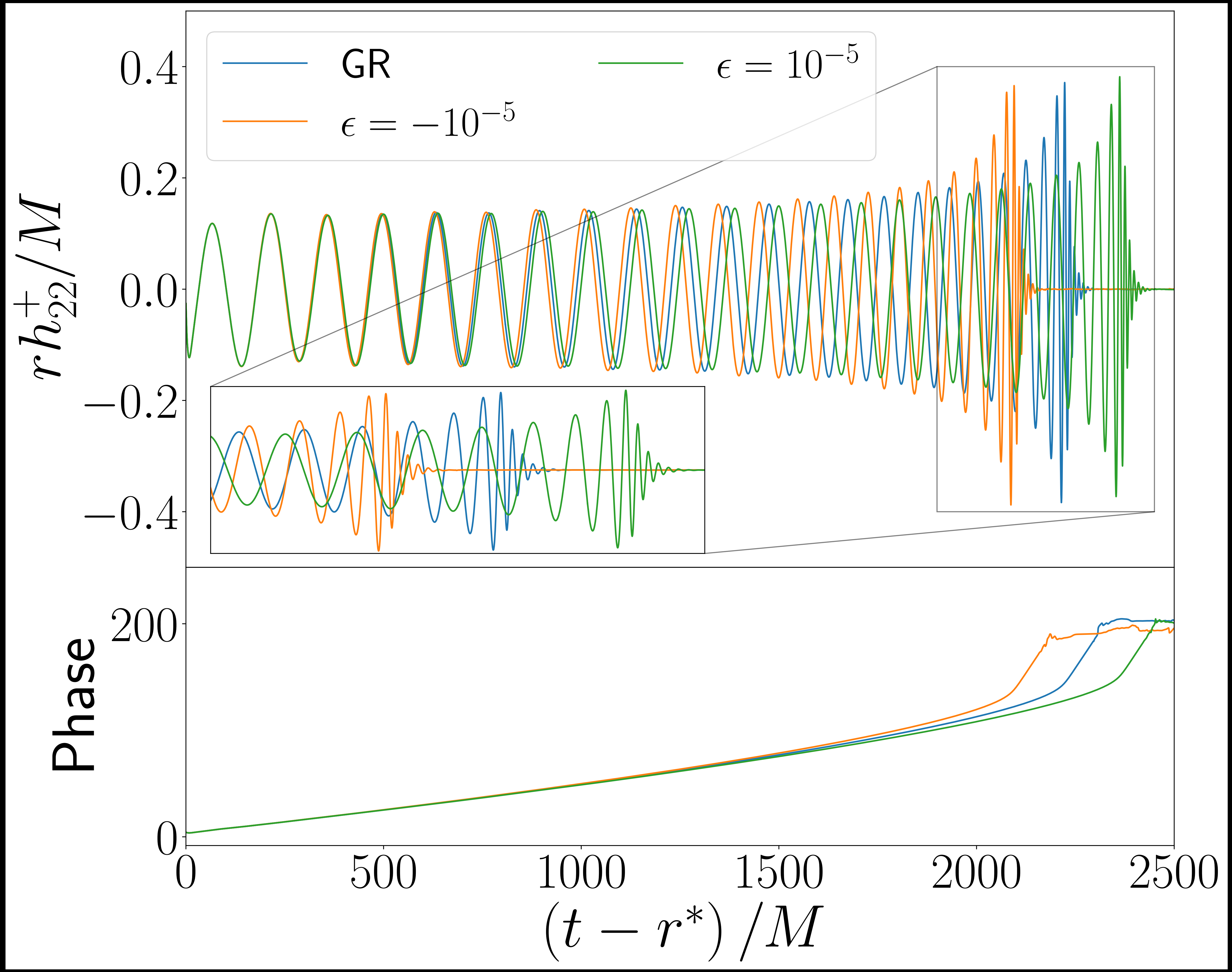
‘Fixing’ higher derivative theories of gravity

- Our solution: $$G_{\mu\nu} = \epsilon\left(4\,\hat{\mathscr{C}}\,C_{\mu}{}^{\alpha\beta\gamma}\,C_{\nu\alpha\beta\gamma} - \frac{1}{2}\,g_{\mu\nu}\,\hat{\mathscr{C}}^2 + 8\,C_{\mu}{}^{\alpha}{}_{\nu}{}^{\beta}\,\nabla_\alpha\nabla_\beta\hat{\mathscr{C}}\right)$$
$$(\partial_t^2 - 2\beta^i\partial_{ti} + \beta^i\beta^j\partial_{ij})\hat{\mathscr{C}} = \frac{1}{\sigma}\left(\mathscr{C} - \hat{\mathscr{C}} - \tau\partial_0\hat{\mathscr{C}}\right)$$
- Reduction of order to replace time derivatives on the RHS
- $\hat{\mathscr{C}} \to \mathscr{C}$ on a time scale set by σ/τ
- Consistency: the IR physics should NOT depend on the parameters σ, τ
- Stationary solutions should agree
- Other ‘fixes’ also work. Can we make a statement à la Geroch?



[Cayuso, PF, França, Lehner ’23]

‘Fixing’ higher derivative theories of gravity: results



Regularisation

[based on 2407.08775, 2505.17986, w/Held Kovacs and Yao]

Higher order EOMs

Regularisation

[PF, Held and Kovacs' 24]

- Observations:

  1. Perturbative field redefinitions can remove (or add) certain terms in the action without affecting the physics [Grosse-Knetter '94; Burgess '20]

  2. The highest derivative terms in the EOMs are the ones that matter the most for well-posedness, but such terms are the least important ones in EFT

- Basic idea:

  1. Perform field redefinitions to modify the highest derivative structure of the PDEs with terms proportional to the 0th order EOMs and their derivatives

  2. The new terms vanish on shell and inherit the good high frequency behaviour of the lower order EOMs (assumed to be well-posed)

Higher order EOMs

Regularisation

- Remarks

    1. The regularised higher derivative equations are only weakly hyperbolic: well-posedness is sensitive to some (but not all) of the lower derivative terms

    2. The relevant lower derivative terms have the right structure (inherited from the 0th order EOMs via field redefinitions) and the regularised EOMs are well-posed

    3. Initial data: it must be consistent with the EFT expansion, otherwise runaway solutions will occur

Classical Abelian-Higgs Model

- UV model:

$$S = -\int d^4x \left[(\partial^a \phi^*)(\partial_a \phi) + V(|\phi|^2)\right], \quad V(|\phi|^2) = \frac{\lambda}{2}\left(\phi^*\phi - \frac{v^2}{2}\right)^2$$

- Global U(1) symmetry: $\phi \rightarrow e^{i\alpha}\phi$

- Vacuum: $\phi^*\phi = \frac{v^2}{2}$ → spontaneous symmetry breaking leading to a spectrum containing a massive field with mass $M^2 = \lambda v^2$ and a massless Goldstone boson

Classical Abelian-Higgs Model

- Introducing $\phi(x) = \frac{v^2}{\sqrt{2}} \left[1 + \rho(x)\right] e^{\mathrm{i}\theta(x)}$ the action becomes

$$\frac{S}{v^2} = -\int d^4x \left[\frac{1}{2}(\partial_a\rho)(\partial^a\rho) + \frac{1}{2}(1+\rho)^2(\partial_a\theta)(\partial^a\theta) + V(\rho)\right] ,$$

$$V(\rho) = \frac{M^2}{2}\left(\rho^2 + \rho^3 + \frac{1}{4}\rho^4\right)$$

- Equations of motion: $(\Box - M^2)\rho = (1+\rho)(\partial_a\theta)(\partial^a\theta) + \frac{M^2}{2}(3\rho^2 + \rho^3)$

$$\Box\theta = -\frac{2}{1+\rho}(\partial_a\rho)(\partial^a\theta)$$

- Global existence [Dong, LeFloch and Wyatt '19]

- For large enough M and assuming suitable boundedness of θ, ρ and their derivatives, ρ can be "integrated out" [Reall and Warnick '21]

- Low energy effective action for the light field θ:

$$\frac{S}{v^2} \approx -\int d^4x \left[\frac{1}{2}(\partial_a\theta)(\partial^a\theta) - \frac{1}{2M^2}\big((\partial_a\theta)(\partial^a\theta)\big)^2 + \frac{2}{M^4}(\partial_a\partial_b\theta)(\partial^a\partial^c\theta)(\partial^b\theta)(\partial_c\theta) + O(M^{-6})\right]$$

Note: if we didn't know the UV theory, the coefficients at each order in the $1/M$ of the action would be unknown

- New (low energy) EOMs:

$$\begin{aligned}\Box\theta = {} & \frac{2}{M^2}\partial_a\left[(\partial\theta)^2\partial^a\theta\right] \\ & + \frac{4}{M^4}\Big[(\partial^a\partial^b\Box\theta)(\partial_a\theta)(\partial_b\theta) + (\Box\theta)(\partial^a\theta)(\partial_a\Box\theta) + (\partial^a\theta)(\partial_a\partial_b\theta)(\partial^b\Box\theta) \\ & \qquad + (\Box\theta)(\partial^a\partial^b\theta)(\partial_a\partial_b\theta) + 2(\partial^a\theta)(\partial^b\partial^c\theta)(\partial_a\partial_b\partial_c\theta)\Big]\end{aligned}$$

$$\Box\theta = \frac{2}{M^2}\partial_a\left[(\partial\theta)^2\partial^a\theta\right]$$

$$+\frac{4}{M^4}\Big[(\partial^a\partial^b\Box\theta)(\partial_a\theta)(\partial_b\theta) + (\Box\theta)(\partial^a\theta)(\partial_a\Box\theta) + (\partial^a\theta)(\partial_a\partial_b\theta)(\partial^b\Box\theta)$$

$$+(\Box\theta)(\partial^a\partial^b\theta)(\partial_a\partial_b\theta) + 2(\partial^a\theta)(\partial^b\partial^c\theta)(\partial_a\partial_b\partial_c\theta)\Big]$$

- 4th order EOMs: how do we construct solutions that are consistent with the EFT expansion?

- Make a perturbative field redefinition up to $O(1/M^4)$:

$$\theta \to \theta + \frac{\alpha_1}{M^2}\left[\Box\,\theta - \tfrac{1}{2}\mathbb{E}_2(\theta)\right] + \frac{\alpha_2}{M^4}\Box^2\,\theta + O(M^{-6})$$

- Regularised action:

$$\frac{S_{reg}}{v^2} \approx -\int d^4x\Big[\,\frac{1}{2}(\partial_a\theta)(\partial^a\theta) - \frac{1}{2M^2}\big((\partial_a\theta)(\partial^a\theta)\big)^2 + \frac{2}{M^4}(\partial_a\partial_b\theta)(\partial^a\partial^c\theta)(\partial^b\theta)(\partial_c\theta)$$
$$-\frac{\alpha_1}{M^2}\theta\,\Box^2\,\theta - \frac{\alpha_2}{M^4}\theta\,\Box^3\,\theta + O(M^{-6})\Big]$$

- New equations of motion for θ up to $O(1/M^4)$:

$$\Box\theta = \frac{2}{M^2}\partial_a\left[(\partial\theta)^2\partial^a\theta\right]$$

$$+\frac{4}{M^4}\left[(\partial^a\partial^b\Box\theta)(\partial_a\theta)(\partial_b\theta) + (\Box\theta)(\partial^a\theta)(\partial_a\Box\theta) + (\partial^a\theta)(\partial_a\partial_b\theta)(\partial^b\Box\theta)\right.$$

$$\left.+(\Box\theta)(\partial^a\partial^b\theta)(\partial_a\partial_b\theta) + 2(\partial^a\theta)(\partial^b\partial^c\theta)(\partial_a\partial_b\partial_c\theta)\right]$$

$$-\frac{2\,\alpha_1}{M^2}\Box^2\theta - \frac{2\,\alpha_2}{M^4}\Box^3\theta$$

- Linear problem:

$$0 = \Box\theta + \frac{2\alpha_1}{M^2}\Box^2\theta + \frac{2\alpha_2}{M^4}\Box^3\theta \;=\; \frac{2\alpha_2}{M^4}(\Box - m_+^2)(\Box - m_-^2)\Box\theta$$

$$m_\pm^2 = \frac{M^2}{2\alpha_2}\left(-\alpha_1 \pm \sqrt{\alpha_1^2 - 2\alpha_2}\right)$$

- Recast the linear system: $\Box\theta \equiv M^2\,\theta^{(0,1)} \quad \Rightarrow \quad 0 = (\Box - m_+^2)(\Box - m_-^2)\theta^{(0,1)}$

$$(\Box - m_-^2)\theta^{(0,1)} \equiv M^2\,\theta^{(0,2)} \quad \Rightarrow \quad 0 = (\Box - m_+^2)\theta^{(0,2)}$$

$$\begin{aligned} \Box\theta &= M^2\,\theta^{(0,1)} \\ (\Box - m_-^2)\theta^{(0,1)} &= M^2\,\theta^{(0,2)} \\ (\Box - m_+^2)\theta^{(0,2)} &= 0 \end{aligned}$$

- Bounded solutions for $m_\pm^2 > 0 \quad \Rightarrow \quad \alpha_1 < 0\,,\ 0 < \alpha_2 < \frac{1}{2}\alpha_1^2$

Full non-linear (higher derivative) problem

- Introduce new variables: $\Box\theta = M^2\,\theta^{(0,1)}\,, \quad (\Box - m_-^2)\theta^{(0,1)} = M^2\,\theta^{0,2)}\,,$

$$\theta_a^{(1,0)} \equiv \partial_a\theta\,, \quad \theta_{ab}^{(2,0)} \equiv \partial_a\partial_b\theta\,, \quad \theta_a^{(1,1)} \equiv \partial_a\theta^{(0,1)} = \frac{1}{M^2}\partial_a\Box\theta$$

- The EOMs can be written as:

$$\begin{aligned}
\Box\theta &= M^2\,\theta^{(0,1)}\,,\\
\Box\theta_a^{(1,0)} &= M^2\,\partial_a\theta^{(0,1)}\,,\\
\Box\theta_{ab}^{(2,0)} &= M^2\,\partial_{(a}\theta_{b)}^{(1,1)}\,,\\
(\Box - m_-^2)\theta^{(0,1)} &= M^2\,\theta^{(0,2)}\,,\\
(\Box - m_-^2)\theta_a^{(1,1)} &= M^2\,\partial_a\theta^{(0,2)}\,,\\
\left(\Box - m_+^2\right)\theta^{(0,2)} &= \frac{1}{\alpha_2}\left[\theta^{(0,1)}\eta^{ab} + \frac{2}{M^2}\,\theta^{(2,0)ab}\right]\theta_a^{(1,0)}\theta_b^{(1,0)}\\
&\quad + \frac{2}{\alpha_2}\left[\frac{1}{M^2}\,\theta^{(1,0)a}\theta^{(1,0)b}\partial_a\theta_b^{(1,1)} + \theta^{(0,1)}\,\theta^{(1,0)a}\,\theta_a^{(1,1)}\right.\\
&\quad + \frac{1}{M^2}\,\theta^{(1,0)a}\,\theta^{(1,1)b}\,\theta_{ab}^{(2,0)} + \frac{1}{M^2}\,\theta^{(0,1)}\,\theta^{(2,0)ab}\,\theta_{ab}^{(2,0)}\\
&\quad \left. + \frac{2}{M^4}\,\theta^{(1,0)a}\,\theta^{(2,0)bc}\,\partial_{(a}\theta_{bc)}^{(2,0)}\right]
\end{aligned}$$

- Diagonal system of wave equations → symmetric hyperbolic
- Degrees of freedom:
  - Original massless field θ
  - Two massive modes, $\theta^{(0,1)}$ and $\theta^{(0,2)}$, with masses $m_\pm \sim O(M)$
  - For $\alpha_1 < 0$ and $0 < \alpha_2 < \frac{1}{2}\alpha_1^2$, we have $m_\pm^2 > 0$
- The new massive modes, $\theta^{(0,1)}$ and $\theta^{(0,2)}$, are ghosts!
- We will provide numerical evidence that global solutions, compatible with the EFT expansion, can be constructed

Numerical experiments

Initial data

- More degrees of freedom → more data → higher time derivatives of θ at $t = 0$

$$\theta^{(0,1)}|_{t=0} = \frac{1}{M^2}\Box\theta|_{t=0} = \frac{1}{M^2}(-\partial_t^2\theta + \partial_x^2\theta)|_{t=0} \quad \Rightarrow \quad \text{need} \quad \partial_t^2\theta|_{t=0}$$

- Arbitrary initial data for the higher derivative fields is expected to lead to runaway solutions

- Initial data for the massive dofs that is consistent with the EFT expansion should be constructed from the IR dofs, namely $\theta|_{t=0}, \partial_t\theta|_{t=0}$

Initial data

- Main idea: impose the EOMs in terms of θ at $t = 0$, and use order reduction to compute, perturbatively in a $1/M$ expansion, the required higher time derivatives

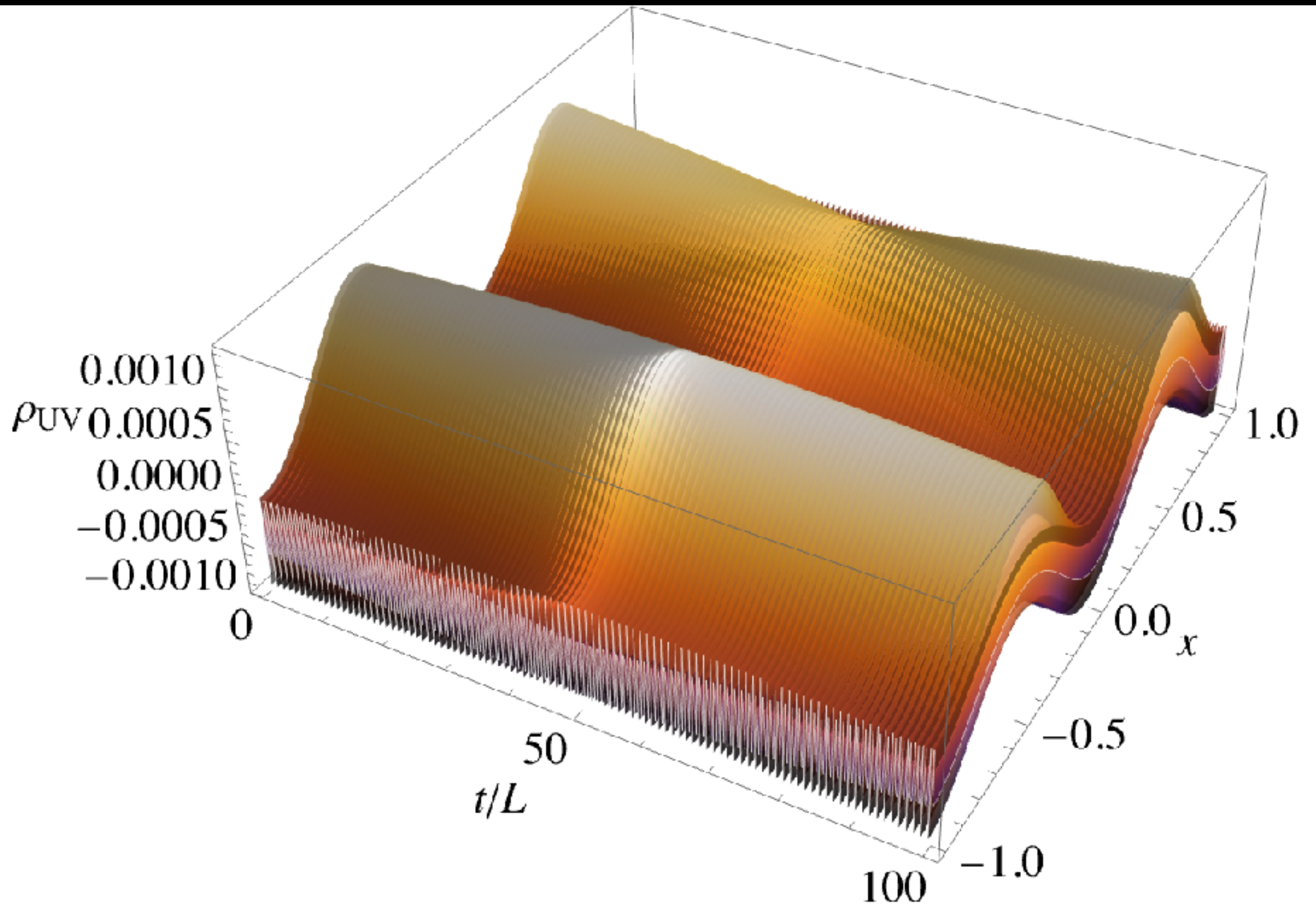
- At $O(1/M^2)$: $\quad \Box\theta = \frac{4}{M^2}(\partial^a\theta)(\partial^b\theta)(\partial_a\partial_b\theta) \quad @ \quad t = 0$

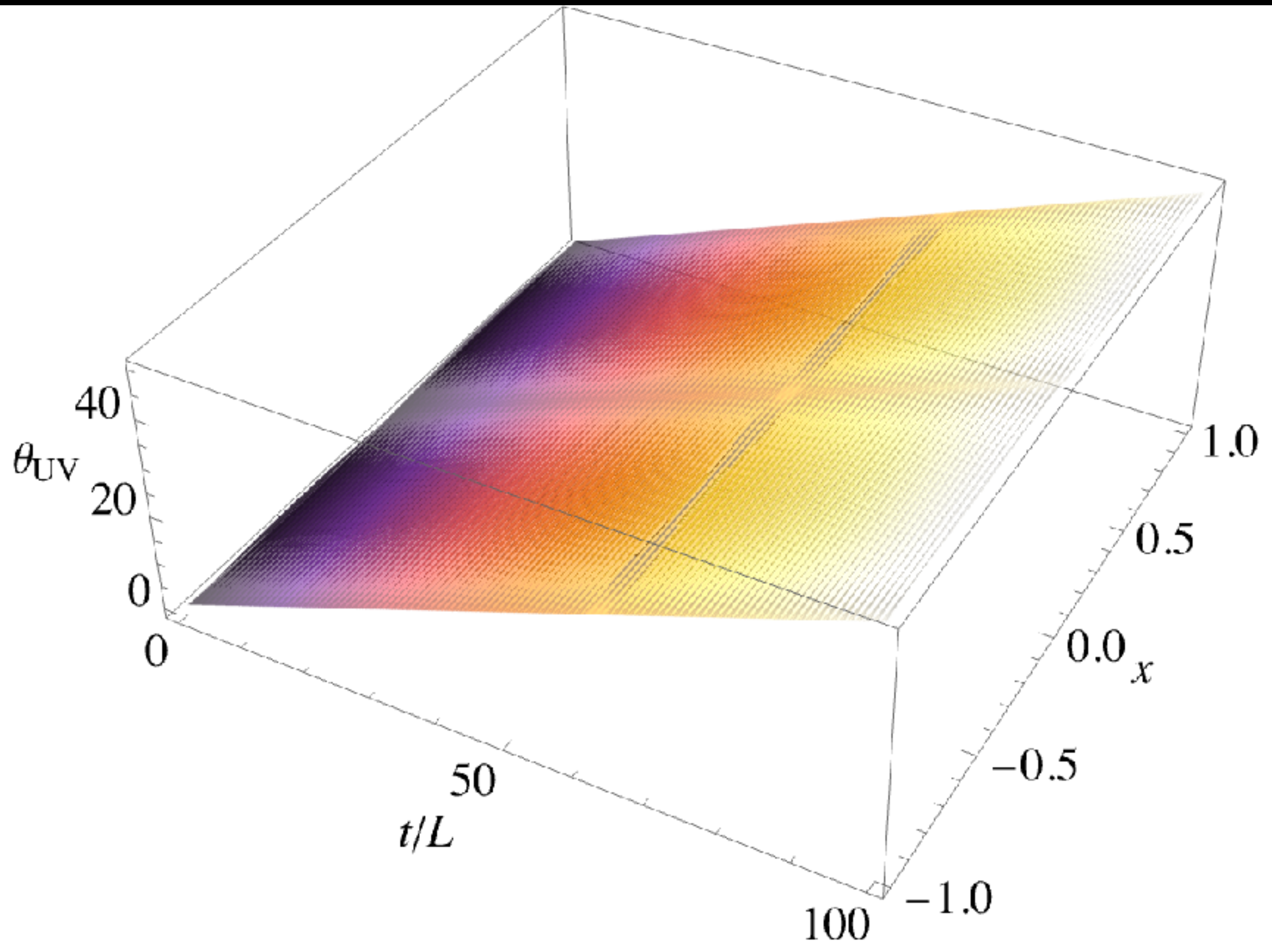
$$\Rightarrow \partial_t^2\theta|_{t=0} = \partial_x^2\theta|_{t=0}$$
$$-\frac{4}{M^2}\left[\partial_x^2\theta|_{t=0}\left((\partial_t\theta|_{t=0})^2 + (\partial_x\theta|_{t=0})^2\right) - 2\,\partial_x\partial_t\theta|_{t=0}\,\partial_t\theta|_{t=0}\,\partial_x\theta|_{t=0}\right]$$

- Covariance is preserved within the accuracy of the EFT [Gavassino, Kovacs and Reall wip]

Examples

$$\theta|_{t=0} = \sin(\pi x)\,, \quad \partial_t\theta|_{t=0} = \frac{1}{2}\cos^2(\pi x)\,, \quad M = 100\,, \quad L = 2$$





Examples

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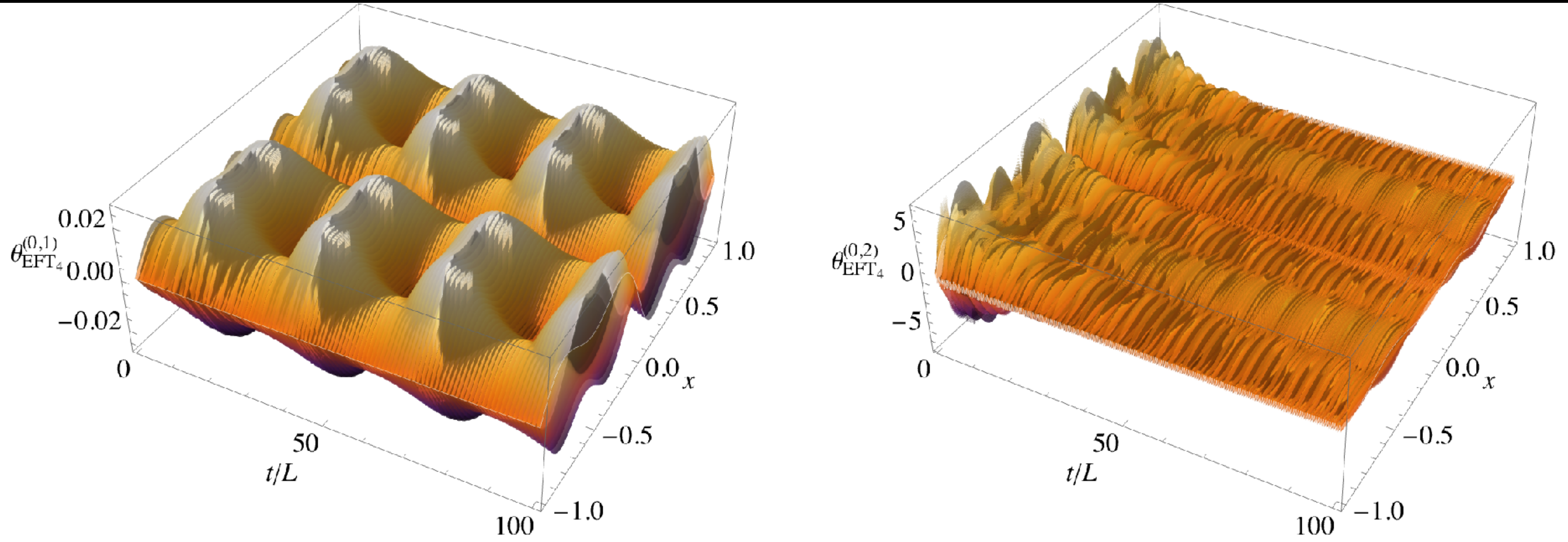

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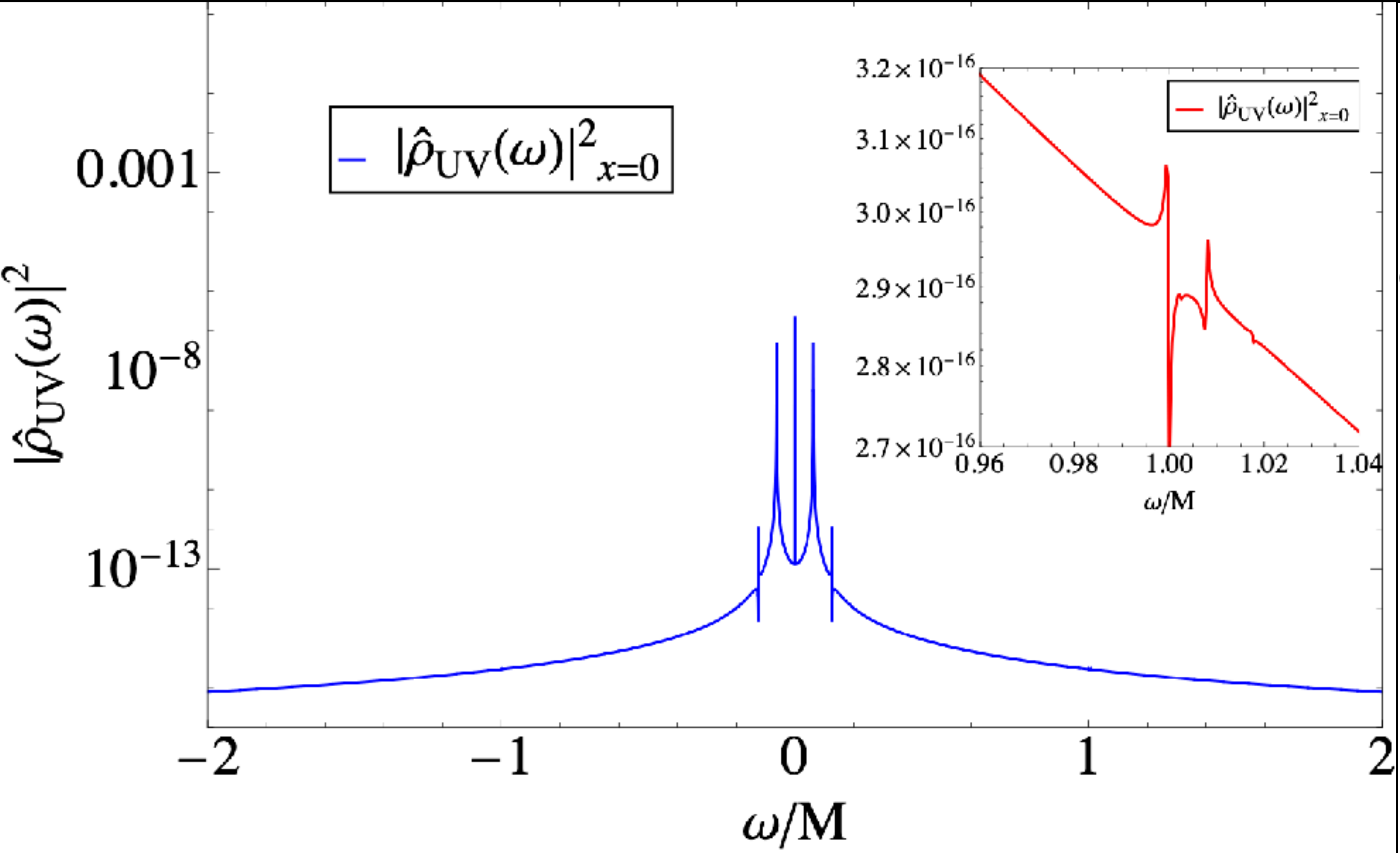

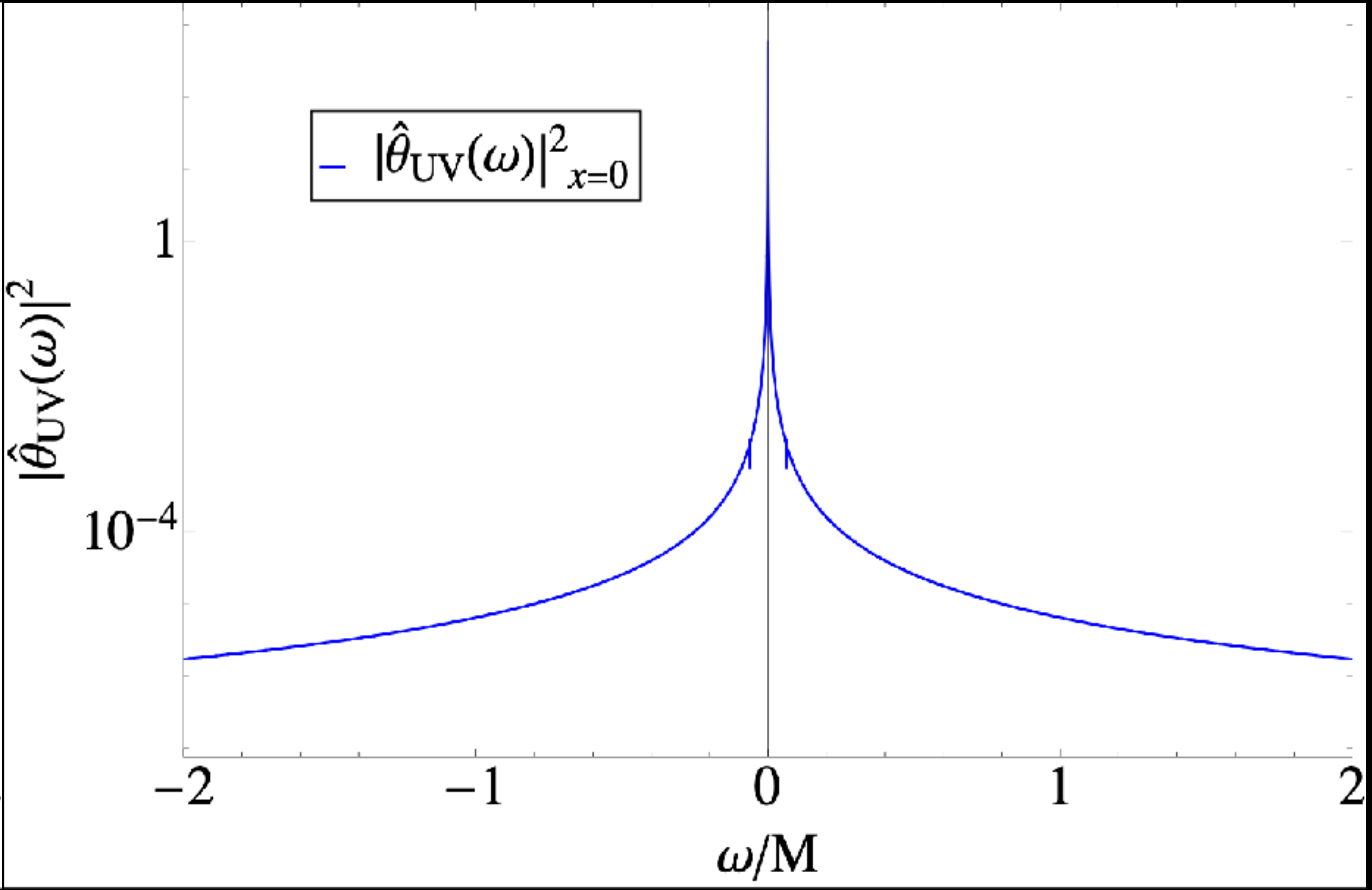

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$$\theta|_{t=0} = \sin(\pi x)\,, \quad \partial_t\theta|_{t=0} = \frac{1}{2}\cos^2(\pi x)\,, \quad M = 100\,, \quad L = 2$$

Breakdown of the EFT

- For simplicity, consider the EFT truncated to $O(1/M^2)$:

$$\frac{S}{v^2} \approx -\int d^4x \left[\frac{1}{2}(\partial_a\theta)(\partial^a\theta) - \frac{1}{2M^2}\big((\partial_a\theta)(\partial^a\theta)\big)^2 + O(M^{-4})\right]$$

- Two EFTs at this order:

$$\Box\theta = \frac{2}{M^2}\left[(\partial_a\theta)(\partial^a\theta)\Box\theta + 2(\partial^a\theta)(\partial^b\theta)\partial_a\partial_b\theta\right]$$

EFT$_1$: $\Rightarrow\ 0 = G^{ab}\partial_a\partial_b\theta = \left[\left(1 - \frac{2}{M^2}(\partial\theta)^2\right)\eta^{ab} - \frac{4}{M^2}(\partial^a\theta)(\partial^b\theta)\right]\partial_a\partial_b\theta$

→ G^{ab} is Lorentzian for $(\partial\theta)^2 < \frac{1}{6}M^2$

Breakdown of the EFT

EFT$_2$:

$$\Box\,\theta = M^2\,\theta^{(0,1)}$$

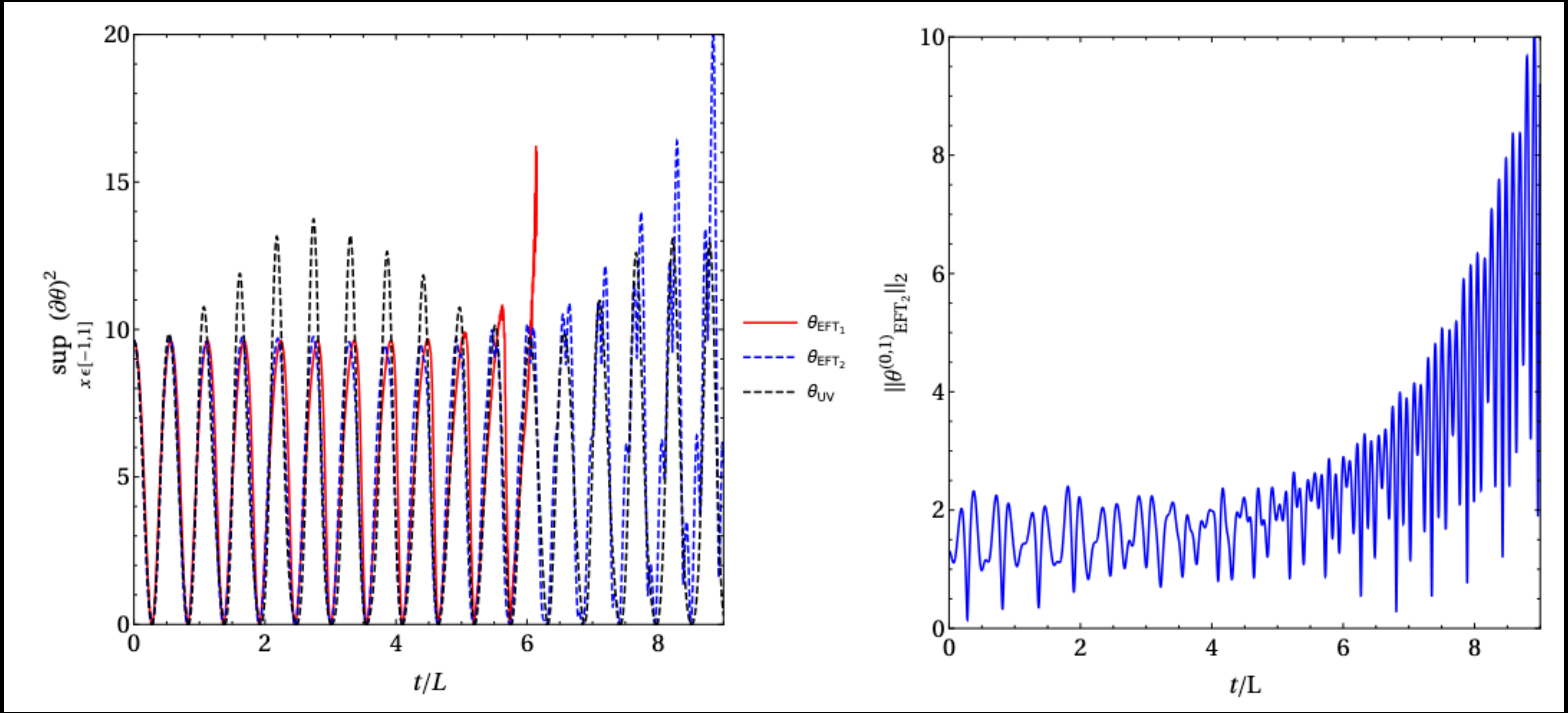
$$\Box\,\theta_a^{(1,0)} = M^2\,\partial_a\theta^{(0,1)}$$

$$\left(\Box + \frac{M^2}{2\,\alpha}\right)\theta^{(0,1)} = \frac{1}{\alpha}\left(\theta_a^{(1,0)}\,\theta^{(1,0)a}\,\theta^{(0,1)} + \frac{2}{M^2}\,\theta^{(1,0)a}\,\theta^{(1,0)b}\,\partial_a\theta_b^{(1,0)}\right)$$

→ the propagation of all modes is controlled by the spacetime metric

Breakdown of the EFT

$$\theta|_{t=0} = \sin(\pi x)\,, \quad \partial_t\theta|_{t=0} = \frac{1}{2}\cos^2(\pi x)\,, \quad M = 10\,, \quad L = 2 \quad \Rightarrow \quad \partial_a\theta \sim O(M)$$



Ghosts

- The regularised EFTs admit a conserved energy:

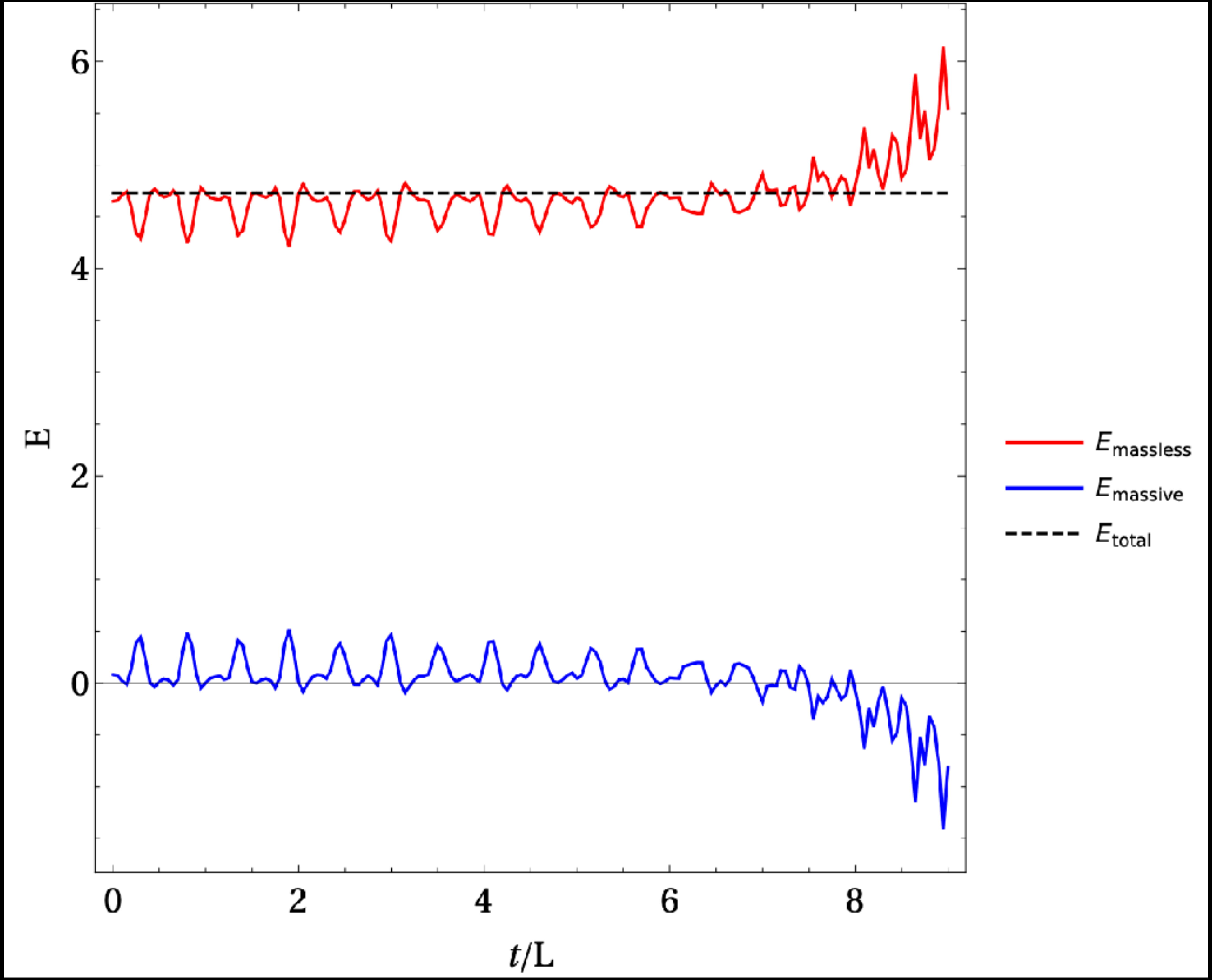
$$\mathscr{E}_{\mathrm{EFT}_2}[\theta](t) = \int \mathrm{d}x \left[\frac{1}{2}(\partial_t\theta)^2 + \frac{1}{2}(\partial_x\theta)^2 - \frac{1}{2M^2}\left(-(\partial_t\theta)^2 + (\partial_x\theta)^2\right)\left(3\,(\partial_t\theta)^2 + (\partial_x\theta)^2\right) + \frac{2\alpha}{M^2}\left((\partial_t\theta)\,\partial_t\theta^{(0,1)} + (\partial_x\theta)\,\partial_x\theta^{(0,1)} + \frac{1}{2}(\theta^{(0,1)})^2\right)\right]$$

Energy for a massless field

Higher derivative correction

Contribution from the regularisation

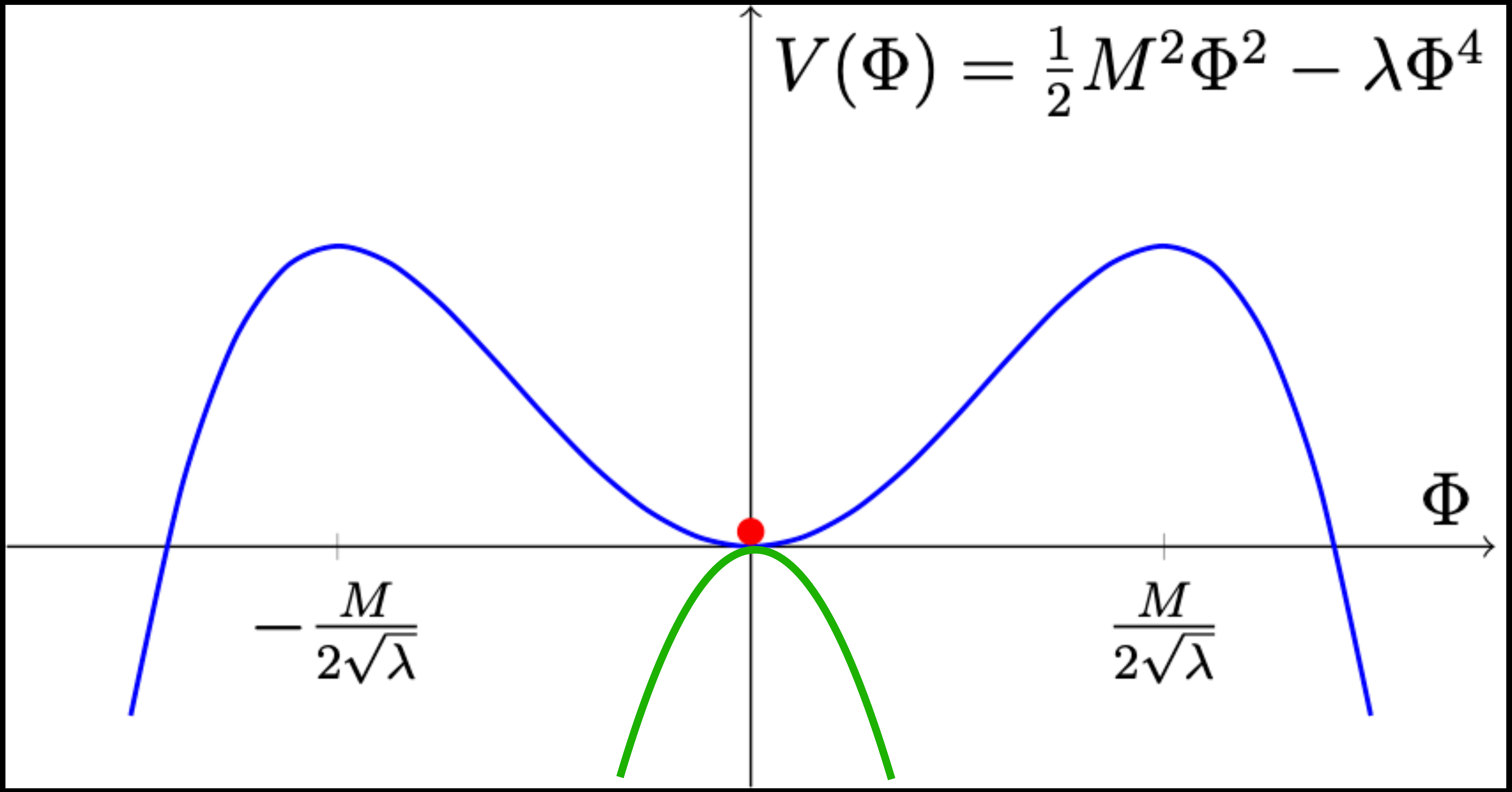
Ghosts



Why ghosts are not a problem?

- Model equation for the ghost fields (higher derivatives):

$$(\Box - M^2)\Phi = -4\lambda\Phi^3$$



- Global existence in 3+1 dimensions for small data and $M = 0$ [Sogge' 95]

- Better behaviour for $M \neq 0$
[Klainerman '85; Le Floch and Ma '14]

I AIN'T AFRAID OF NO GHOST

Regularised higher derivative theories of gravity

General result

[PF, Held and Kovacs '24]

- **Theorem**: any higher derivative theory of pure gravity that derives from a Lagrangian of the form

$$\mathscr{L} = R + \sum_{m\geq 2}^{N} \ell^m \mathscr{L}_m$$

where $\mathscr{L}_m$ is any scalar built out of the metric (and its derivatives) and the volume form, admits a well-posed initial value problem after a field redefinition that adds a regularising Lagrangian of the form

$$\mathscr{L}_{reg} = \sum_{k=0}^{n} \alpha_k \ell^{2k+2} R^{ab} \Box^k G_{ab}$$

Scalar-tensor theories of gravity

[to appear PF, Kovacs and Yao]

- Field redefinition by 0th order EOM → new action:

$$S_{reg} = \int d^4x\sqrt{-g}\Big\{ \frac{M_{Pl}^2}{2} R - \frac{1}{2}(\nabla_a\phi)(\nabla^a\phi)$$
$$+\ell^2\Big(M_{Pl}\,G(\phi)\,\mathscr{L}_{GB} + M_{Pl}\,F(\phi)\,\tilde{R}_{abcd}\,R^{abcd} + \frac{1}{4\,M_{Pl}^2}\,H(\phi)\big((\nabla_a\phi)(\nabla^a\phi)\big)^2\Big)$$
$$+\alpha\,\ell^2\Big[\big(M_{Pl}^2\,R^{ab} - 2(\nabla^a\phi)(\nabla^b\phi)\big)G_{ab} + \frac{1}{2\,M_{Pl}^2}\big((\nabla_a\phi)(\nabla^a\phi)\big)^2\Big]$$
$$+\gamma\,\ell^2\,\phi\,\Box^2\,\phi\Big\},$$

- Define $\phi_a^{(1,0)} \equiv \nabla_a \phi$. Then, the equations of motion can be written as

$$
\begin{aligned}
R_{ab}[g] &= R_{ab} \\
\Box \phi &= \phi^{(0,1)} \\
\Box \phi_a^{(1,0)} &= \nabla_a \phi^{(0,1)} + R_{ab} \phi^{(1,0)b} \\
\left(\Box + \frac{1}{2\gamma \ell^2} \right) \phi^{(0,1)} &= \mathscr{F}_\phi(\phi^{(0,1)}, \phi^{(1,0)}, \partial\phi^{(1,0)}, g, R, W) \\
\left(\Box + \frac{1}{2\,\alpha\,\ell^2} \right) R_{ab} &= \mathscr{F}_R(\phi^{(0,1)}, \phi^{(1,0)}, \partial\phi^{(1,0)}, g, R, W) \\
\mathscr{L}_n E &= -\varepsilon D B + \mathscr{F}_E(\gamma, K, E, B, R) \\
\mathscr{L}_n B &= +\varepsilon D E + \mathscr{F}_B(\gamma, K, E, B, R)
\end{aligned}
$$

➡ wave equations for g_{ab} , ϕ (massless) and $\Box \phi$, R_{ab} (massive)

➡ Auxiliary variables: $\phi_a^{(1,0)}, E_{ab}, B_{ab}$

➡ Need to impose $\alpha < 0, \gamma < 0$

Cubic gravity

[to appear PF, Kovacs and Yao]

- Leading order higher derivative correction to Einstein's gravity in 3+1 dimensions:

$$S = \int d^4x\sqrt{-g}\left[R + \ell^4\left(c_3 R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + \tilde{c}_3 R_{ab}{}^{cd} R_{cd}{}^{ef} \tilde{R}_{ef}{}^{ab}\right)\right]$$

→ 4th order EOMs

- Field redefinition by 0th order EOM → new action:

$$S_{reg} = \int d^4x\sqrt{-g}\left[R + \ell^4\left(c_3 R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab} + \tilde{c}_3 R_{ab}{}^{cd} R_{cd}{}^{ef} \tilde{R}_{ef}{}^{ab}\right)\right.$$
$$\left. + \alpha_0 \ell^2 R^{ab} G_{ab} + \alpha_1 \ell^4 R^{ab} \Box G_{ab}\right]$$

Cubic gravity

$$
\begin{aligned}
R_{ab}[g] &= R_{ab} \\
\left(\Box - m_-^2\right) R_{ab} &= \ell^2 R_{ab}^{(0,1)} \\
\left(\Box - m_+^2\right) R_{ab}^{(0,1)} &= \mathscr{F}_{(0,1)}(g, \partial g, R, R^{(1,0)}, \partial R^{(1,0)}, R^{(0,1)}, W, \partial W) \\
\Box R_{abc}^{(1,0)} &= \mathscr{F}_{(1,0)}(g, \partial g, R, R^{(1,0)}, \partial R^{(0,1)}) \\
\Box W_{abcd} &= \mathscr{F}_W(g, \partial g, R, R^{(1,0)}, \partial R^{(1,0)}, R^{(0,1)}, W)
\end{aligned}
$$

➡ EOMs: wave equations for g_{ab} (massless) and R_{ab} , $R_{ab}^{(0,1)}$ (2 massive scalars + 2 massive spin 2 tensors)

➡ Auxiliary variables: $R_{abc}^{(1,0)} \equiv \nabla_a R_{bc}$, W_{abcd}

➡ m_+ and m_- depend on the α_k's and can always be chosen s.t. $m_+^2, m_-^2 > 0$

Summary and Conclusions

Summary and Conclusions

- We have various methods to simulate black hole binary mergers in large classes of physically interesting higher derivative theories of gravity
- New proposal: Regularisation
- The higher derivatives are packed in new massive degrees of freedom that are ghosts
- For initial data consistent with the EFT expansion, runaway solutions don't occur and the regularised EFT provides a consistent description of the long distance physics
- Regularisation can be generalised to non-relativistic theories and it does not require an action [wip w/ R. M. Wald]

Thank you!

Sketch of proof

- For n sufficiently large (depending on N), the equations of motion for the regularised theory are

$$\Box^{n+1} R_{ab} = F_{ab}$$

where F_{ab} is a sum of monomials built out of the metric, $W^{(l)}_{abcde_1\dots e_l} \equiv \nabla_{e_1}\dots\nabla_{e_l} W_{abcd}$ and $R^{(p,q)}_{abc_1\dots c_p} \equiv \nabla_{c_1}\dots\nabla_{c_p}\Box^q R_{ab}$ with $0 \leq l \leq n, p+q \leq n+1$ and $q \leq n-1$

- This system can be written as $g^{\alpha\beta}\partial_\alpha\partial_\beta \nu_A = F_A(\nu, \partial\nu)$, where F_A is a set of tensor-valued polynomials of its arguments and ν stands for $g_{\alpha\beta}$, $W^{(k)}$ with $0 \leq k \leq n-1, R^{(p,q)}$ with $p+q \leq n$ and $q \leq n$

- One uses the Bianchi identities to derive wave equations for $W^{(k)}$ and $R^{(p,q)}$ with $p \geq 1$ and $q < n$:

$$\Box W^{(k)} = F_W^{(k)}$$

$$\Box R^{(p,q)} = R^{(p,q+1)} + F_R^{(p,q)}$$

- For the variables $R^{(p,q)}$ with $p = 0$ and $q < n$ we simply have

$$\Box R^{(0,q)} = R^{(0,q+1)}$$

- $R^{(0,n)}$ is evolved with the original equation: $\Box^{n+1} R_{ab} = F_{ab}$

- Initial data consists of $(\Sigma, \gamma_{ij}, K_{ij}, \rho_{ij}^{(0)}, \ldots, \rho_{ij}^{(2n+1)})$ where $\rho_{ij}^{(m)} \equiv (\mathscr{L}_n)^m R_{ij}$

- Constraints propagate