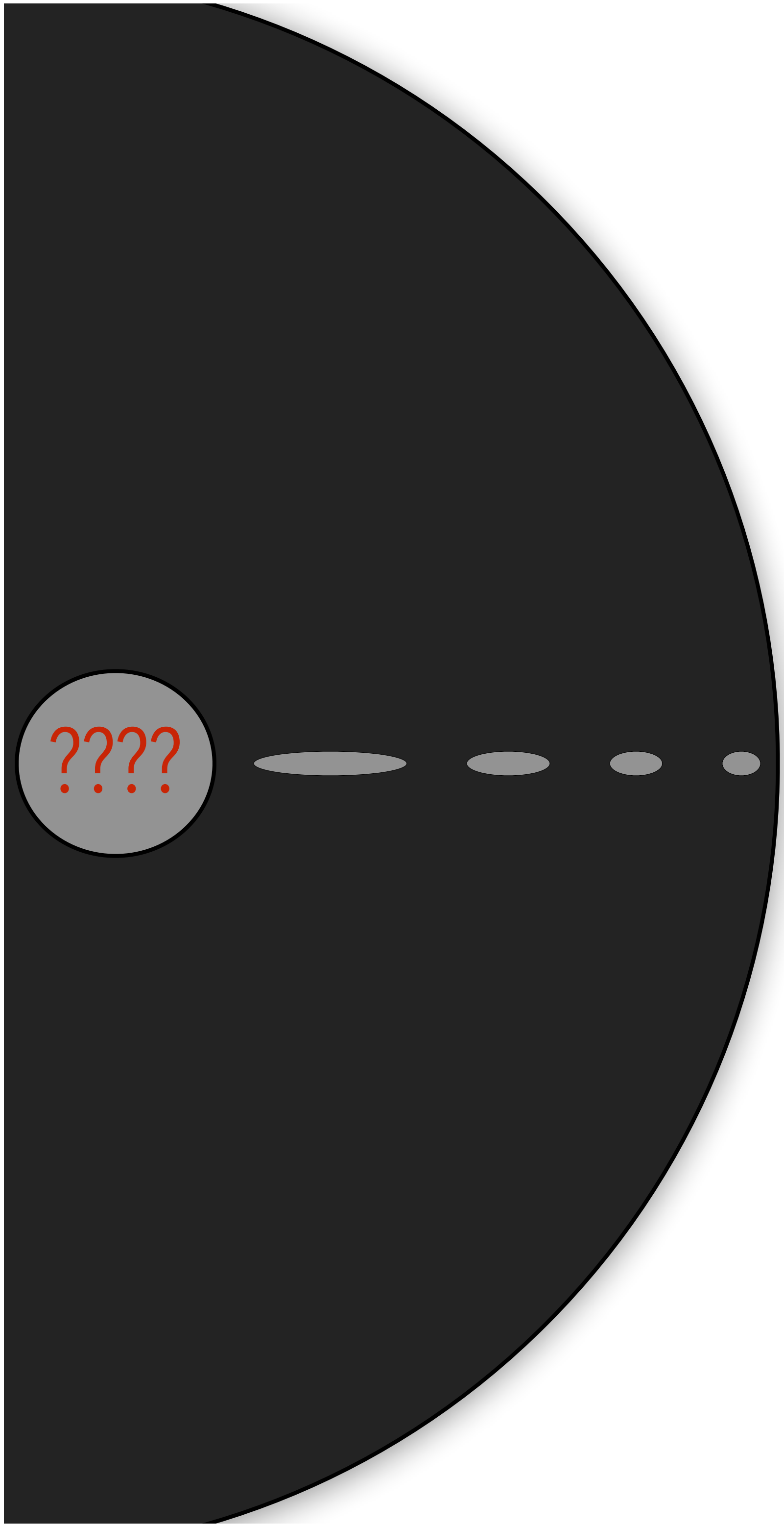


Black Hole Geometry and Evolution:

the stability operator and birfucation theory

DBHM & GWG, Trieste

Ivan Booth
Graham Cox
Mary Chiamaka Okpala (MSc)

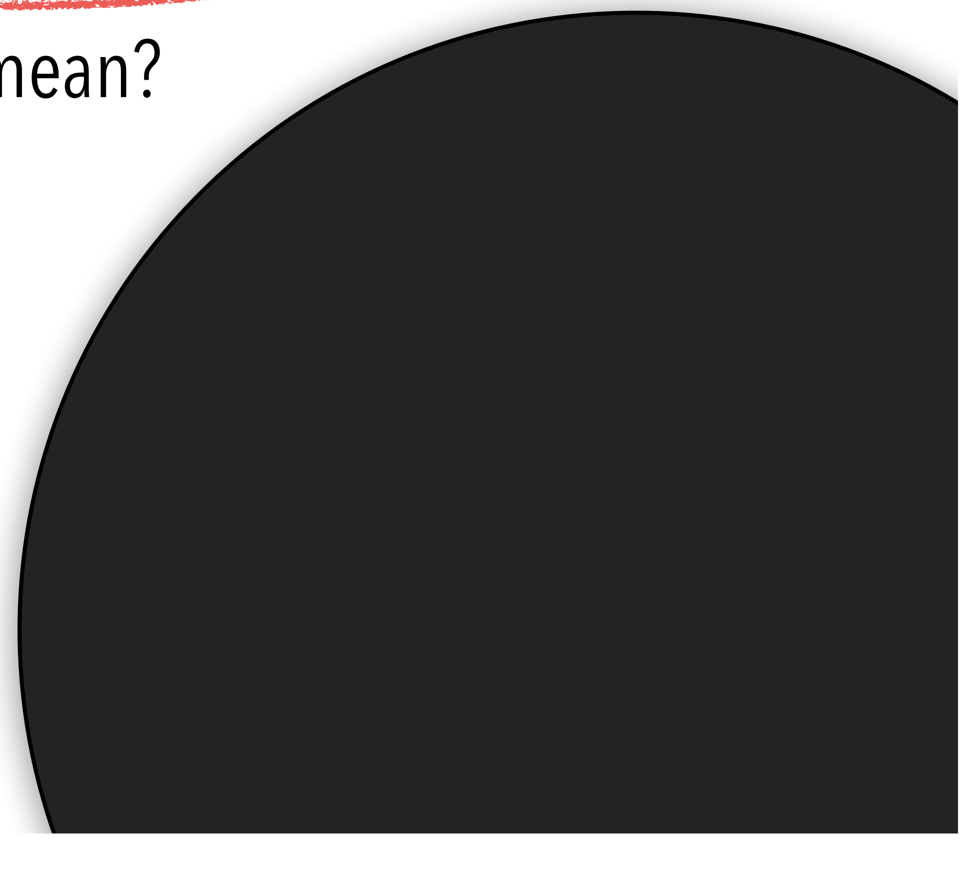
MEMORIAL UNIVERSITY

NSERC CRSNG

Also: R.Hennigar, Hari Kunduri, Sarah Muth, Lucy Newhook, Saikat Mondal, Daniel Pook-Kolb, Billy Sievers (Chan)

Physics Questions

- During a merger what happens to the original black holes?
- What else happens inside after the final apparent horizon forms?
- More specifically : Do black holes end? How?
- What does this question even mean?



How do black holes end?

How do

black holes

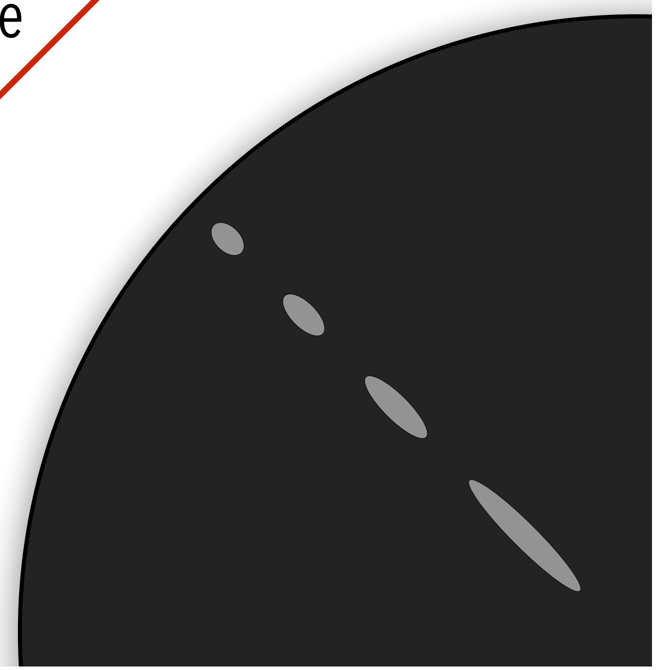
end?

Should be:

i. gravitationally bound region from which light can't escape

~~ii. distinguishable from surrounding spacetime~~

- no longer distinct



How do black holes end?

How do

- ``external fields'' become so strong that the BH can no longer be distinguished

black holes

Should be:

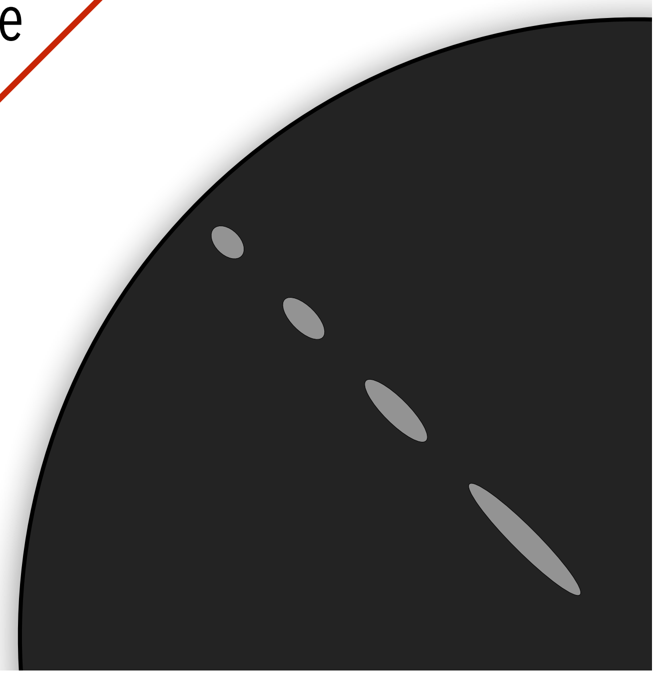
i. gravitationally bound region from which light can't escape

~~ii. distinguishable from surrounding spacetime~~

end?

- no longer distinct

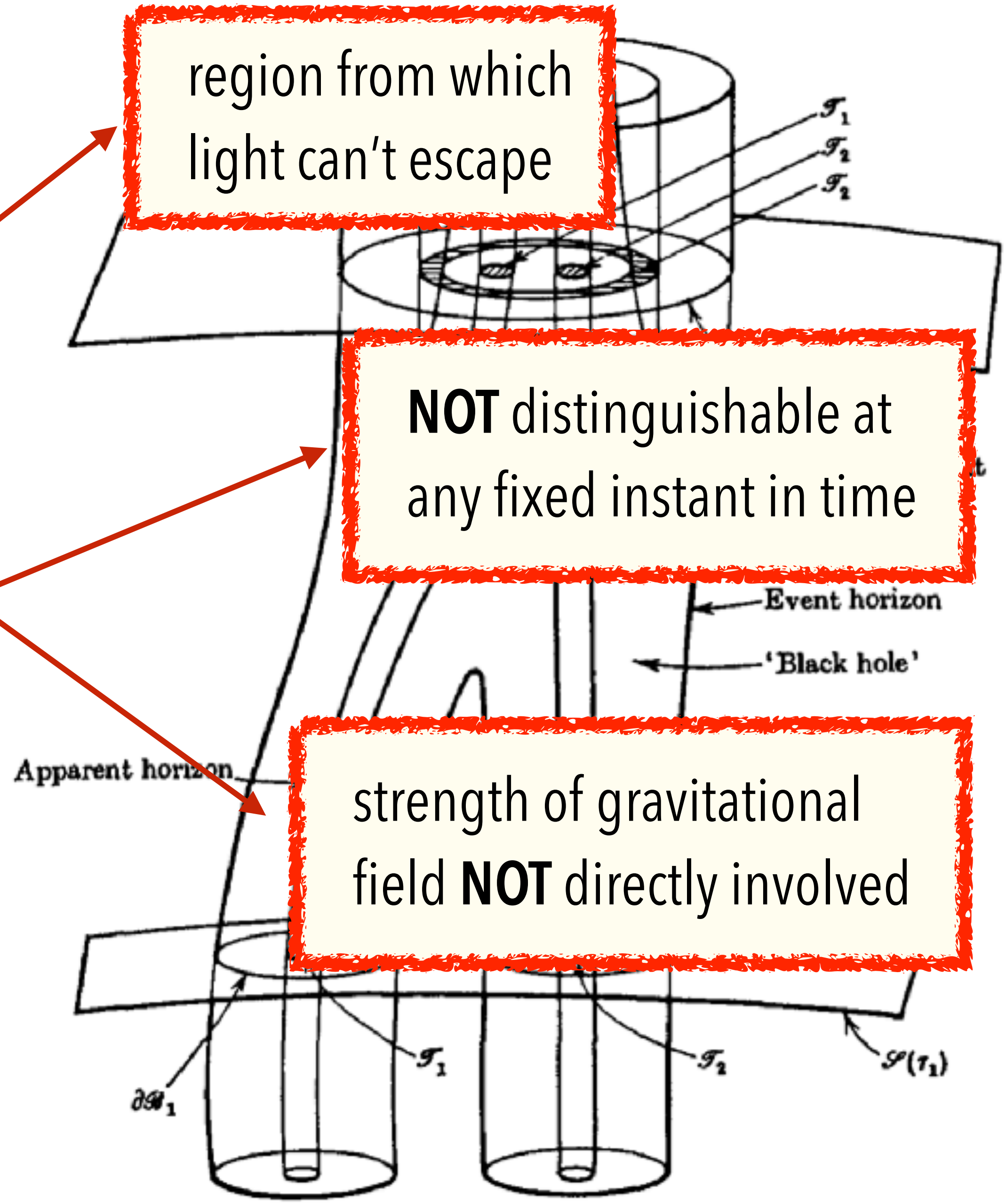
``Dissolved''



Black hole mergers and event horizons

Causal black holes

- regions of spacetime that cannot send signals to $\mathscr{I}^+$
- event horizon is a null surface determined by future boundary conditions

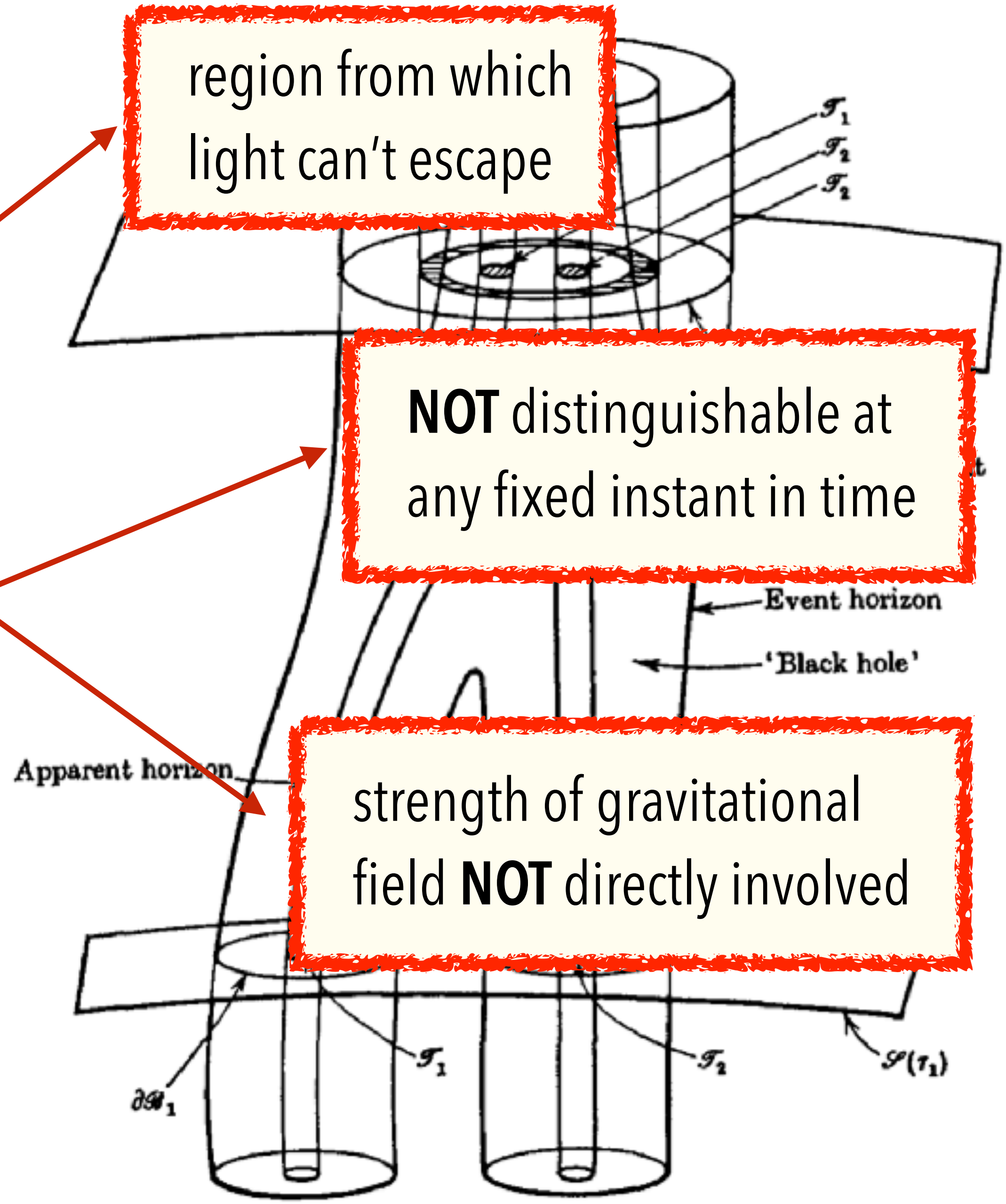


Hawking-Ellis 1973

Black hole mergers and event horizons

Causal black holes

- regions of spacetime that cannot send signals to $\mathscr{I}^+$
- event horizon is a null surface determined by future boundary conditions
- don't tell us anything about internal structure



Hawking-Ellis 1973

Black hole mergers and event horizons

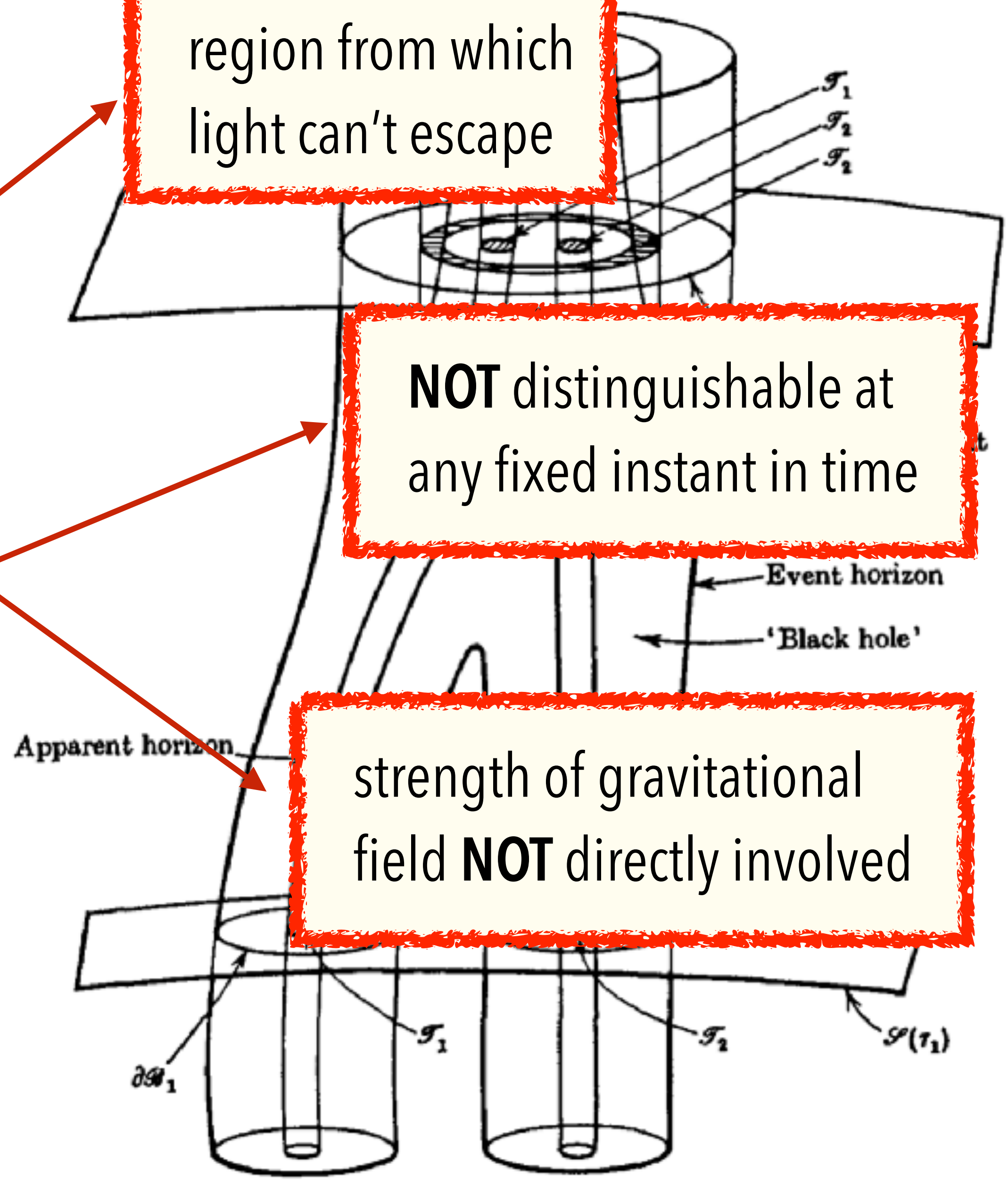
Causal black holes

- regions of spacetime that cannot send signals to $\mathscr{I}^+$
- event horizon is a null surface determined by future boundary conditions
- don't tell us anything about internal structure
- interesting but picture is definitely incomplete

region from which light can't escape

NOT distinguishable at any fixed instant in time

strength of gravitational field **NOT** directly involved



Hawking-Ellis 1973

Black hole mergers and event horizons

Causal black holes

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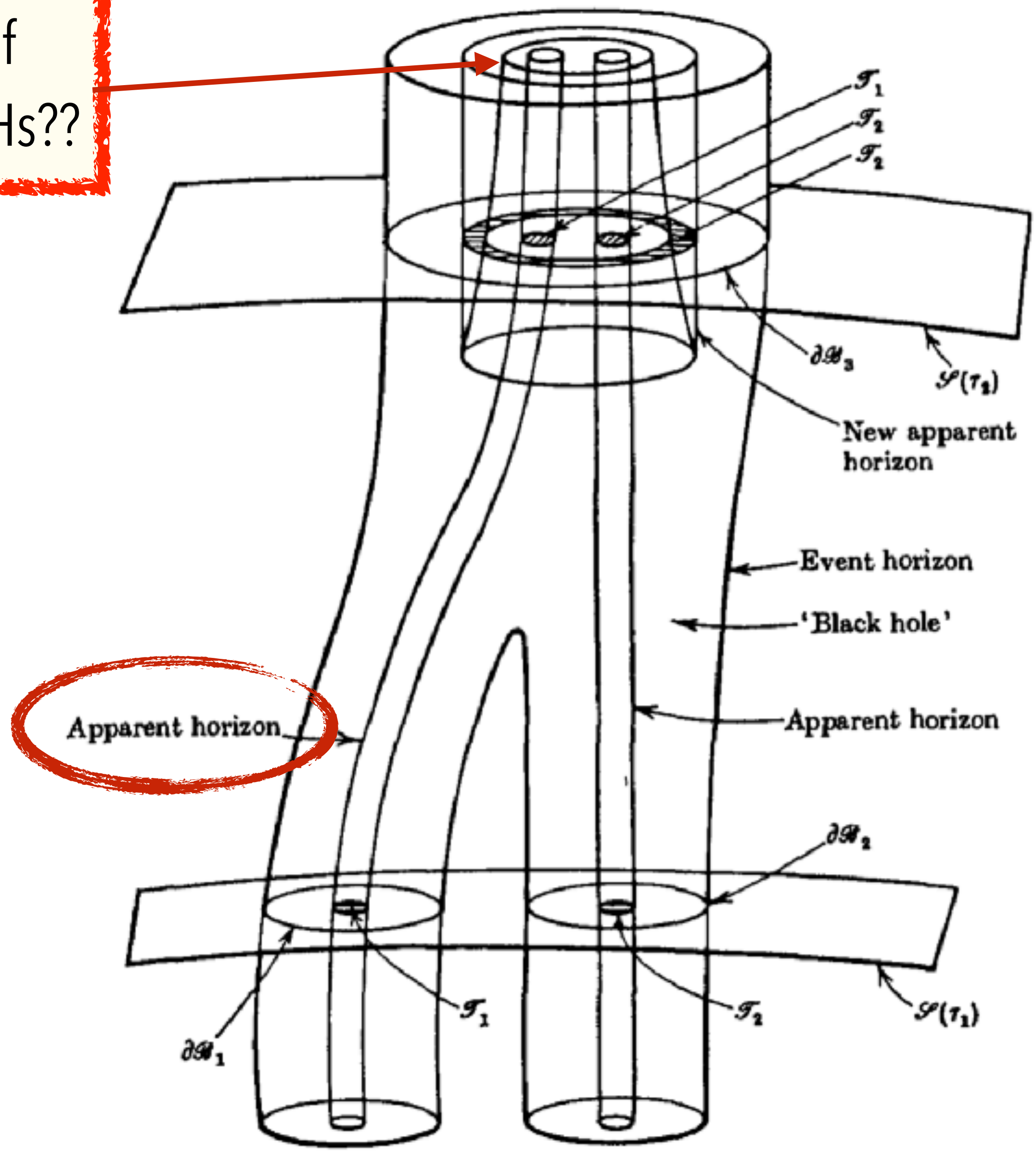
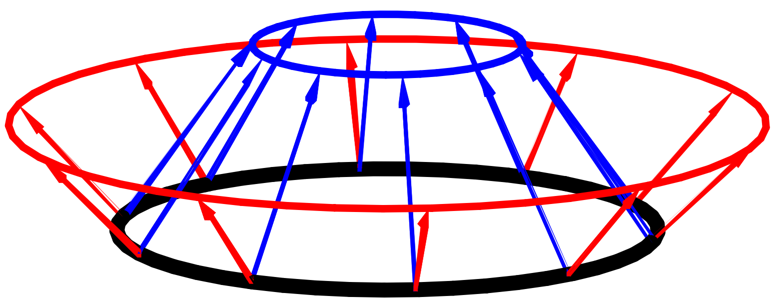
- event horizon is a null surface determined by future boundary conditions
- don't tell us anything about internal structure
- interesting but picture is definitely incomplete

final fate of original AHs??



Hawking-Ellis 1973

How do you know if you are inside a black hole?



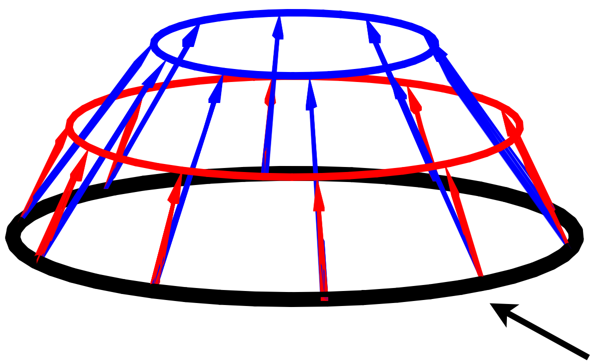
ℓ_+ - outward null normal

ℓ_- - inward null normal

spacelike two-surface

- "Regular" convex surface (ie sphere): $\theta_+ > 0$ and $\theta_- < 0$

How do you know if you are inside a black hole?



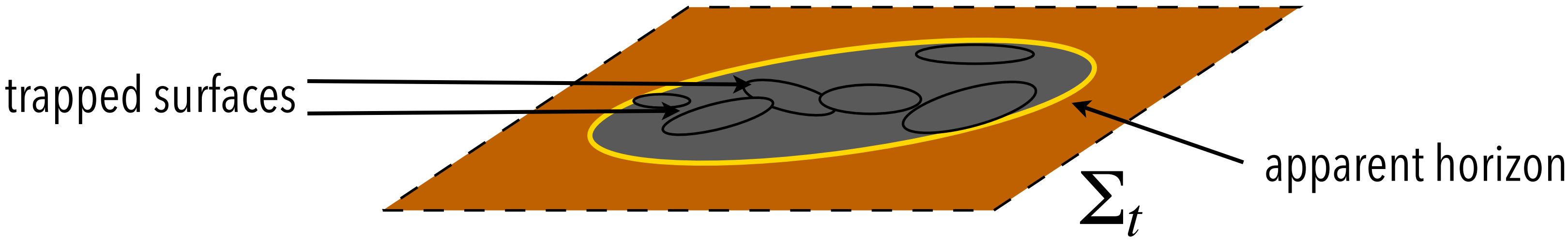
ℓ_+ - outward null normal

ℓ_- - inward null normal

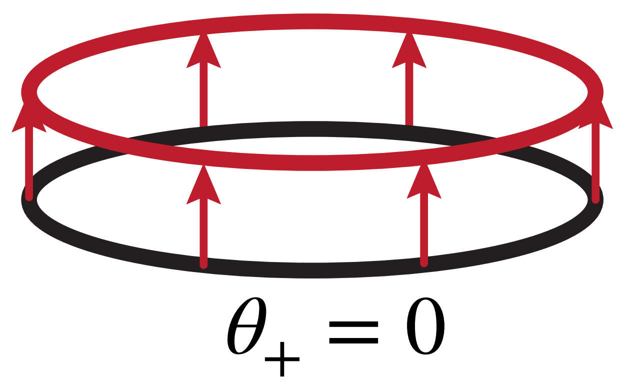
- "Regular" convex surface (ie sphere): $\theta_+ > 0$ and $\theta_- < 0$
- Trapped surface: $\theta_+ < 0$ and $\theta_- < 0$ (everything falls inwards!)
- Trapped surfaces imply the existence of singularities "inside" and (if asymptotically flat) event horizons "outside" (Penrose PRL 65, Nobel 2020)

(Outer) trapped surfaces are inside black holes

Apparent horizons and MOTS



- The *apparent horizon* bounds the (outer) trapped region
- It is a *marginally outer trapped surface* (MOTS)



- Coincides with or is inside the event horizon

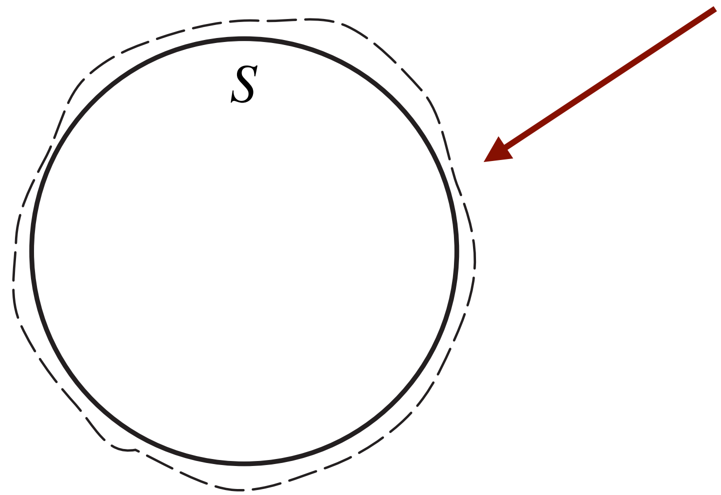
Schwarzschild, RN, Kerr event horizons are MOTS and apparent horizons

RN and Kerr inner horizons as well as cosmo horizons are MOTS

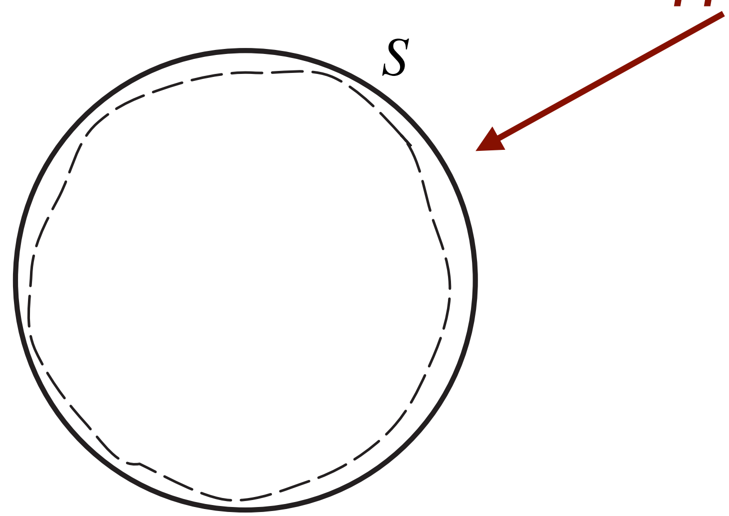
A local characterization of apparent horizons

A MOTS S in a time slice Σ_t is a (local) boundary and ***strictly stably outermost*** if:

1) Small outward deformations become ***outer untrapped***



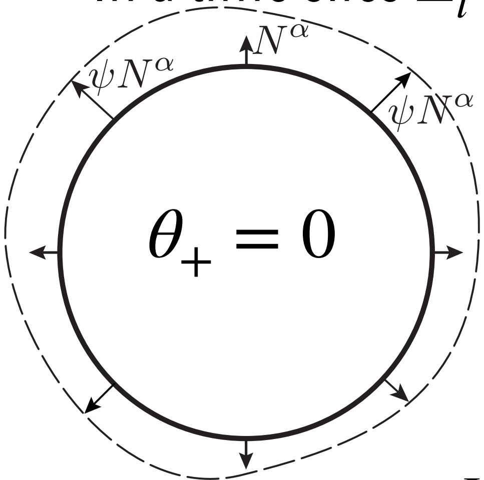
2) Small inward deformations become ***outer trapped***



Newman 1987
Hayward 1994
Andersson, Mars, Simon 2005
Pook-Kolb, Hennigar, Booth 2021

3D Stability Operator for (non-rotating) MOTS

- In a time slice Σ_t with N unit normal to S and future outward null $l_+ = u + N$:



variation of expansion | stability operator | 2D surface Laplacian | Gauss curvature | square of null shear | matter term

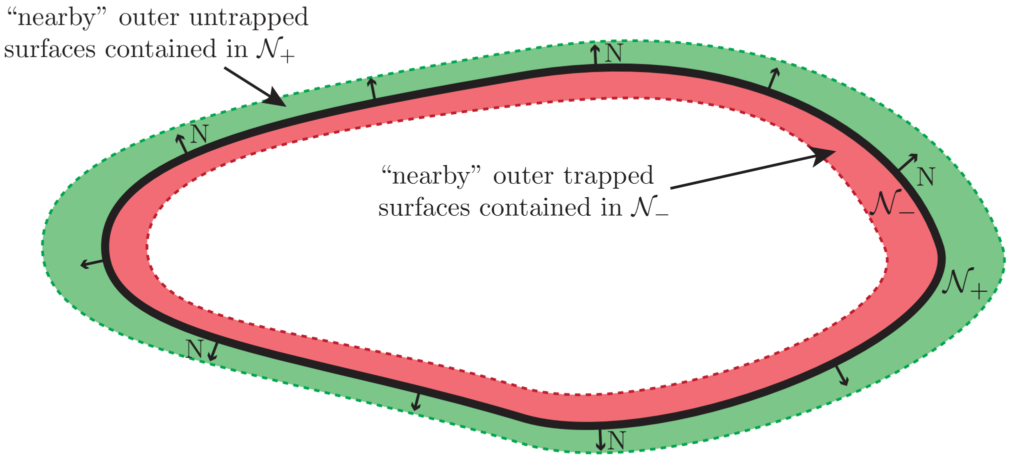
$$\delta_{\psi N}\theta_+ = L_S(l_+, N)\psi = -\mathscr{D}^2\psi + \left(\mathscr{K} - 2\|\sigma_+\|^2 - G_{++}\right)\psi$$

(Andersson, Mars, Simon, 2005, PRL 111102)

(Andersson, Mars, Simon, Adv.Theor.Math.Phys. 12 (2008) 4, 853-888)

- $L_S(l_+, N)\psi = \delta_{\psi N}\theta_+$ is a self-adjoint linear elliptic operator on ψ
- Eigenvalue spectrum is discrete and real
- There is a smallest **principal eigenvalue** λ_o. All other eigenvalues are larger.
- Principal eigenfunction $\psi_o > 0$
- Define (n_0, n_-)–stability: $n_0 =$ # of vanishing eigenvalues
  $n_- =$ # of negative eigenvalues
- Strictly stable $\lambda_o > 0 \iff$ all eigenvalues positive
  $\iff$ outward deformations outer untrapped
  $\iff$ inward deformations outer trapped

Stability: MOTSs as local boundaries

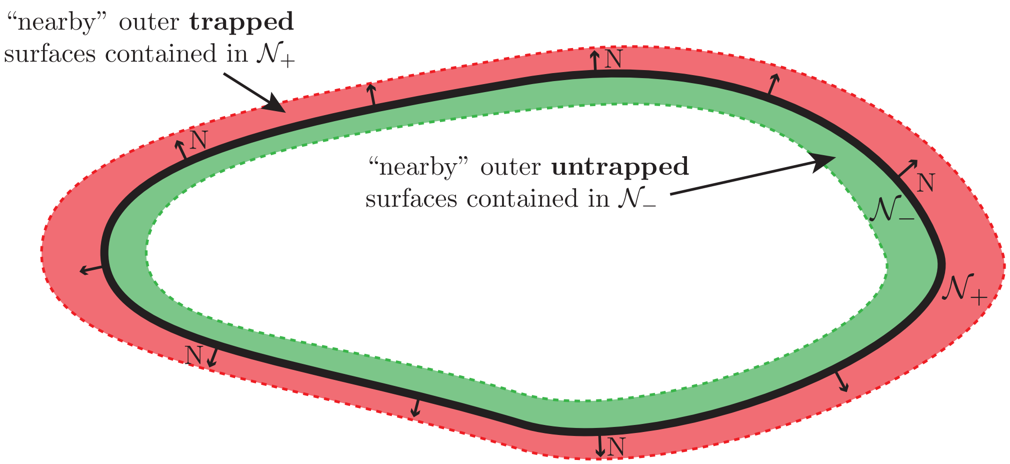


Strictly stable ($\lambda_o > 0$)

$\Rightarrow$ outer untrapped outside,

outer trapped inside

Local apparent horizon



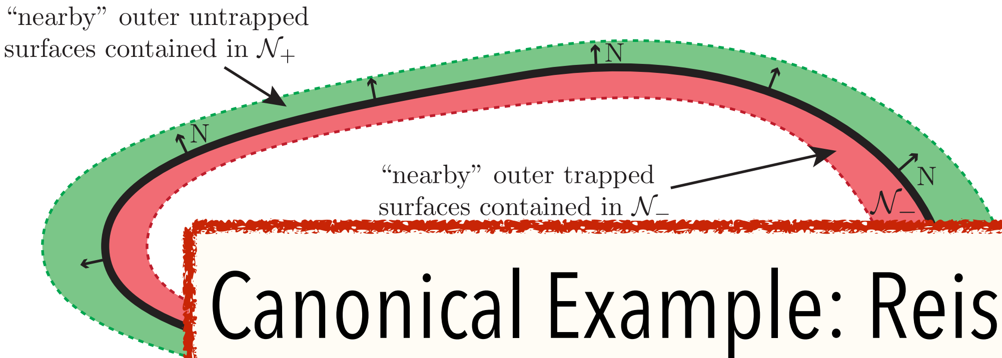
Strictly unstable ($\lambda_o < 0$)

$\Rightarrow$ outer untrapped outside,

outer untrapped inside

Local inner horizon

Stability: MOTSs as local boundaries



Strictly stable ($\lambda_o > 0$)

⇒ outer untrapped outside,

e

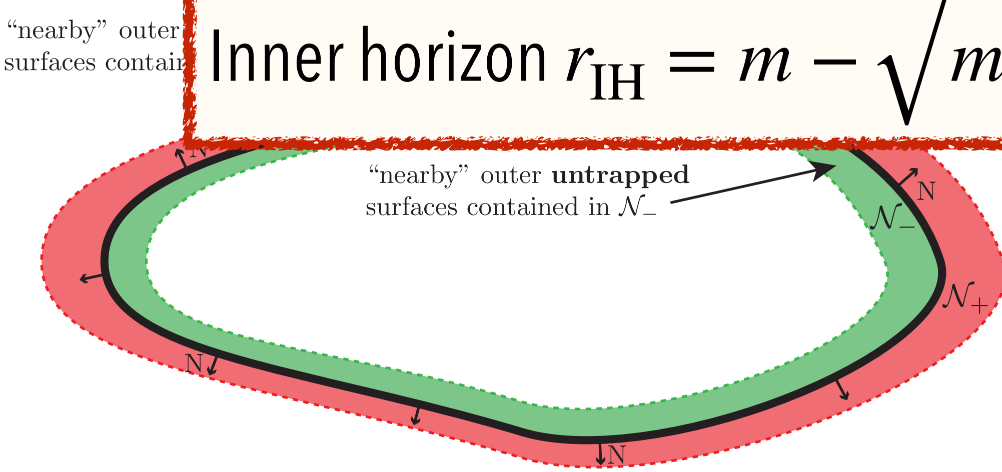
on

Canonical Example: Reissner-Nordström

Outer horizon $r_{\mathrm{OH}} = m + \sqrt{m^2 - q^2}$: strictly stable

Inner horizon $r_{\mathrm{IH}} = m - \sqrt{m^2 - q^2}$: strictly unstable

< 0)



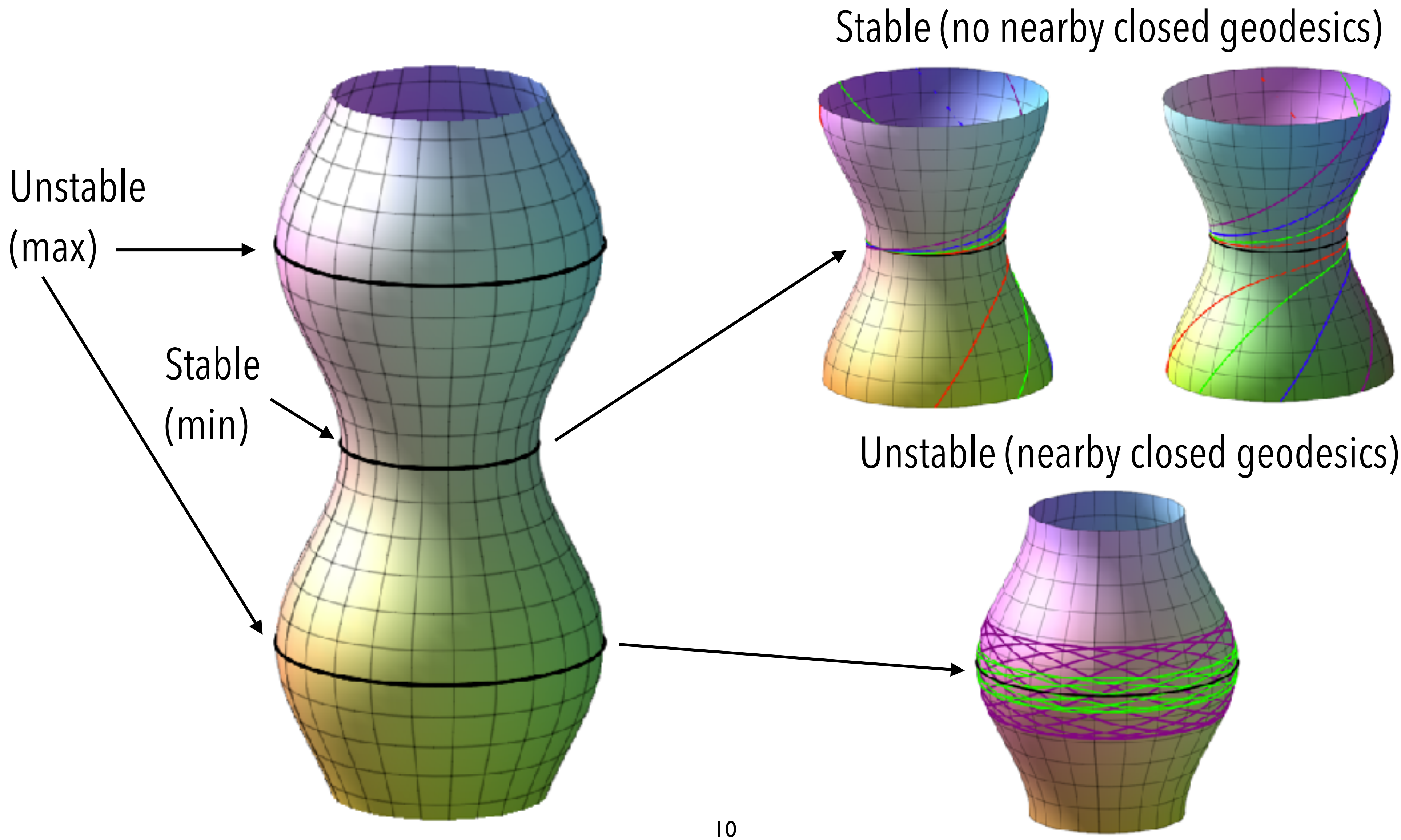
⇒ outer untrapped outside,

outer untrapped inside

Local inner horizon

What does "stable" mean?

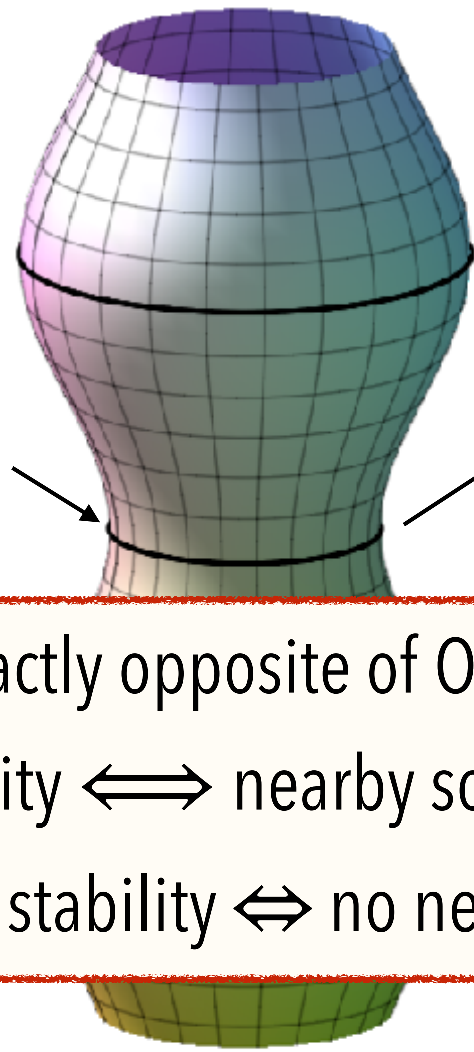
This is *geometric stability* in the sense of minimal surfaces/geodesics.

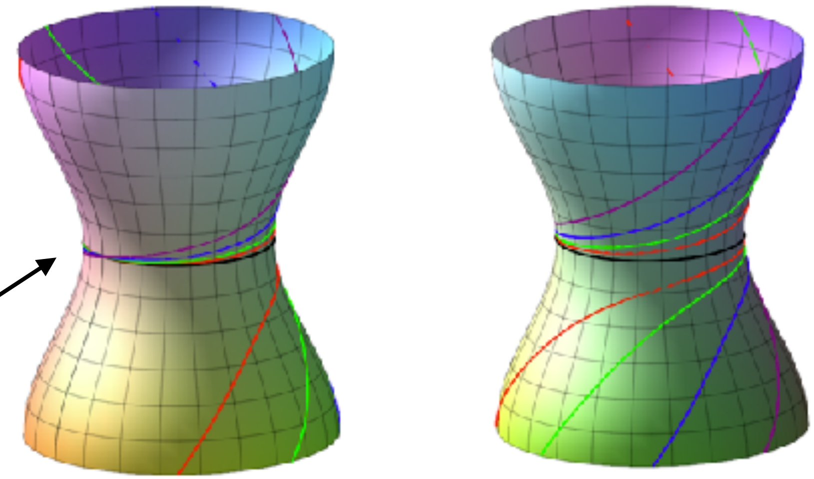


An aside…

What does "stable" mean?

This is *geometric stability* in the sense of minimal surfaces/geodesics.

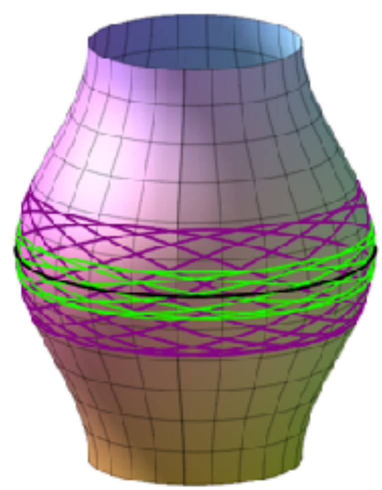




Almost exactly opposite of ODE stability!

ODE stability ⟺ nearby solutions

geometric stability ⇔ no nearby solutions

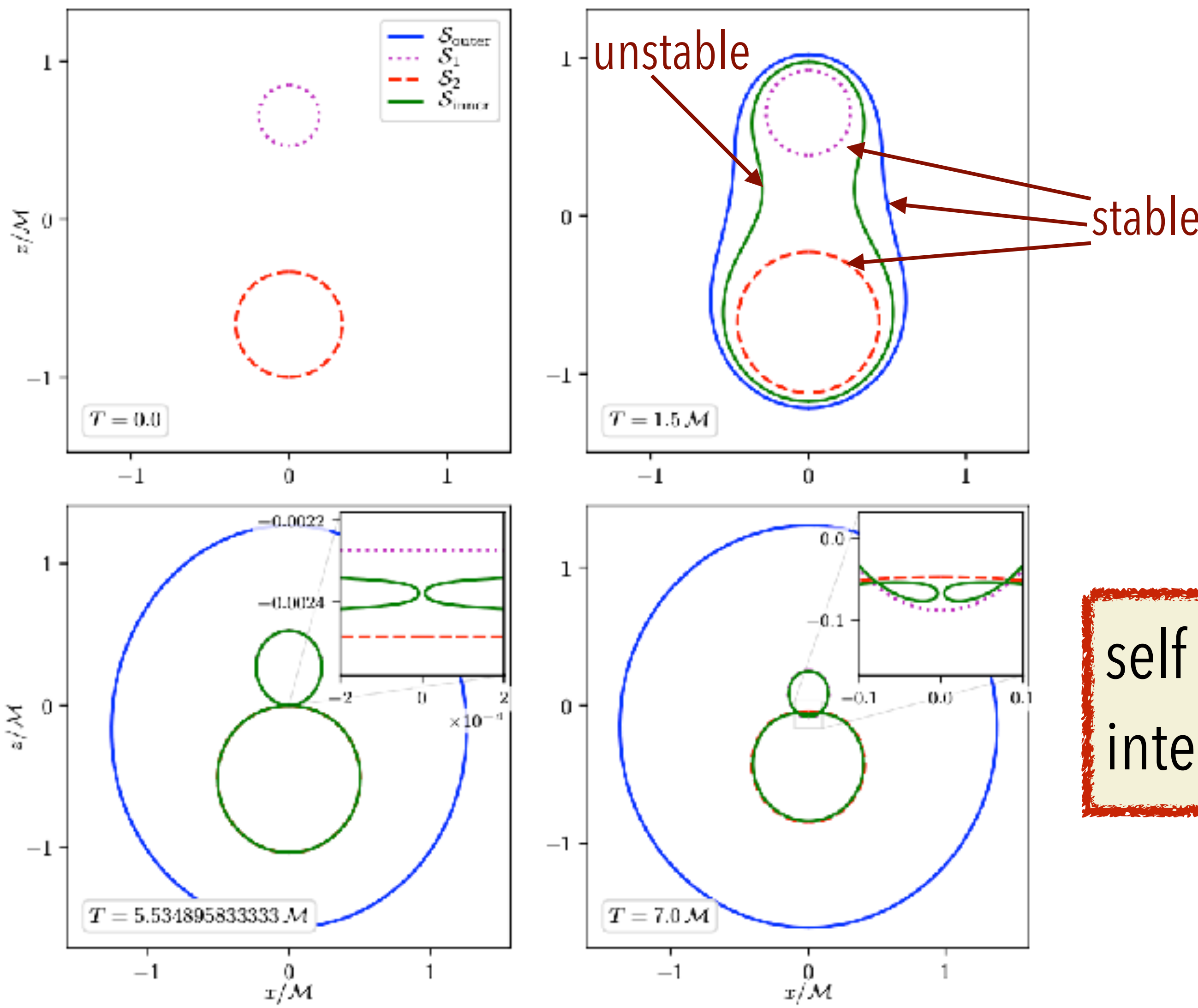


MOTSs can have "exotic" properties

Self-intersecting marginally outer trapped surfaces

Daniel Pook-Kolb,[1,2] Ofek Birnholtz,[3] Badri Krishnan,[1,2] and Erik Schnetter[4,5,6]

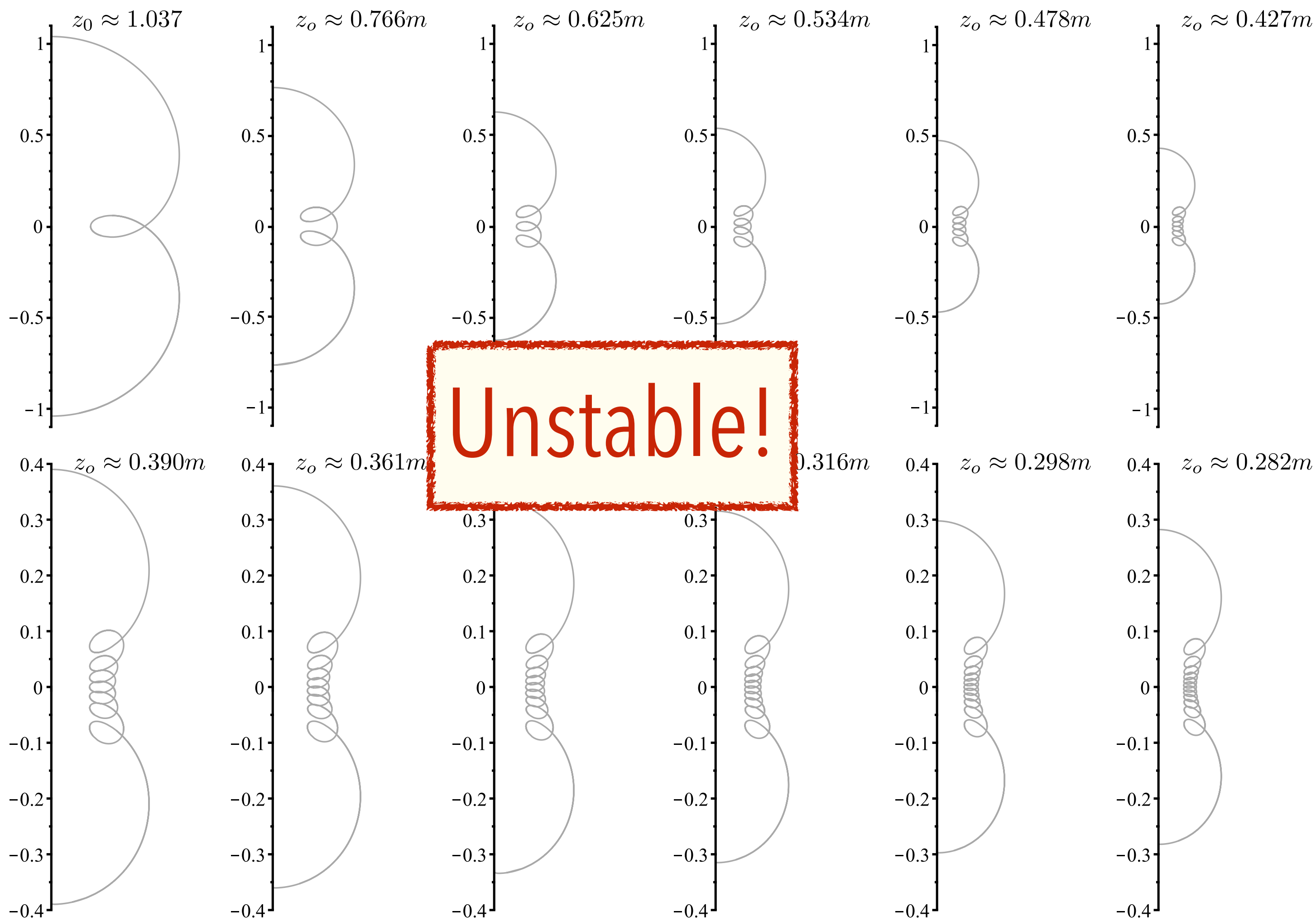
Phys. Rev. D 100, 084044 (2019)



self intersections!

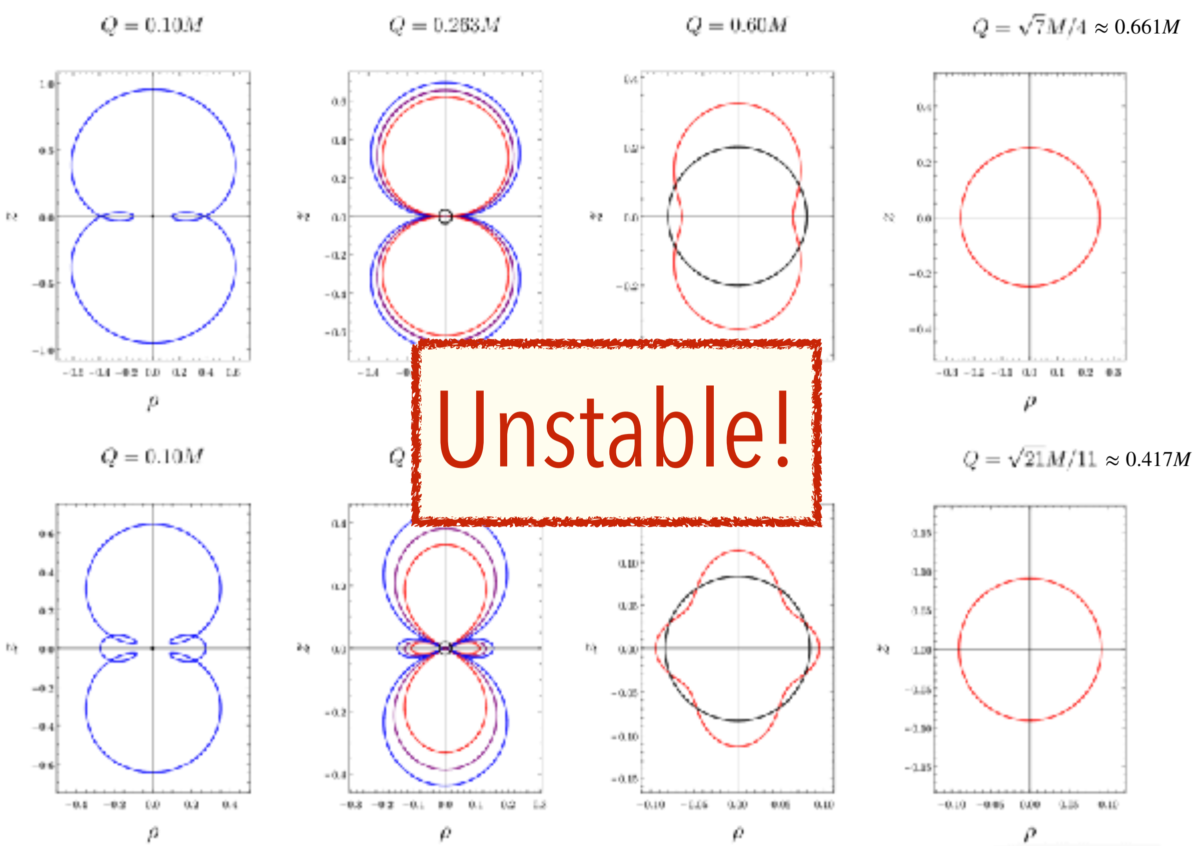
Most MOTS are not horizons/boundaries…

Schwarzschild



IB, R.Hennigar, S.Mondal (PRD 2020)

Reissner-Nordström (around inner horizon)

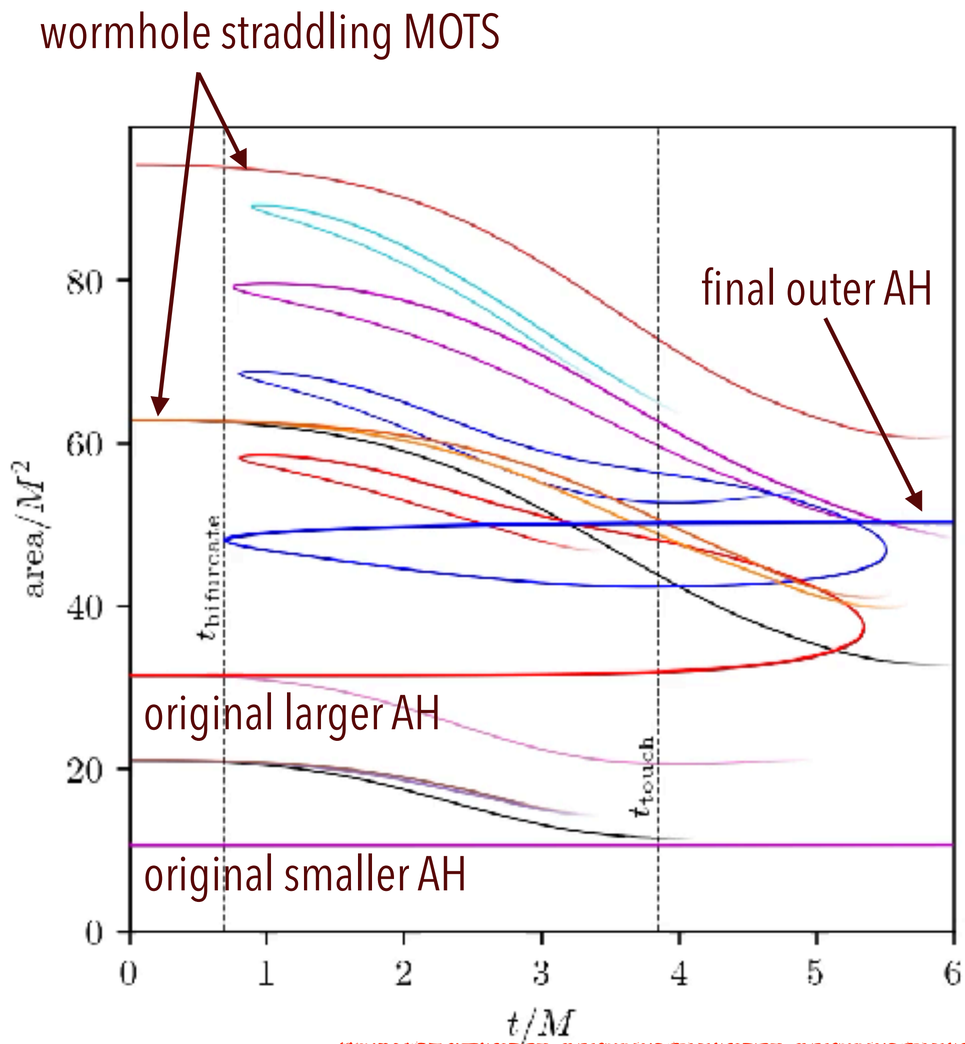


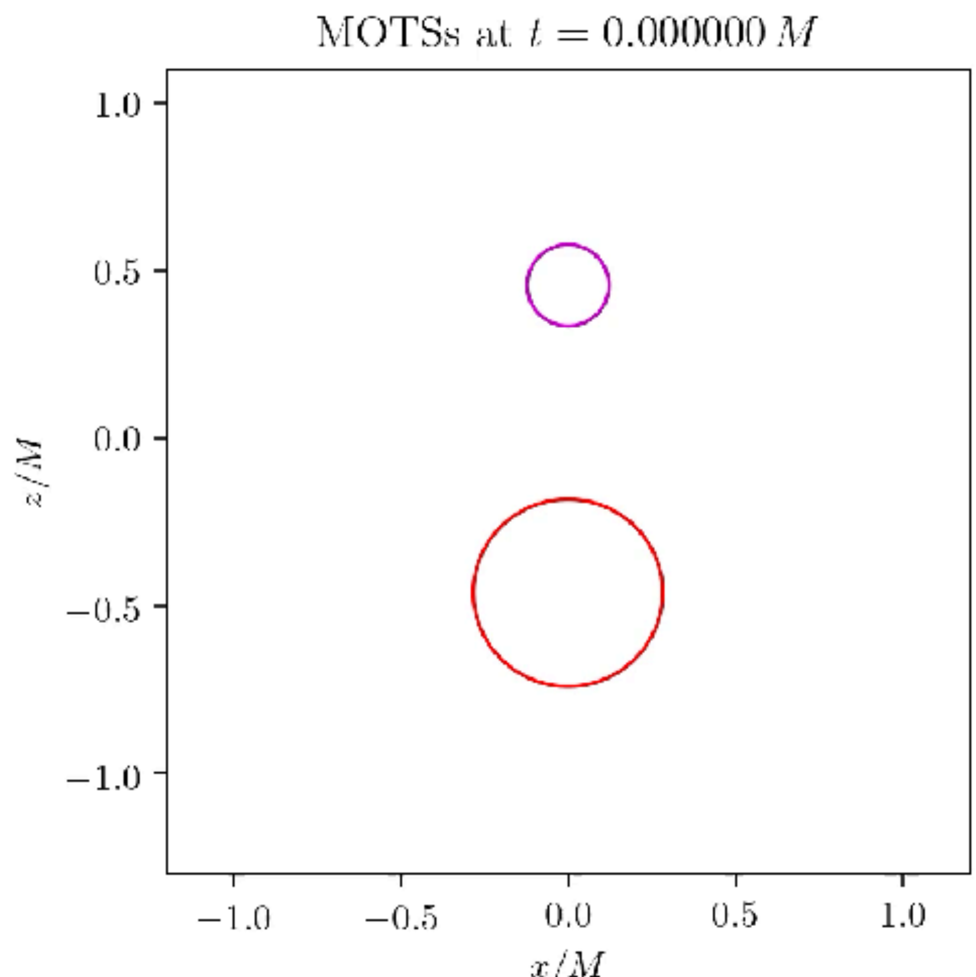
R.Hennigar, B.Chan/Sievers, L.Newhook, IB PRD 2022

Existence is robust across coordinate systems and solutions
(B.Sievers, R.Hennigar, H.Kunduri, S.Muth, L.Newhook, IB)

Mergers are complicated…

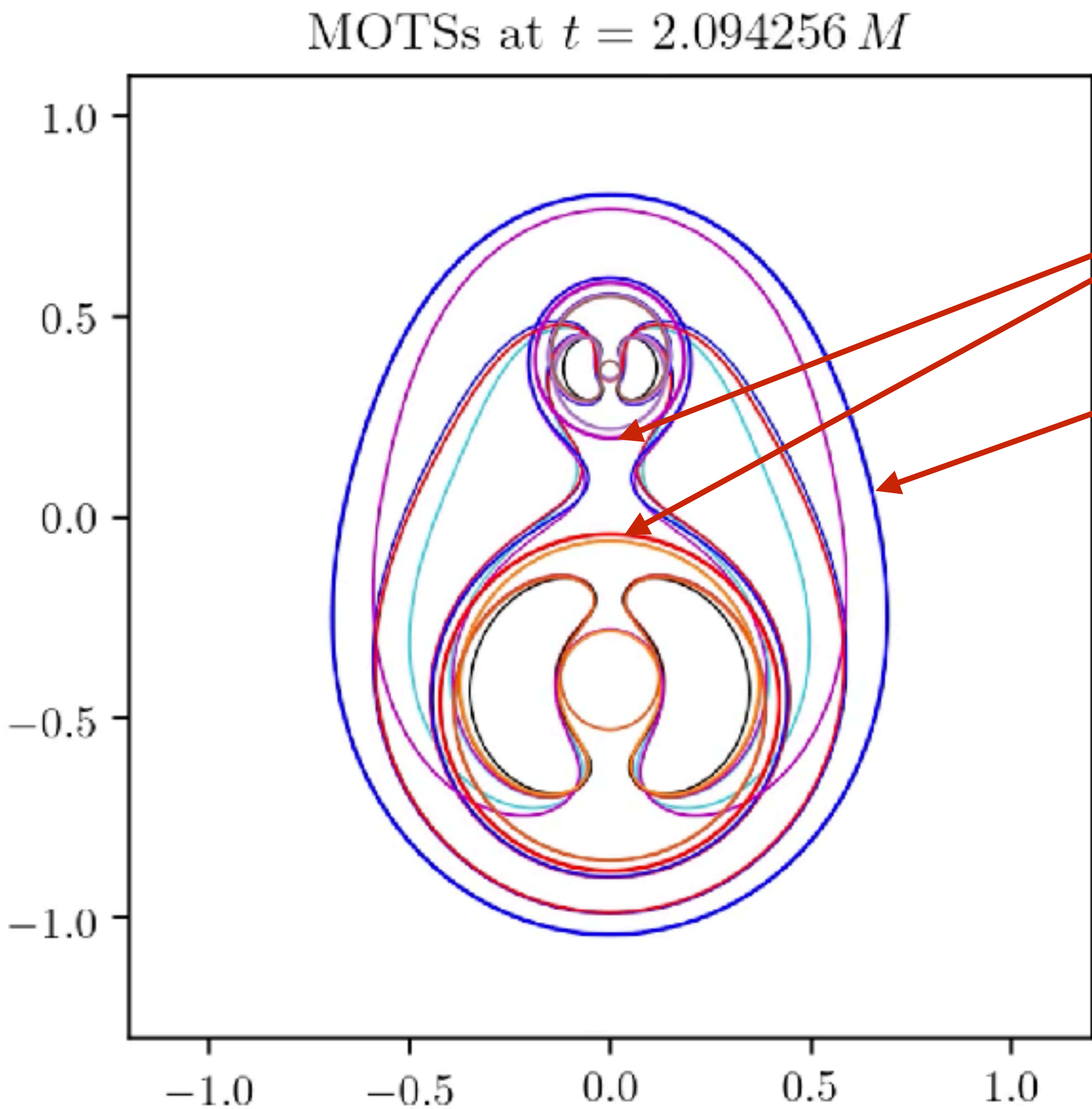
Pook-Kolb, Hennigar, Booth PRL 2021 (Summary)
Pook-Kolb, Hennigar, Booth PRD 2021 (Theory)
Booth, Hennigar, Pook-Kolb PRD 2021 (Numerical Results)





Most MOTS do not bound trapped regions

During a merger



The original and final apparent horizons are stable.

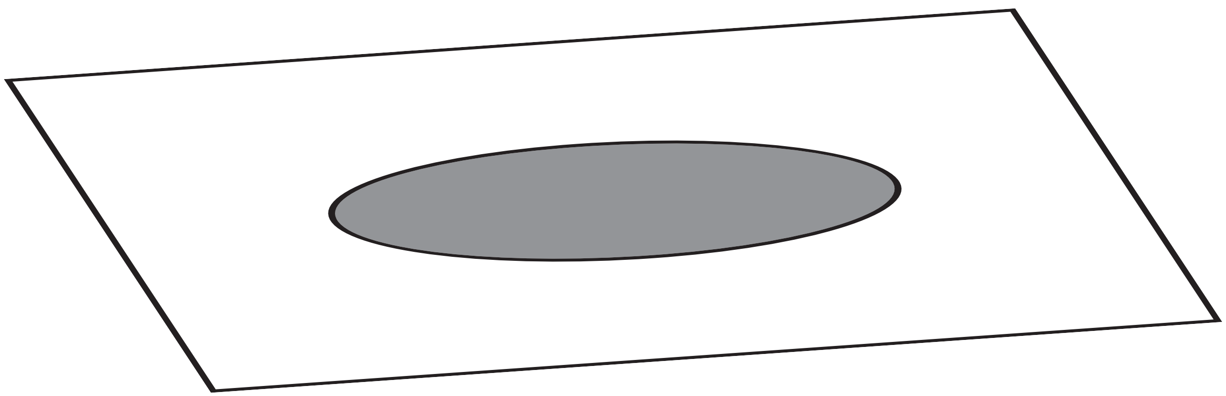
All other MOTS are unstable.

Stability Operator and Evolution/Deformation

examples: time evolution, q in RN, Λ is dS, anything else

Problem: δ_α = one-parameter deformation of the geometry (h_{ij}, K_{ij}) of a slice Σ_t

Can we perturb S in Σ_t so that it remains a MOTS?



Second application of stability operator

Stability Operator and Evolution/Deformation

examples: time evolution, q in RN, Λ is dS, anything else

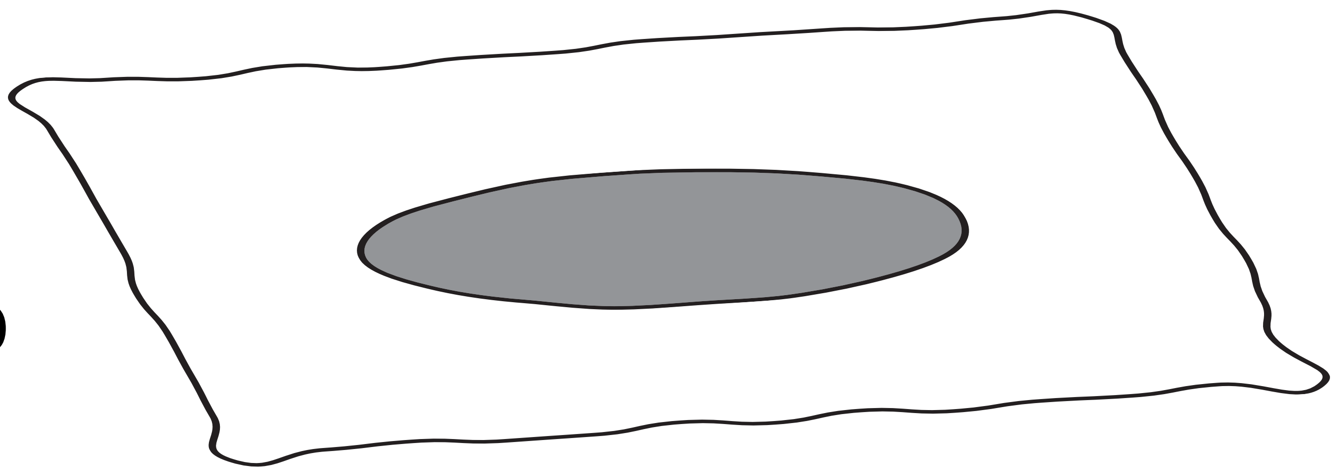
Problem: δ_α = one-parameter deformation of the geometry (h_{ij}, K_{ij}) of a slice Σ_t

Can we perturb S in Σ_t so that it remains a MOTS?

distorted

(h_{ij}, K_{ij})

$\theta_+\big|_S \neq 0$



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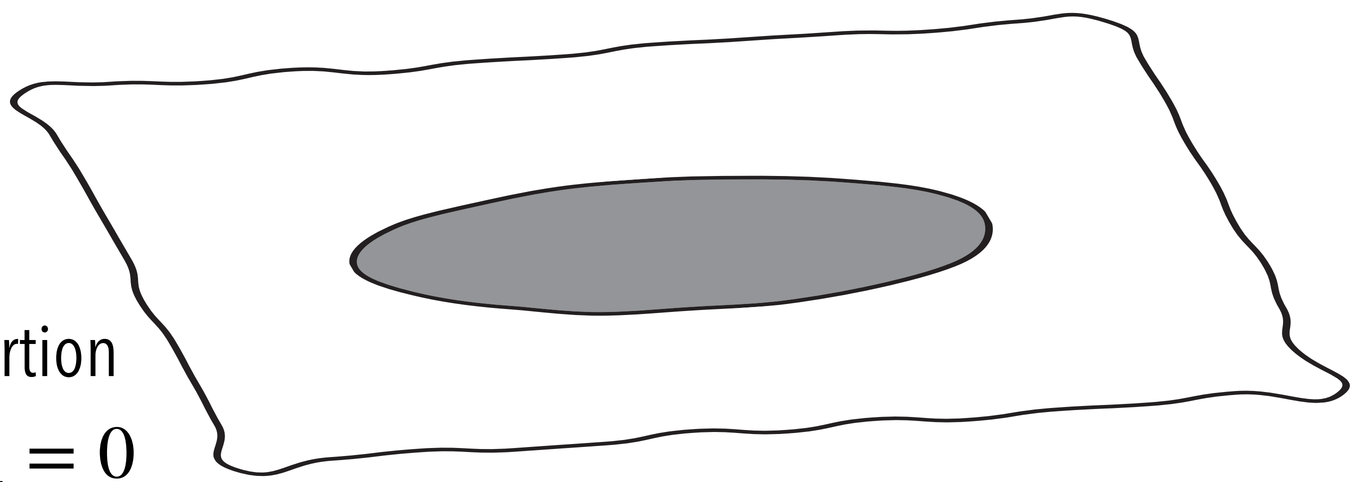


small distortion of $S \to \theta_+ = 0$

Stability Operator and Evolution/Deformation

examples: time evolution, q in RN, Λ is dS, anything else

Problem: δ_α = one-parameter deformation of the geometry (h_{ij}, K_{ij}) of a slice Σ_t

Can we perturb S in Σ_t so that it remains a MOTS?

total change in expansion; change due to (h, K) deformation

Solution: Look for a ψ for which: $0 = \delta\theta_+ = \delta_\alpha\theta_+ + \delta_{\psi N}\theta_+$

change from S perturbation

$$\iff 0 = \delta_\alpha\theta_+ + L_S(l_+, N)\psi$$

$$\iff \psi = -L_S^{-1}(l_+, N)[\delta_\alpha\theta_+]$$

ψ is unique if $L_S(l_+, N)$ is invertible $\iff L_S(l_+, N)$ has no vanishing eigenvalues

Strict stability ($\lambda_o > 0$) is sufficient but not necessary for uniqueness

Non-uniqueness: If ψ_0 is an eigenfunction with $\lambda_0 = 0$ and ψ is a particular solution then

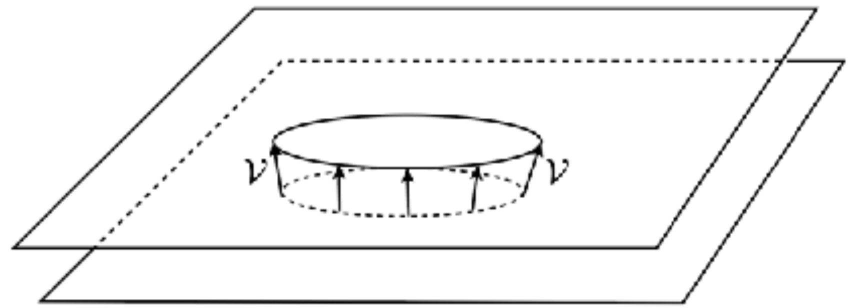
$$\tilde{\psi} = \psi + k\psi_0$$

is also a solution.

Example: time evolution

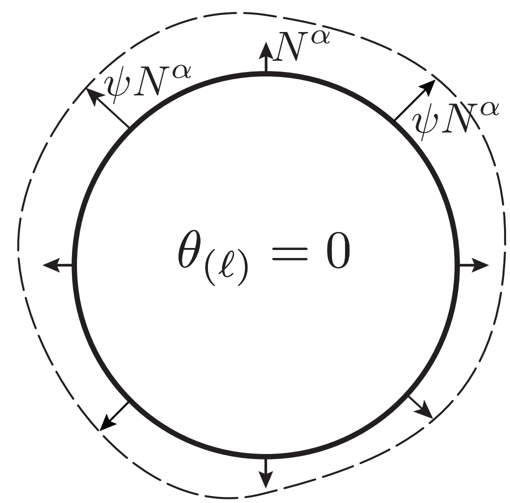
- Time evolution of a MOTS:

$$0 = \delta_t \theta_+ + \delta_{\psi N} \theta_+$$
$$\iff \psi = -L^{-1}(l_+, N)[\delta_t \theta_+]$$



$\mathscr{V} = t + \psi N$

- Unique if there are no vanishing eigenvalues



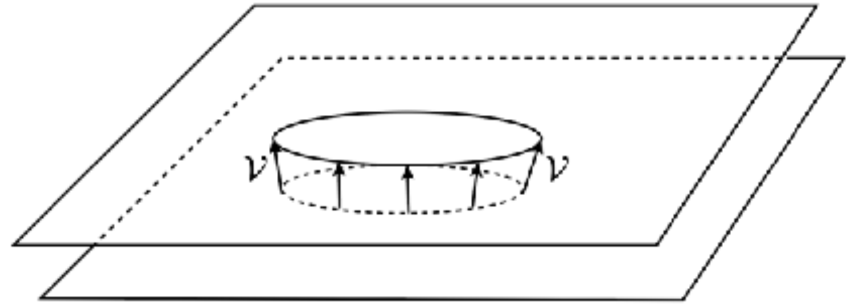
- If the MOTS is **marginally stable** then there exists a function ψ_0 such that: $\delta_{\psi_0 N} \theta_+ = L_\Sigma \psi_0 = 0$

Andersson Mars Simon, PRL 05

Example: time evolution

- Time evolution of a MOTS:

$$0 = \delta_t \theta_+ + \delta_{\psi N} \theta_+$$
$$\Longleftrightarrow \psi = - L^{-1}(l_+, N)[\delta_t \theta_+]$$

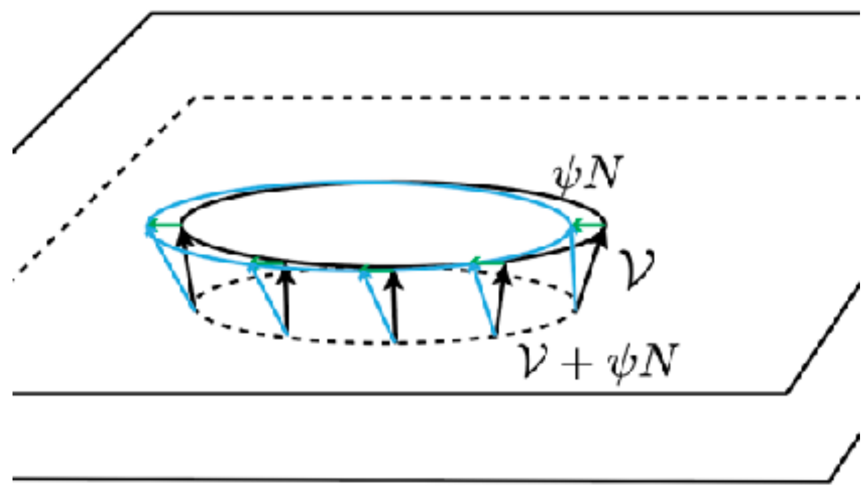


$\mathscr{V} = t + \psi N$

- Unique if there are no vanishing eigenvalues

- If the MOTS is **marginally stable** then there exists a function ψ_0 such that: $\delta_{\psi_0 N} \theta_+ = L_\Sigma \psi_0 = 0$

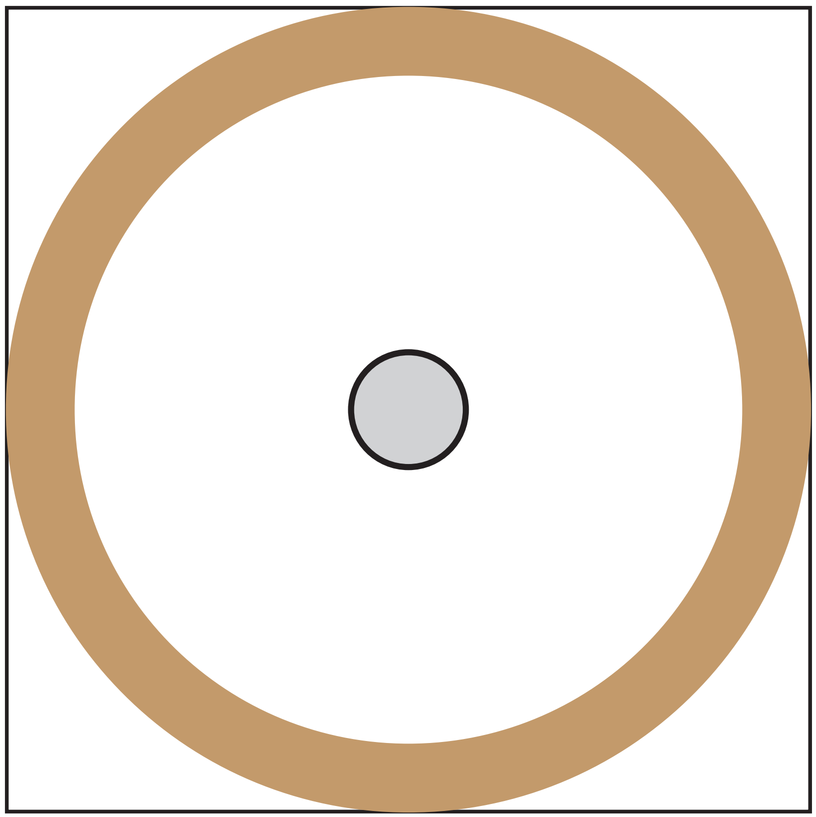
- Then: $\delta_{(\mathscr{V} + \psi_0 N)} \theta_+ = 0$ so:
  $\mathscr{V} + \psi_0 N$ is an equally good "time-evolution" vector



Andersson Mars Simon, PRL 05

MOTS time evolution - an example

Huge shell of dust falls into black hole

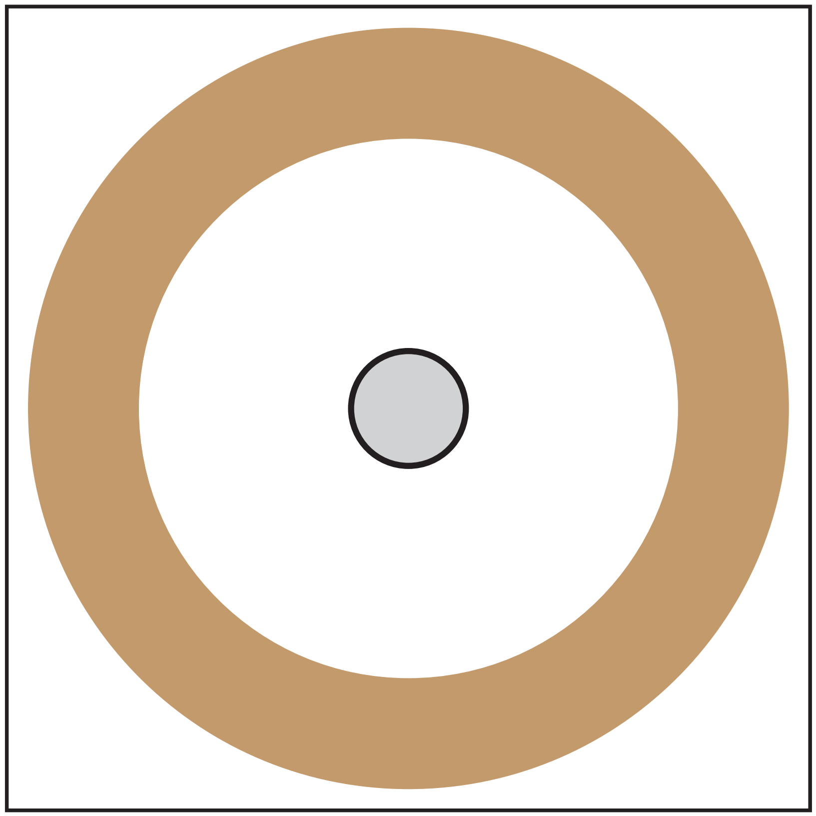


t_1

Booth, Brits, Gonzalez, Van Den Broeck
CQQ 2006

MOTS time evolution - an example

Huge shell of dust falls into black hole

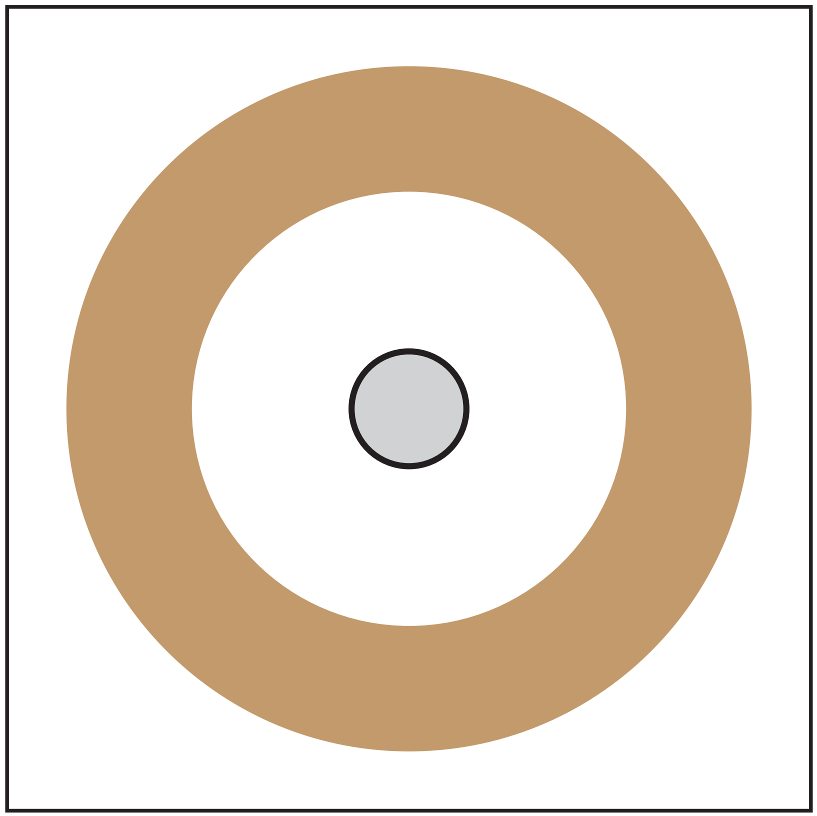


t_2

Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example

Huge shell of dust falls into black hole

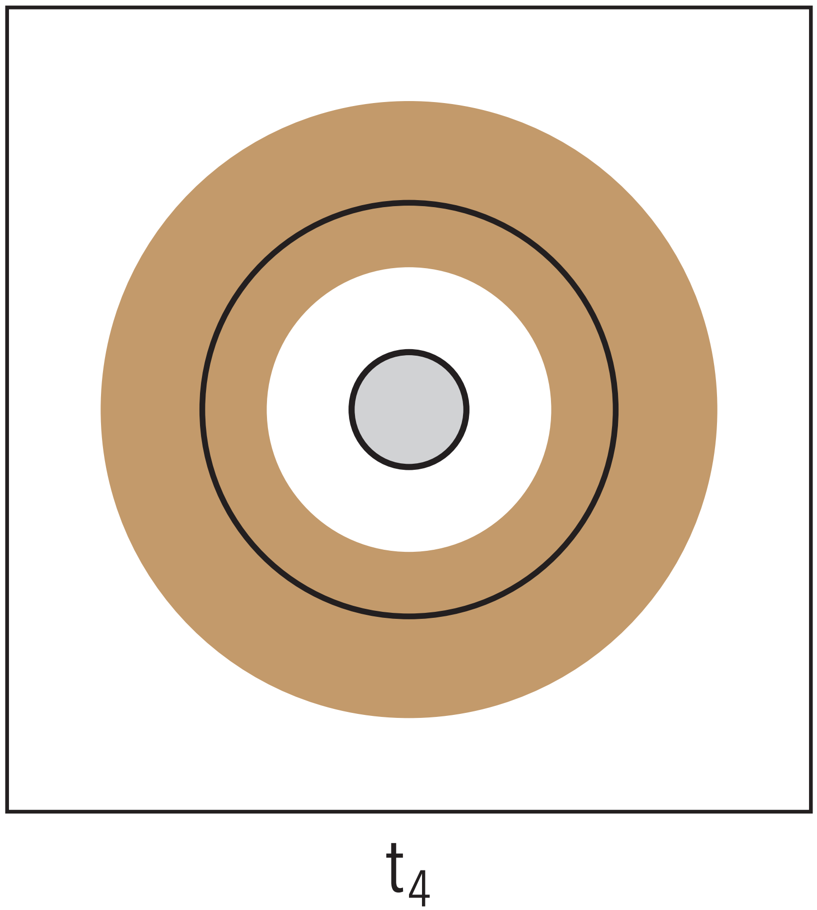


t_3

Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example

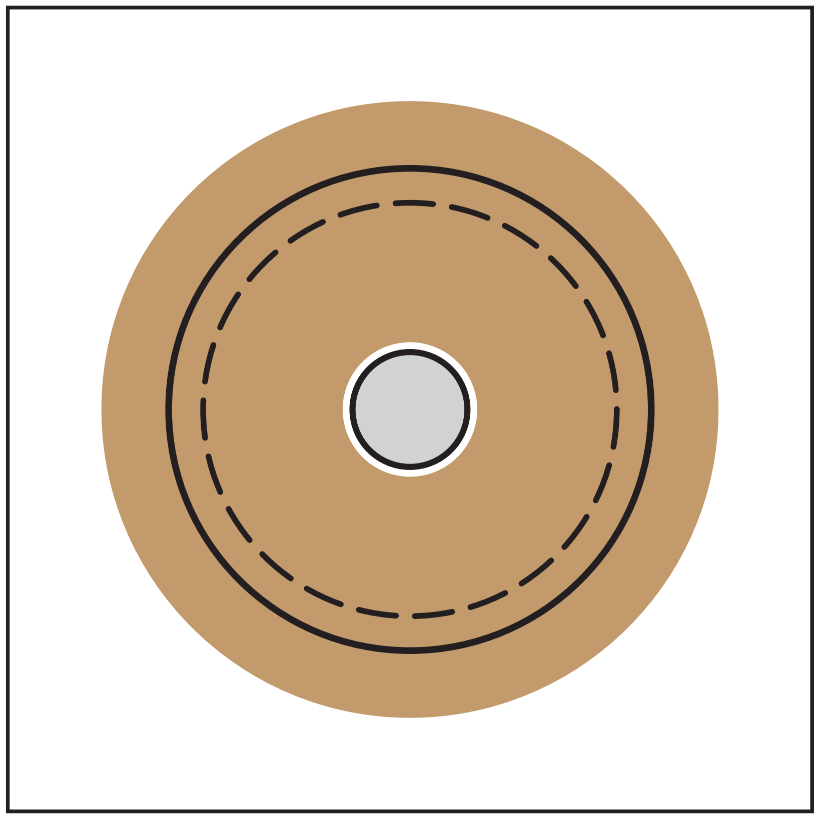
New marginally stable outer MOTS forms



Booth, Brits, Gonzalez, Van Den Broeck
CQQ 2006

MOTS time evolution - an example

Bifurcates into unstable and stable MOTS

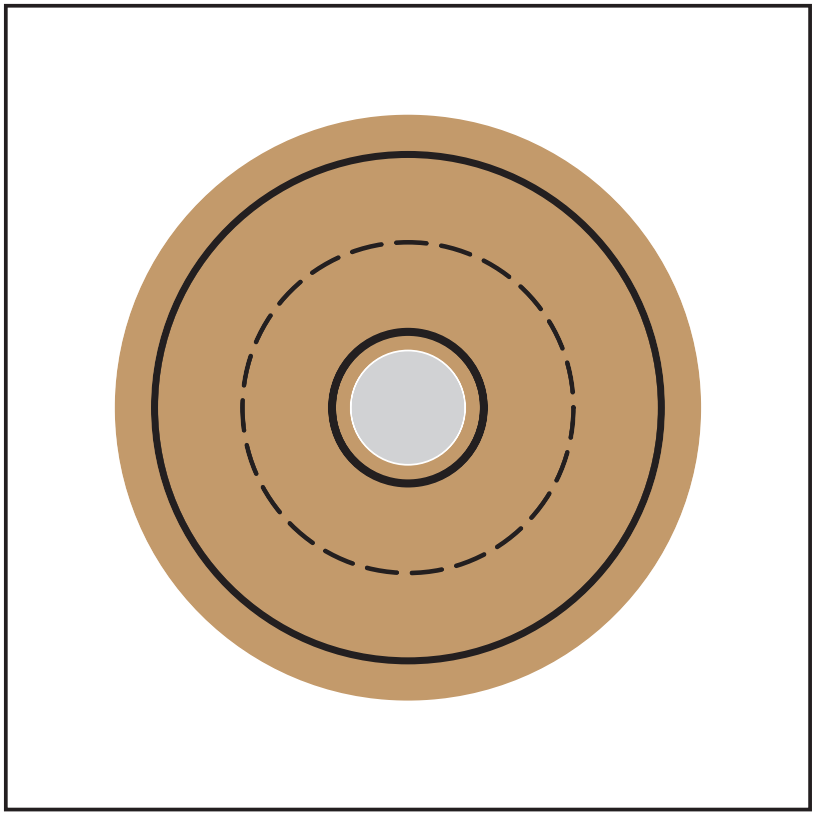


t_5

Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example

Bifurcates into unstable and stable MOTS

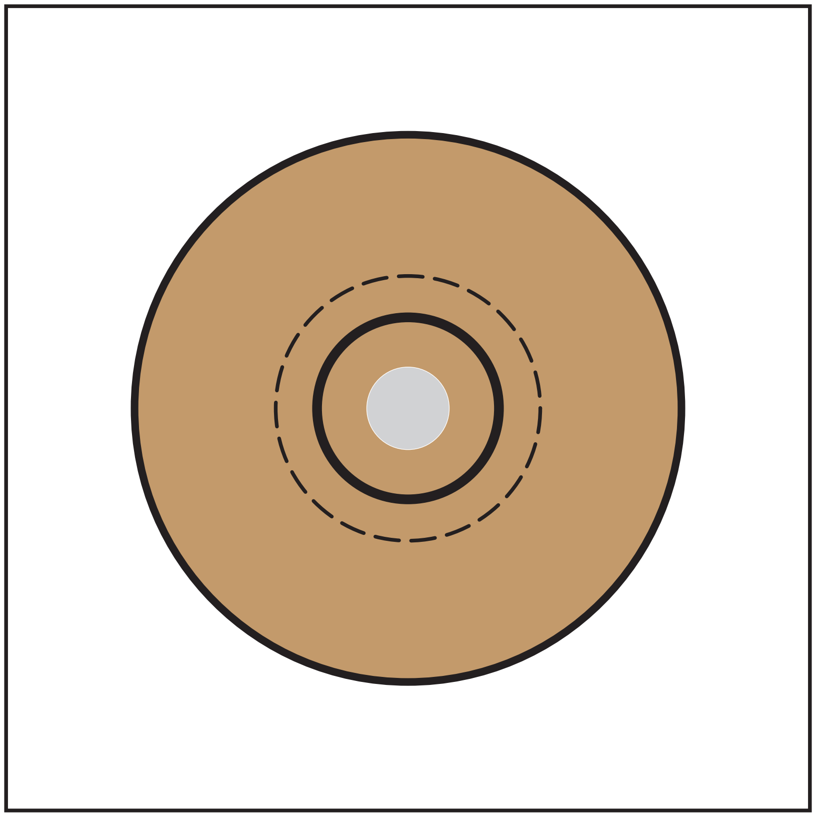


t_6

Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example

Unstable MOTS approaches inner stable

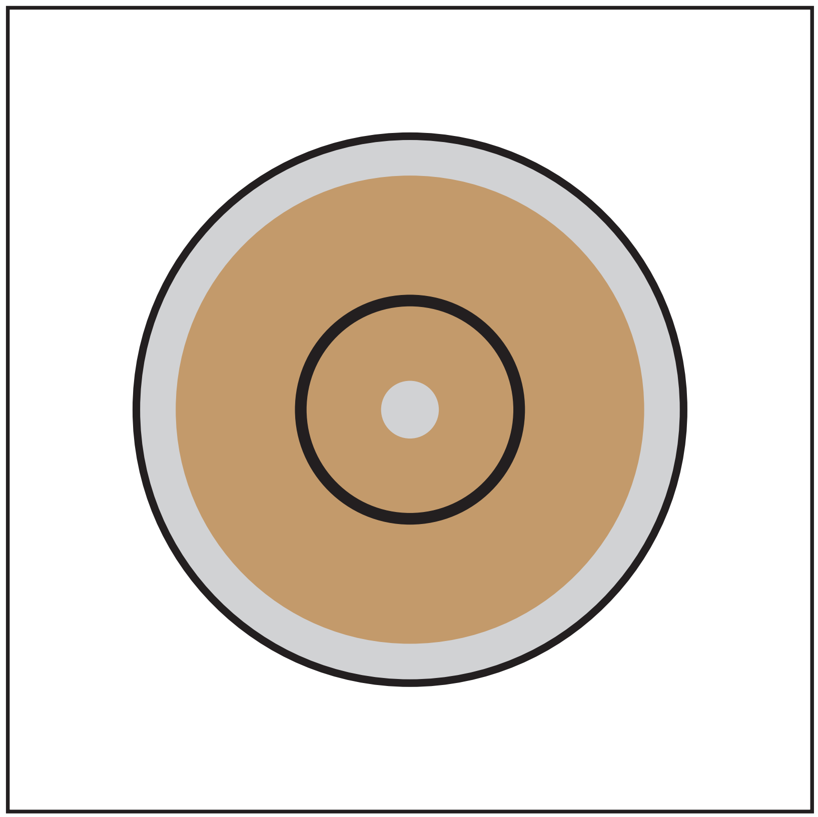


t_7

Booth, Brits, Gonzalez, Van Den Broeck
CQQ 2006

MOTS time evolution - an example

Inner MOTS marginally stable

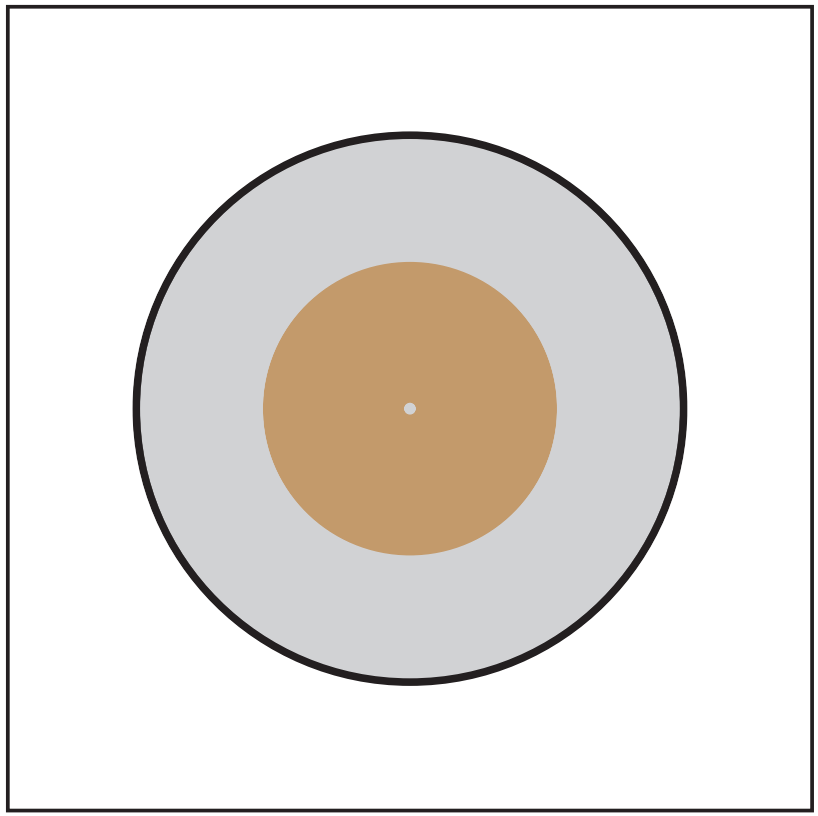


t_8

Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example

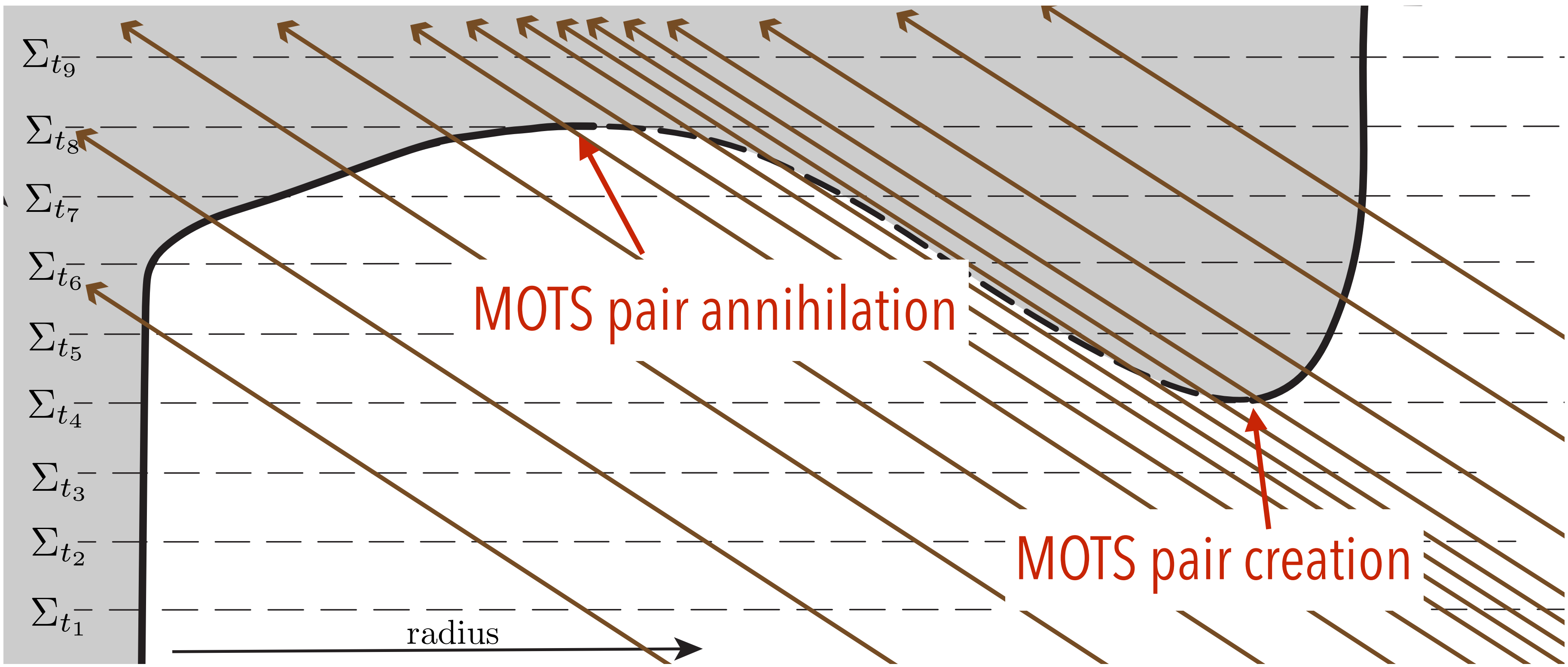
Inner MOTS (original black hole) “dissolves”



t_9

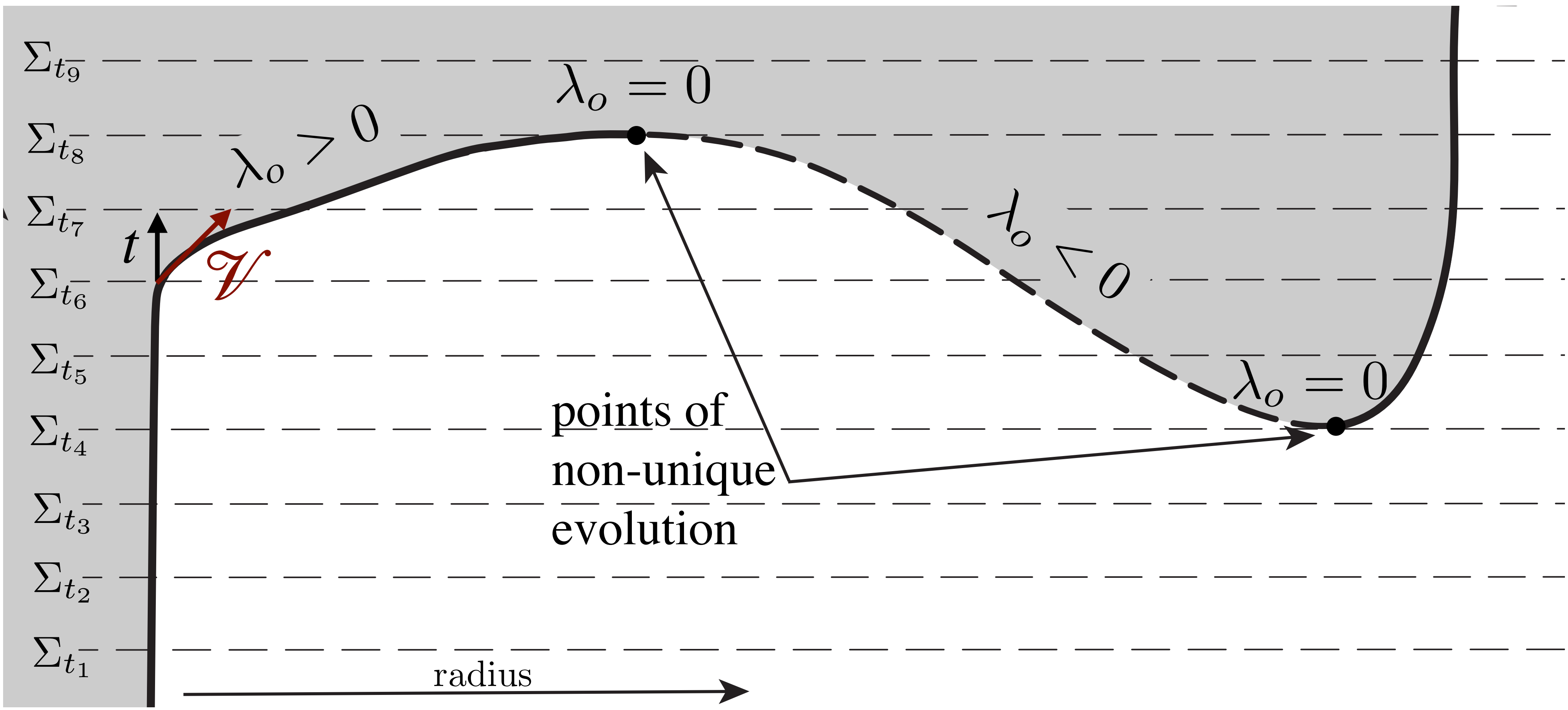
Booth, Brits, Gonzalez, Van Den Broeck
CQG 2006

MOTS time evolution - an example



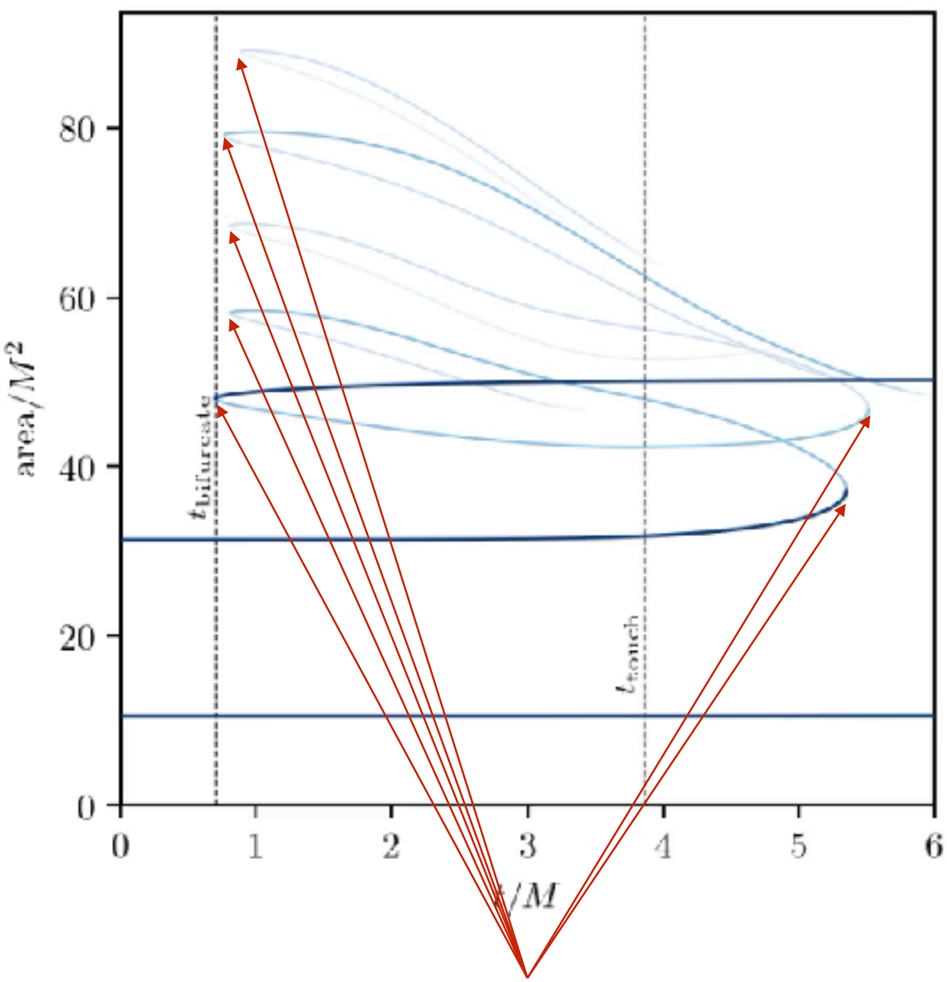
Booth, Brits, Gonzalez, Van Den Broeck
CQQ 2006

MOTS evolution and the stability operator

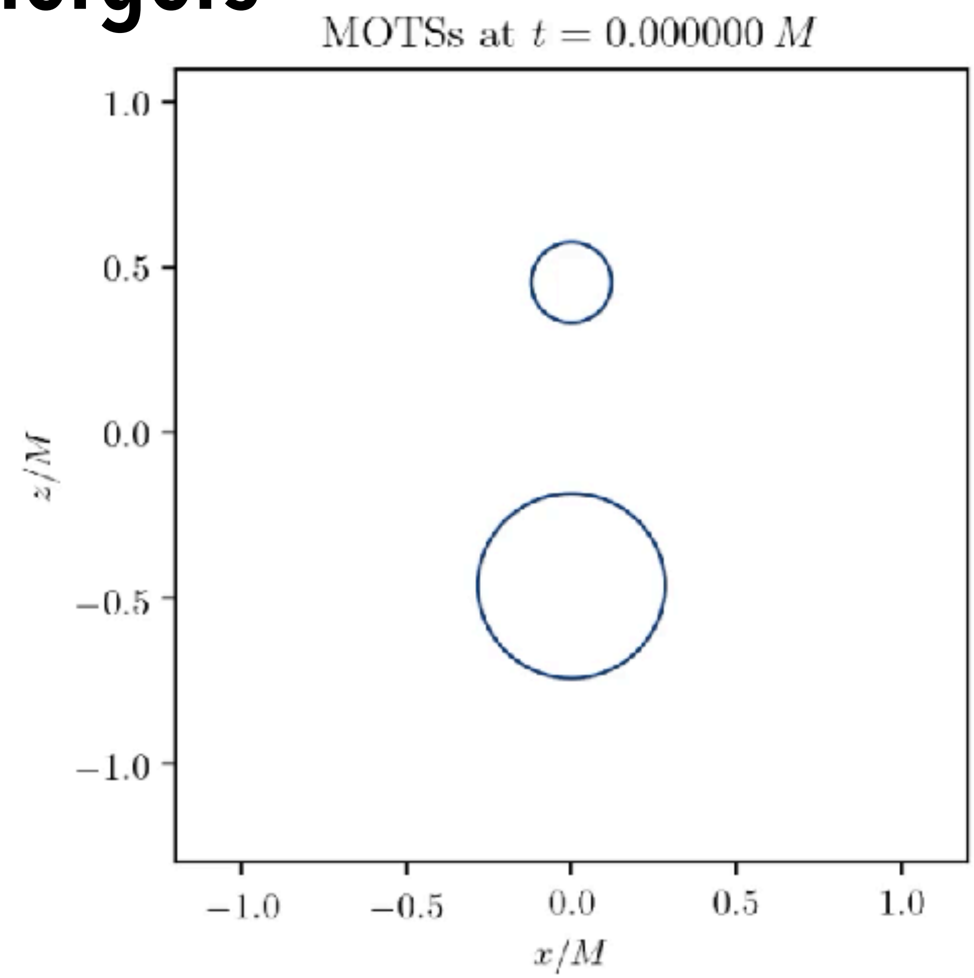


Pook-Kolb, Hennigar, Booth PRL 2021 (Summary)
Pook-Kolb, Hennigar, Booth PRD 2021 (Theory)
Booth, Hennigar, Pook-Kolb PRD 2021 (Numerical Results)

MOTS during black hole mergers



L_Σ has a vanishing eigenvalue at creation and annihilation points



Black = stable MOTS = apparent horizons

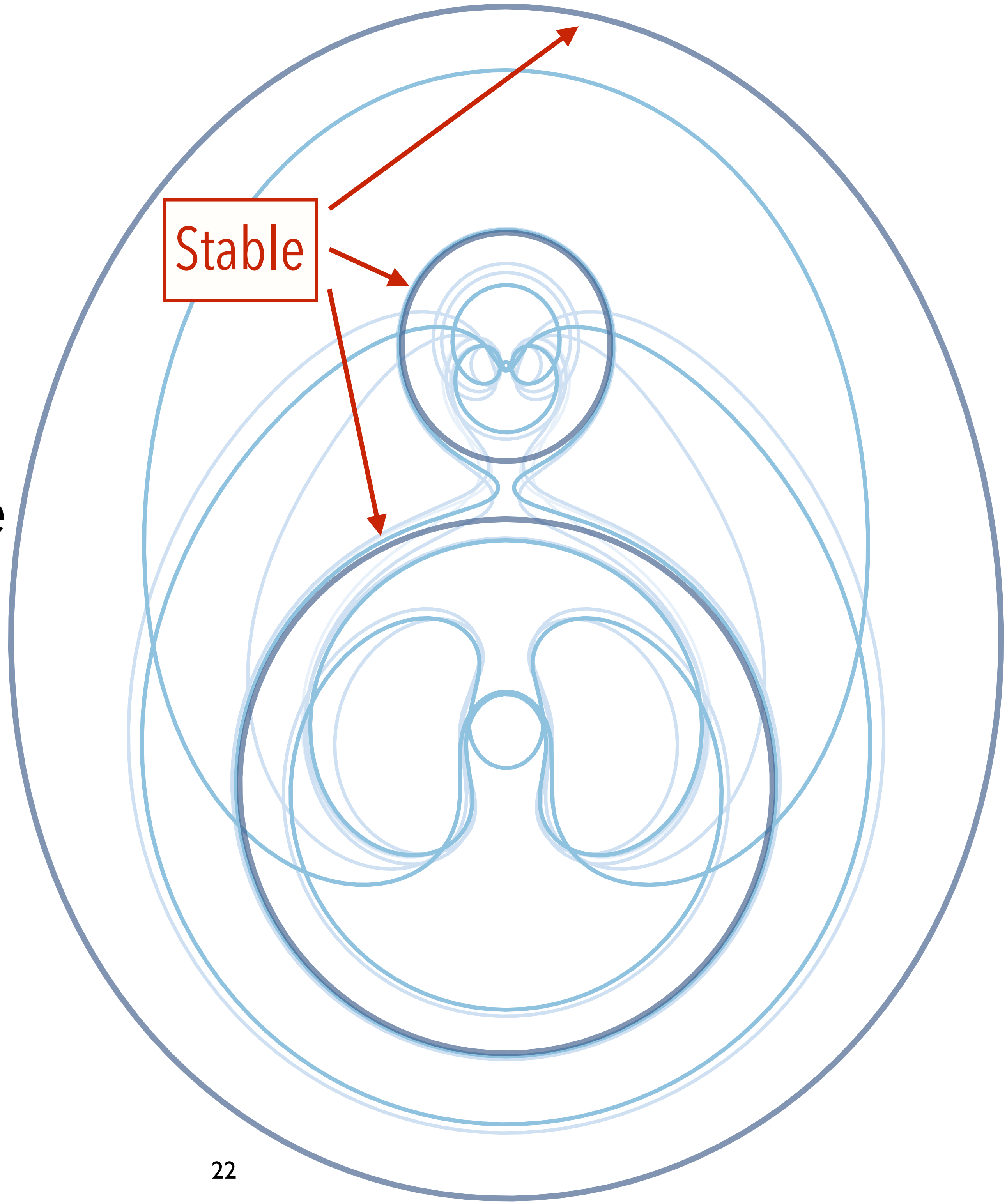
Blue = unstable MOTS

Fading = lost for numerical reasons

(Pook-Kolb, Booth, Hennigar 2021)

Binary Mergers

- Only the initial and final outer AHs are stable
- Other MOTS are unstable and do not bound trapped regions
- Eigenvalues vanish at creation/annihilation events



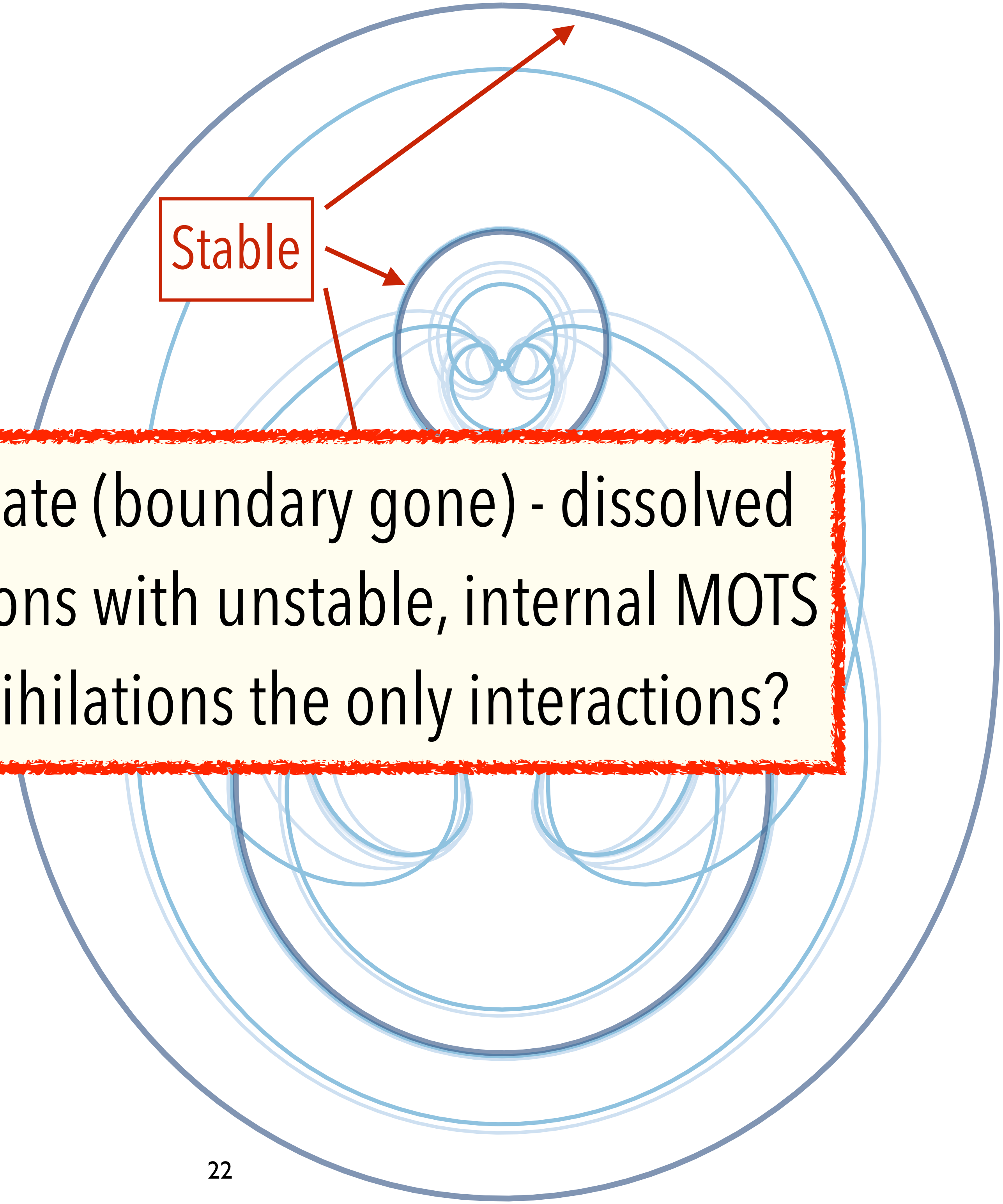
Binary Mergers

- Only the initial and final outer AHs are stable

-

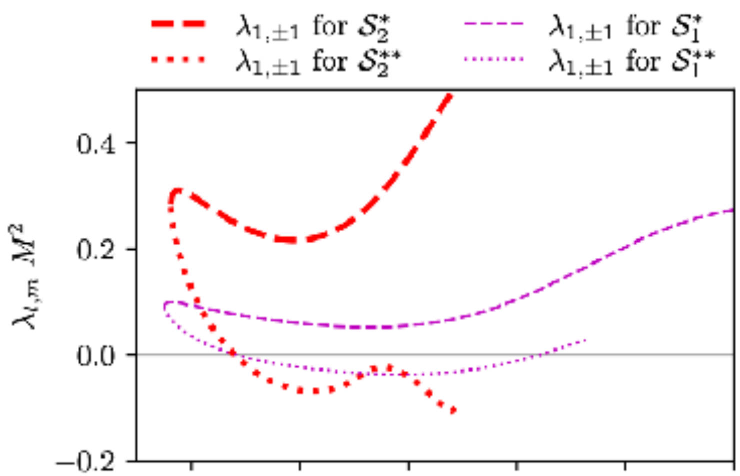
- MOTS can annihilate (boundary gone) - dissolved
- Involves interactions with unstable, internal MOTS
- Are creations/annihilations the only interactions?

- Eigenvalues vanish at creation/annihilation events



Clue 1: Vanishing eigenvalues, smooth evolution

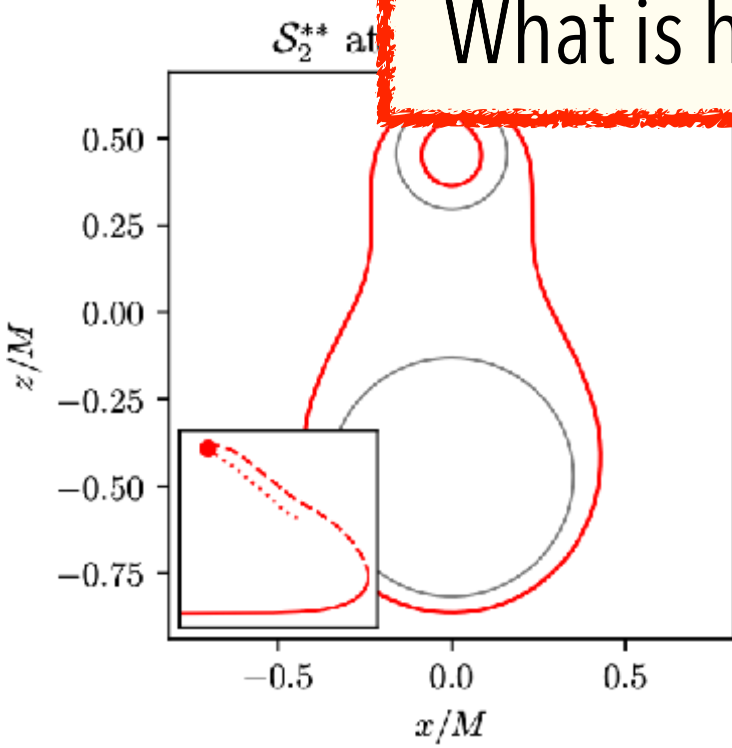
Pook-Kolb, Booth, Hennigar, PRD 2021 "Ultimate fate…II.."

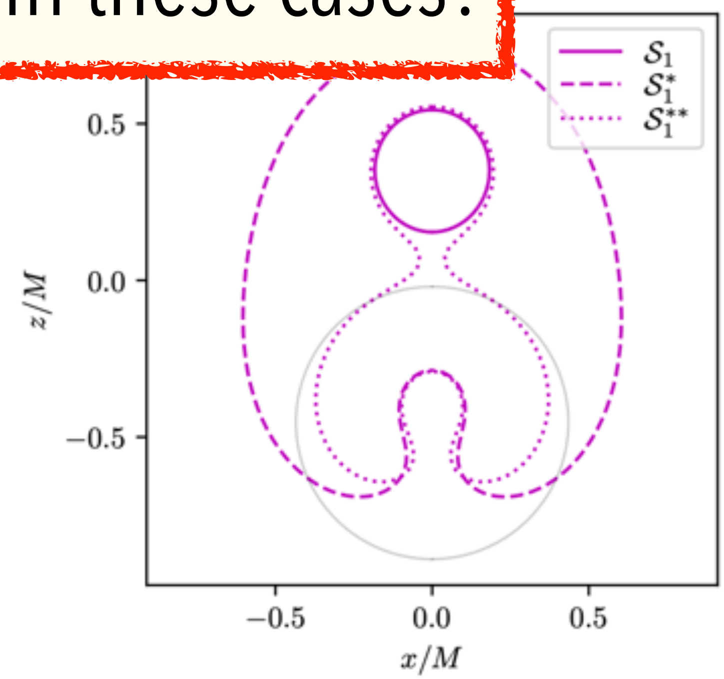


spacetime deformation

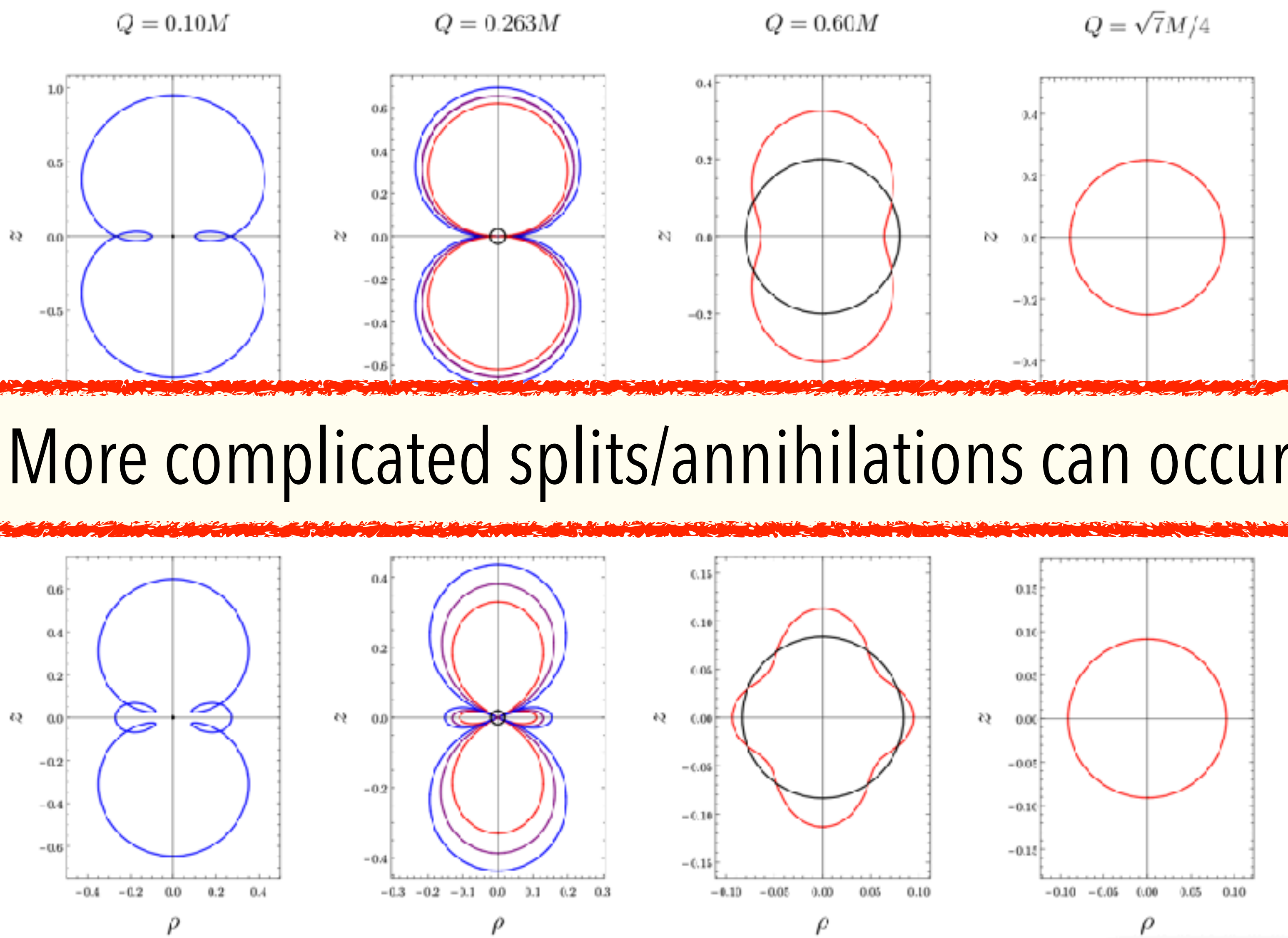
Non-axisymmetric eigenvalues

What is happening in these cases?





Clue 2: Tri-furcations in Reissner-Nordström???



More complicated splits/annihilations can occur

Hennigar, Chan, Newhook, Booth PRD 2022 (Interior MOTS…)

Recall bifurcation theory for dynamical systems…

Fixed points for differential equations

$$\dot{x} = \alpha + x^2$$

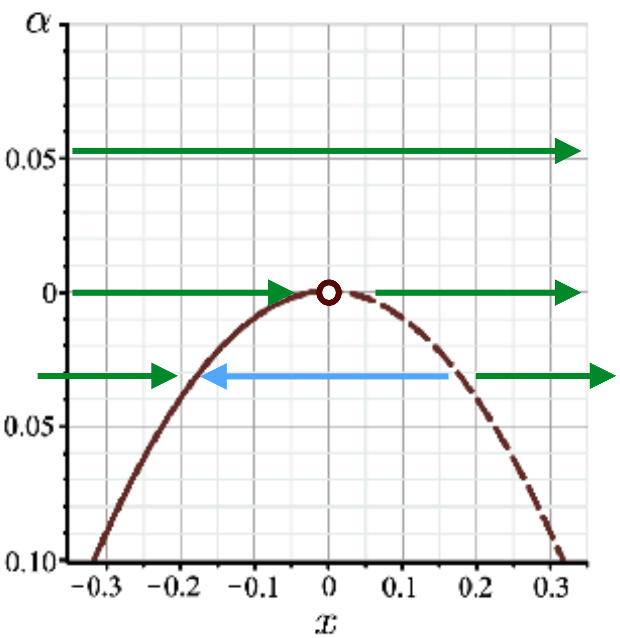
$\alpha < 0$ has fixed points:
$x_o = \pm\sqrt{|\alpha|}$

$$\dot{x} = x(\alpha + x)$$

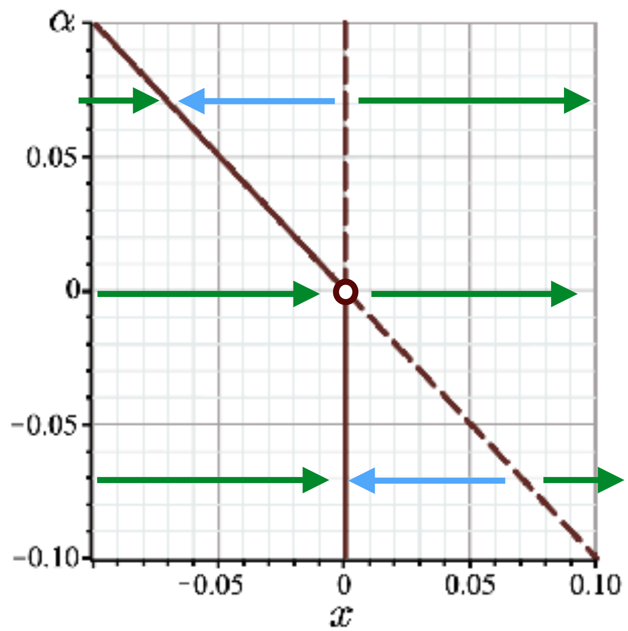
fixed points:
$x_o = 0, -\alpha$

$$\dot{x} = x\left(\alpha + x^2\right)$$

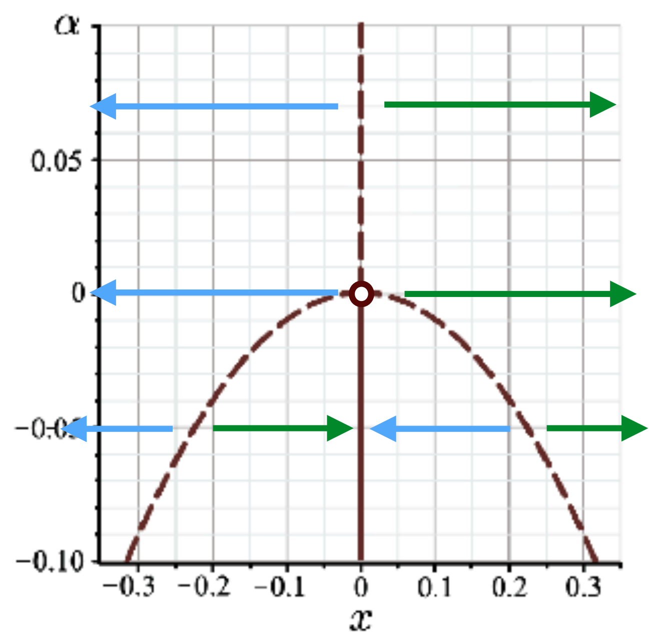
fixed points: $x = 0$,
$\alpha < 0 : x_o = \pm\sqrt{|\alpha|}$



a) Saddle-node



b) Transcritical



c) Pitchfork

Recall bifurcation theory for dynamical systems…

Fixed points for differential equations

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$$\dot{x} = x(\alpha + x)$$

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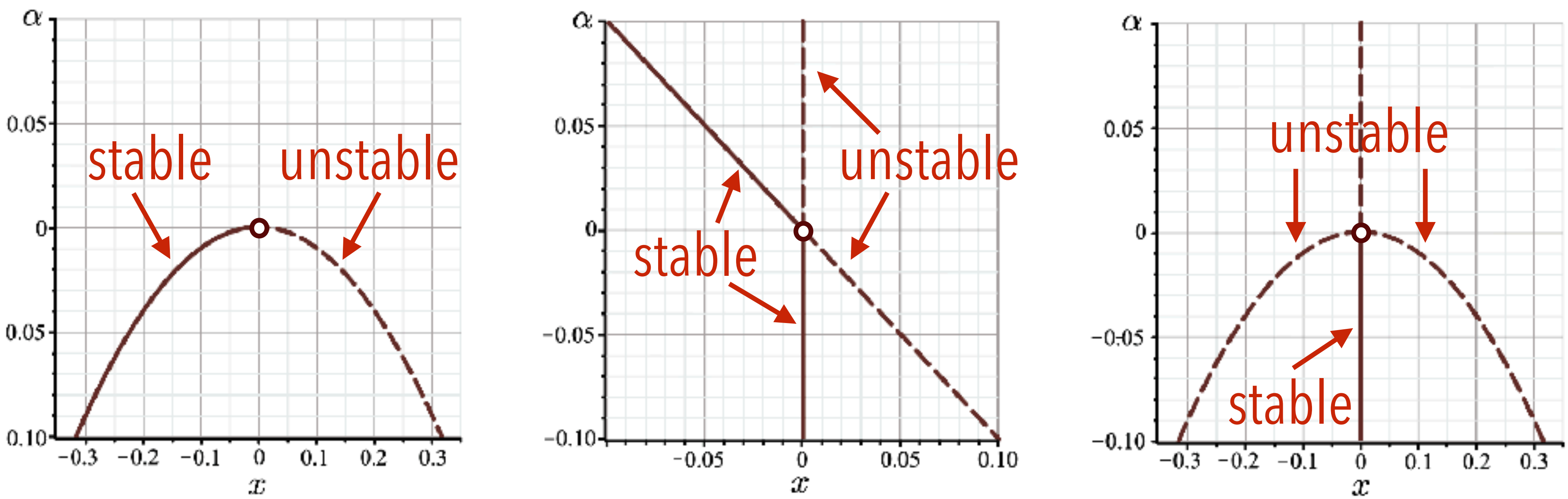


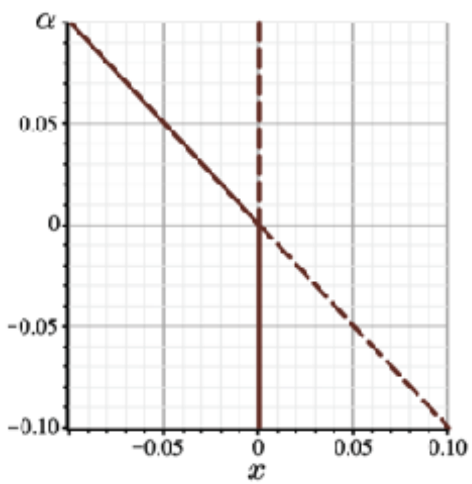
a) Saddle-node b) Transcritical c) Pitchfork

More generally:

- $\dot{x} = f(\alpha, x) \implies$ for a fixed point $f(\alpha, x_o) = 0$
- Near fixed point: $x = x_o + \delta x \implies \delta\dot{x} = f_x\,\delta x$
- Stable $f_x(\alpha, x_o) < 0$

  Unstable $f_x(\alpha, x_o) > 0$

$$f = x(x + \alpha)$$
$$\implies f_x = 2x + \alpha$$
$$\implies f_\alpha = x$$



b) Transcritical

- DE for a curve of fixed points $X(\alpha)$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\big(f(\alpha, X(\alpha))\big) = f_\alpha + f_x X_\alpha = 0$$

Unique solution if $X_\alpha = -\dfrac{f_\alpha}{f_x} = -\dfrac{x}{2x + \alpha}$

At bifurcation point $(\alpha, x) = (0{,}0)$, X_α is unconstrained

Even more generally…

- $\dot{x}^i = f^i(\alpha, x) \implies$ for a fixed point $f^i(\alpha, x_o^j) = 0$
- Near fixed point: $x^i = x_o^i + \delta x^i \implies \delta\dot{x}^i = f^i_{,j}\,\delta x^j$
- Stable $f^i_{,j}$ all negative eigenvalues

  Unstable $f^i_{,j}$ one or more positive eigenvalues

- DE for a curve of fixed points $X^i(\alpha)$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(f^i(\alpha, X^j(\alpha))\right) = f^i_\alpha + f^i_{,j} X^j_\alpha = 0 \implies X^i_\alpha = -\,[f^i_{,j}]^{-1}\, f^j_\alpha$$

if inverse exists

- If $f^i_{,j}$ has a vanishing eigenvalue, X^j_α is unconstrained in that direction.
- Vanishing eigenvalue $\iff$ non-invertible $\iff$ bifurcation point

And now: bifurcation theory for MOTSs

	ODEs	MOTSs
Phase space	coordinates : x^i (finite dimensional)	surfaces S : $X^\alpha(\theta, \phi)$ (infinite dimensional)
Fixed points	$f^j(\alpha, x^i) = 0$	$\theta_\ell(X^\alpha(\theta, \phi)) = 0$
Linearization	$x^i \to x^i + \epsilon \Delta x^i$ $f_\alpha \Delta\alpha + f^j_{,i} \Delta x^i = 0$	$X^\alpha \to X^\alpha + \epsilon \psi N^\alpha$ $\delta_\alpha \theta_+ + L_S \psi = 0$
Stability	all eigenvalues of $f^i_{,j}$ <u>negative</u>	all eigenvalues of L_S <u>positive</u>
Birfurcations	vanishing λ_ν of $f^i_{,j}$	vanishing λ_ν of L_S
"Direction"of bifurcation	eigenvector v_ν	eigenfunction ψ_ν

opposite name conventions!!!

Spherical Symmetry

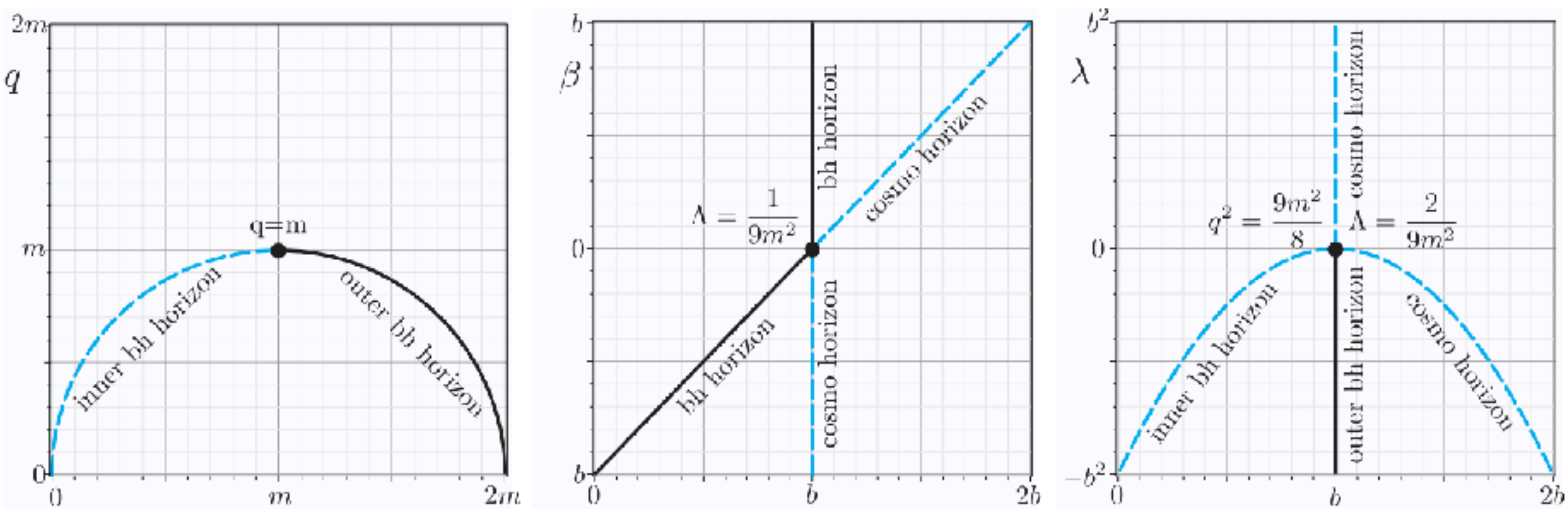
Reissner-Nordström-deSitter

$$ds^2 = -F\,dv^2 + 2dvdr + r^2 d\Omega^2$$

$$F = -\frac{\Lambda}{3}r^2 + 1 - \frac{2m}{r} + \frac{q^2}{r^2}$$ for functions $\Lambda(\alpha), m(\alpha), q(\alpha)$

MOTSs bifurcations are **exactly** ODE bifurcations (modulo naming) with

$$\dot{r} = \frac{dr}{dv} = f = \frac{F}{2}$$ (radial null geodesics)



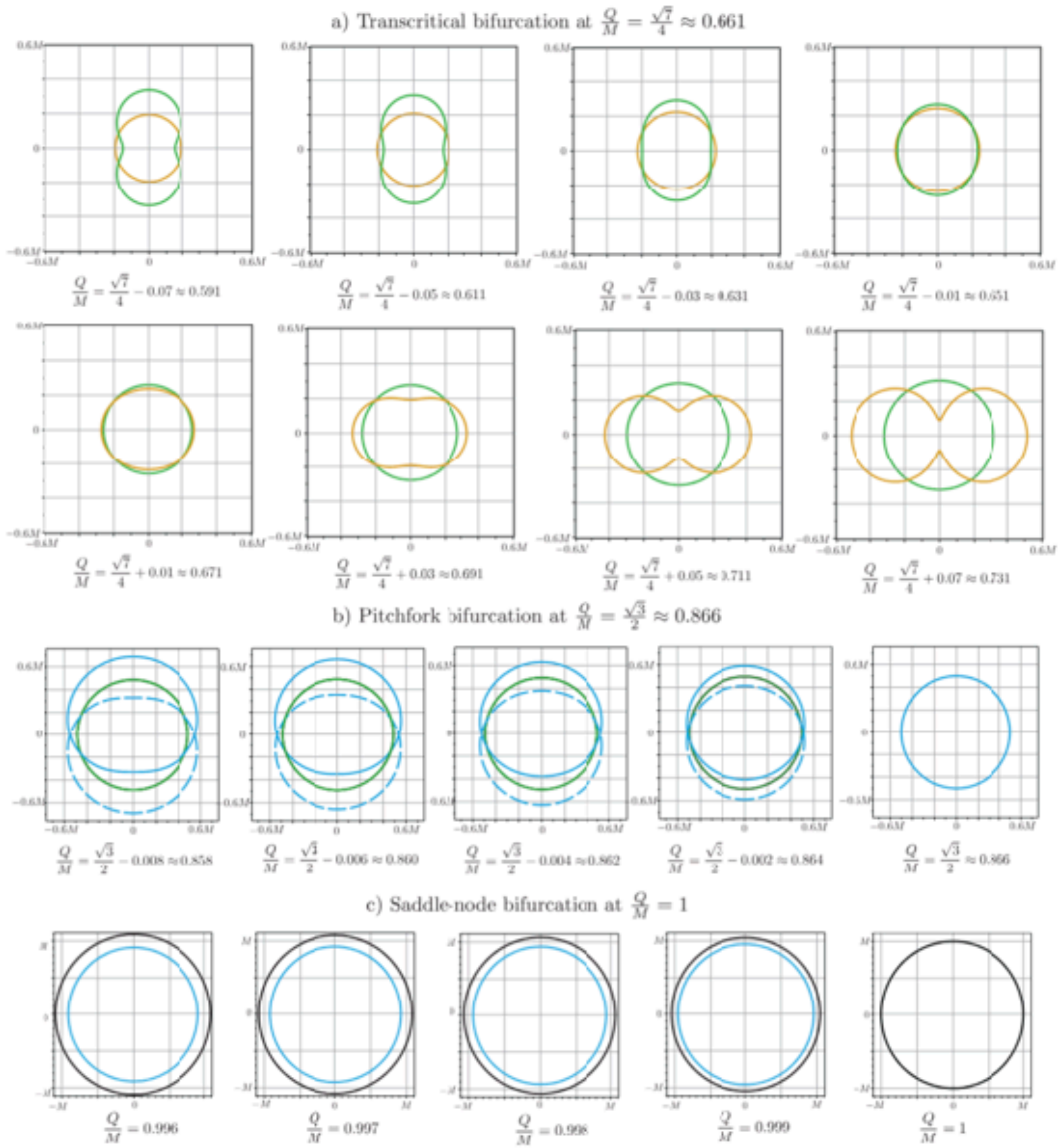
a) Reissner-Nordström: saddle-node

b) Schwarzschild-deSitter: transcritical

c) Reissner-Nordström-deSitter: pitchfork

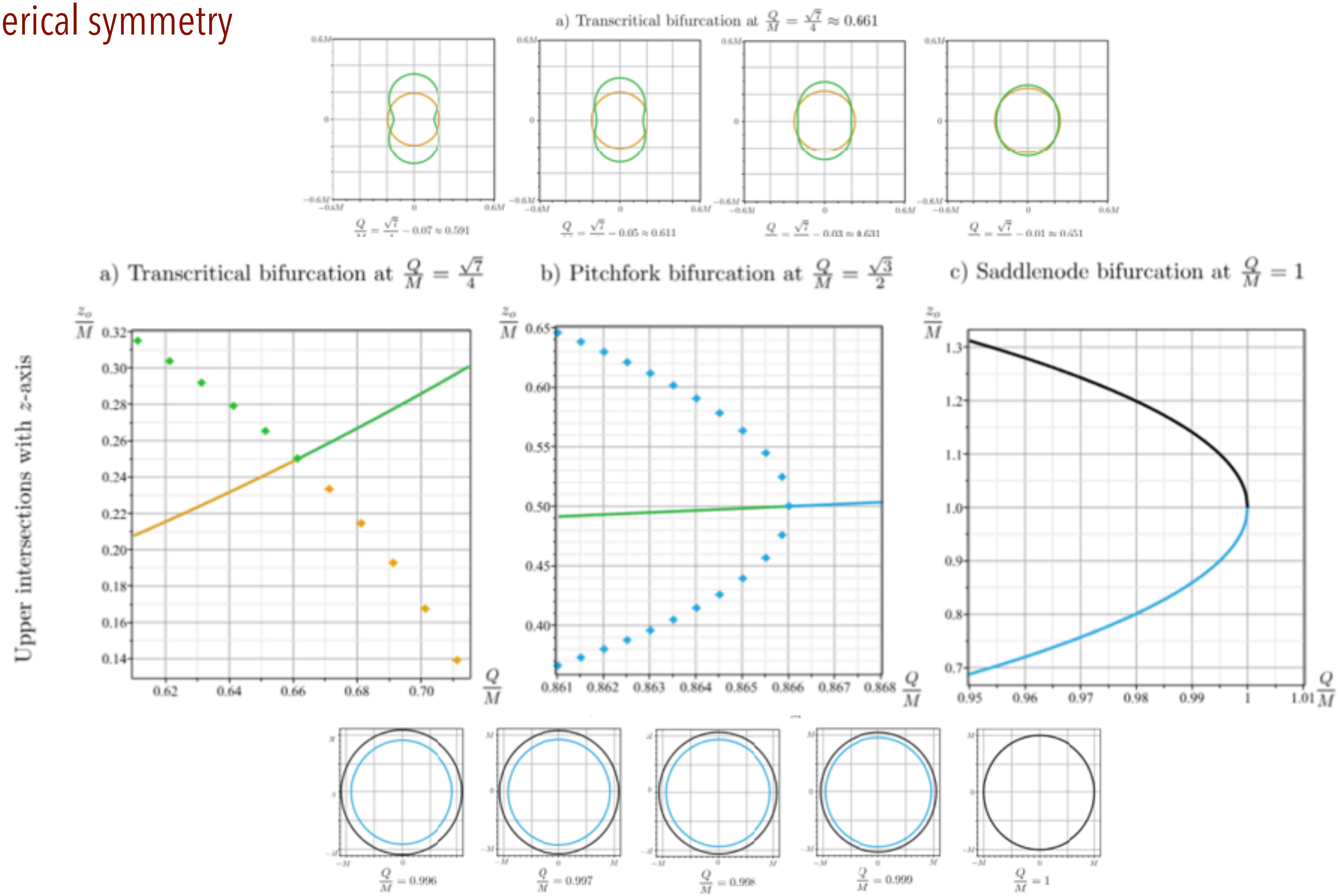
Axi-symmetric bifurcations from spherical symmetry

Back to Reissner-Nordström: order emerges



Axi-symmetric bifurcations from spherical symmetry

Back to Reissner-Nordström: order emerges



Only axisymmetry

Weyl-Schwarzschild

- All static, axisymmetric metrics can be written in the form

$$ds^2 = -e^{U} dt^2 + e^{-2U+2V}\left(dz^2 + d\rho^2\right) + e^{-2U}\rho^2 d\phi^2$$

where $U = U(\rho, z)$ and $V = V(\rho, z)$.

- These are vacuum solutions if U is a solution of the flat space Laplace equation:

$$\frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{\partial^2 U}{\partial z^2} = 0$$

and $\displaystyle \frac{\partial V}{\partial \rho} = \rho\left(\left[\frac{\partial U}{\partial \rho}\right]^2 + \left[\frac{\partial U}{\partial z}\right]^2\right) \ , \ \frac{\partial V}{\partial z} = 2\rho\frac{\partial U}{\partial \rho}\frac{\partial U}{\partial z} \quad .$

- U solutions can be classified in the usual way: monopole, dipole, quadrupole, octopole etc.

Weyl-distorted Schwarzschild

$$ds^2 = -e^{2U}\left(1-\frac{2m}{r}\right) + e^{-2U+2V}\left(\frac{dr^2}{1-\frac{2m}{r}} + r^2 d\theta^2\right) + e^{-2U} r^2 \sin^2\theta d\phi^2$$

- If we demand no conical singularities on the horizon and deform from infinity, then the metric on the horizon at r=2m is:

$$dS^2 = 4m^2 e^{-2U}\left(e^{4U-4u_o} d\theta^2 + \sin^2\theta d\phi^2\right)$$

where $U(2m,\theta) = \sum_{i=0}^{\infty} \alpha_i P_i(\cos\theta)$ and $V(2m,\theta) = 2U(2m,\theta) - 2u_o$

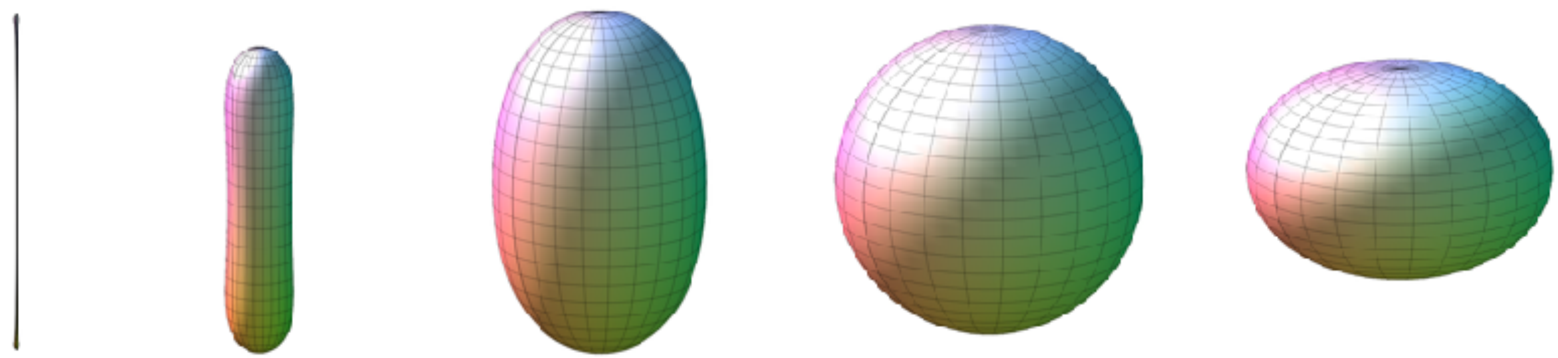
with $U(2m,0) = U(2m,\pi) = u_o \ , \ \sum_{k=1}^{\infty} \alpha_{2k-1} = 0 \ , \ \sum_{k=1}^{\infty} \alpha_{2k} = u_o$

Focus on quadrupole case

Weyl Distorted Schwarzschild

T.Pilkington, A.Melanson, J. Fitzgerald, IB
CQG 28:125018,2011

- Quadrupole solutions give a one-parameter (α) family of distortions
- $r = 2m$ bends but it doesn't break!

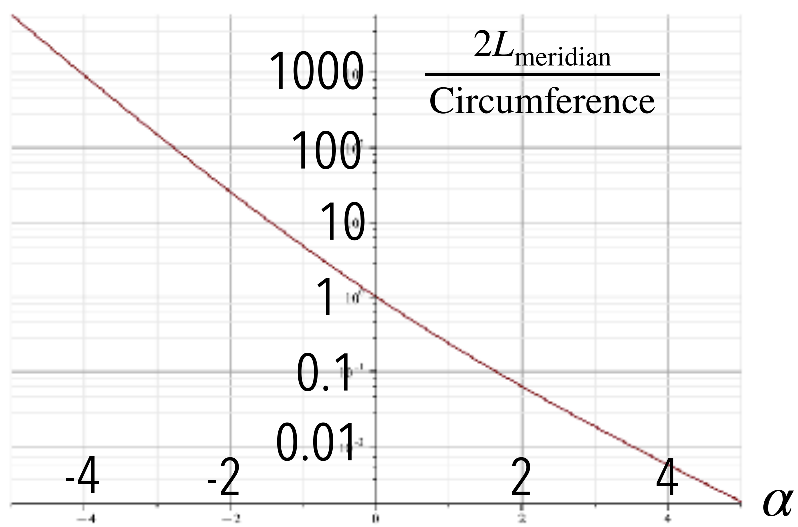


$\alpha = -3.2$ $\alpha = -0.8$ $\alpha = -0.2$ $\alpha = 0$ $\alpha = 0.124$

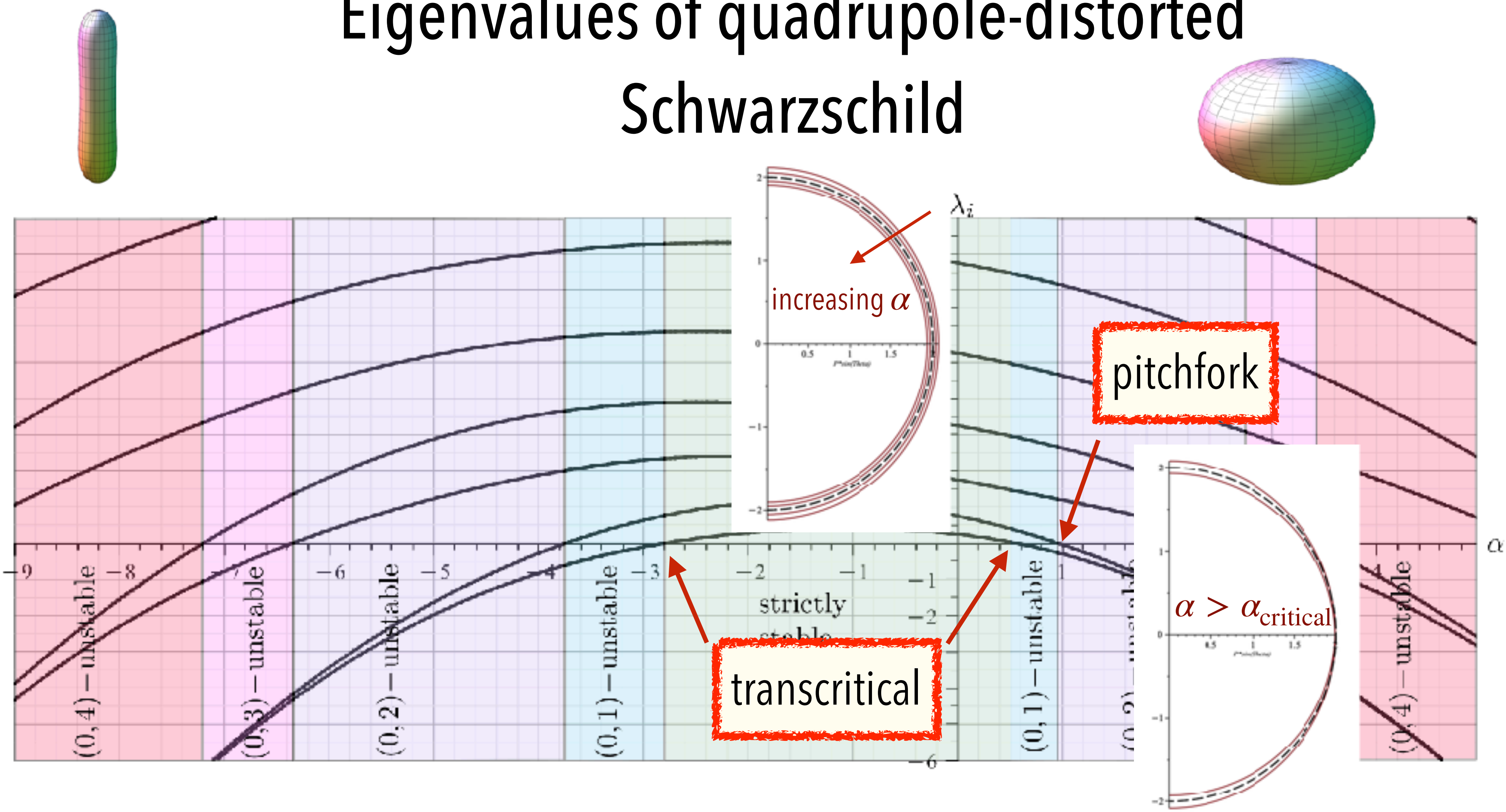
Not embeddable
$\alpha \geq 0.125$

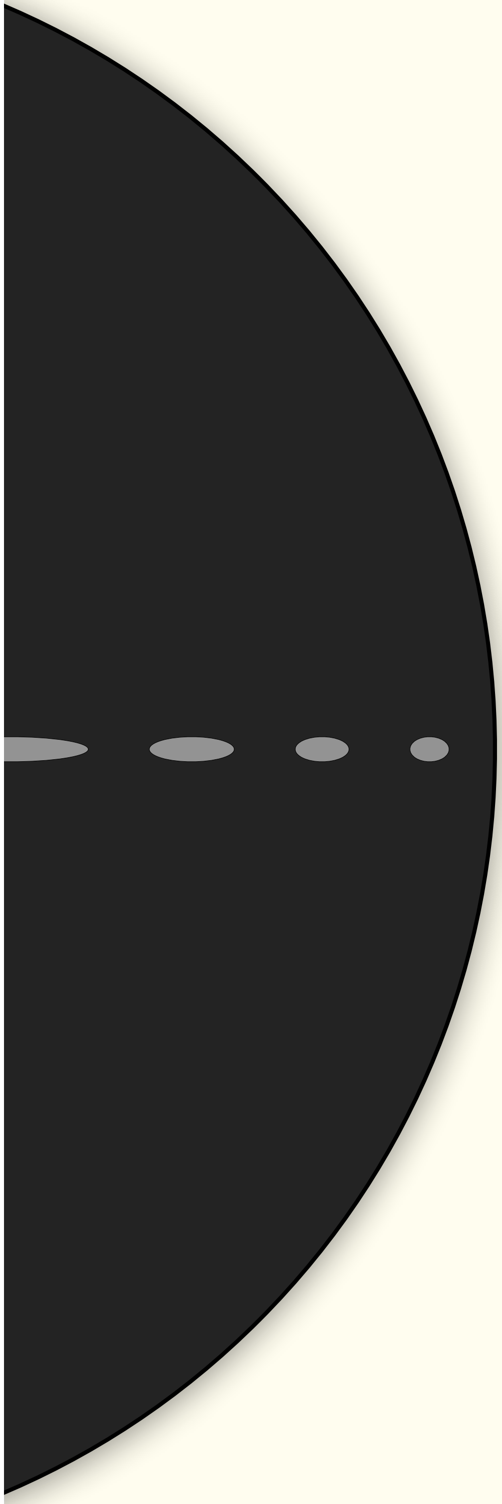
Area$=16\pi^2 m^2 e^{-2\alpha}$

Circumference $= 4\pi m$



Eigenvalues of quadrupole-distorted Schwarzschild





Take aways

- Most MOTS are not boundaries of trapped regions: they simply identify regions of (very) strong gravity
- Rich set of unstable MOTS inside all black holes studied
- Strictly stable MOTS are (at least local) boundaries
- Stability operator also determines MOTS evolution
- Evolution is unique *except* at stability transitions
- These include the well-known pair-creations and pair-annihilations (principal eigenvalues, saddle-node)
- These involve interactions with internal (unstable) MOTS
- Other possibilities include transcritical and pitchfork
- Possible bifurcations are universal