

Beyond Circles

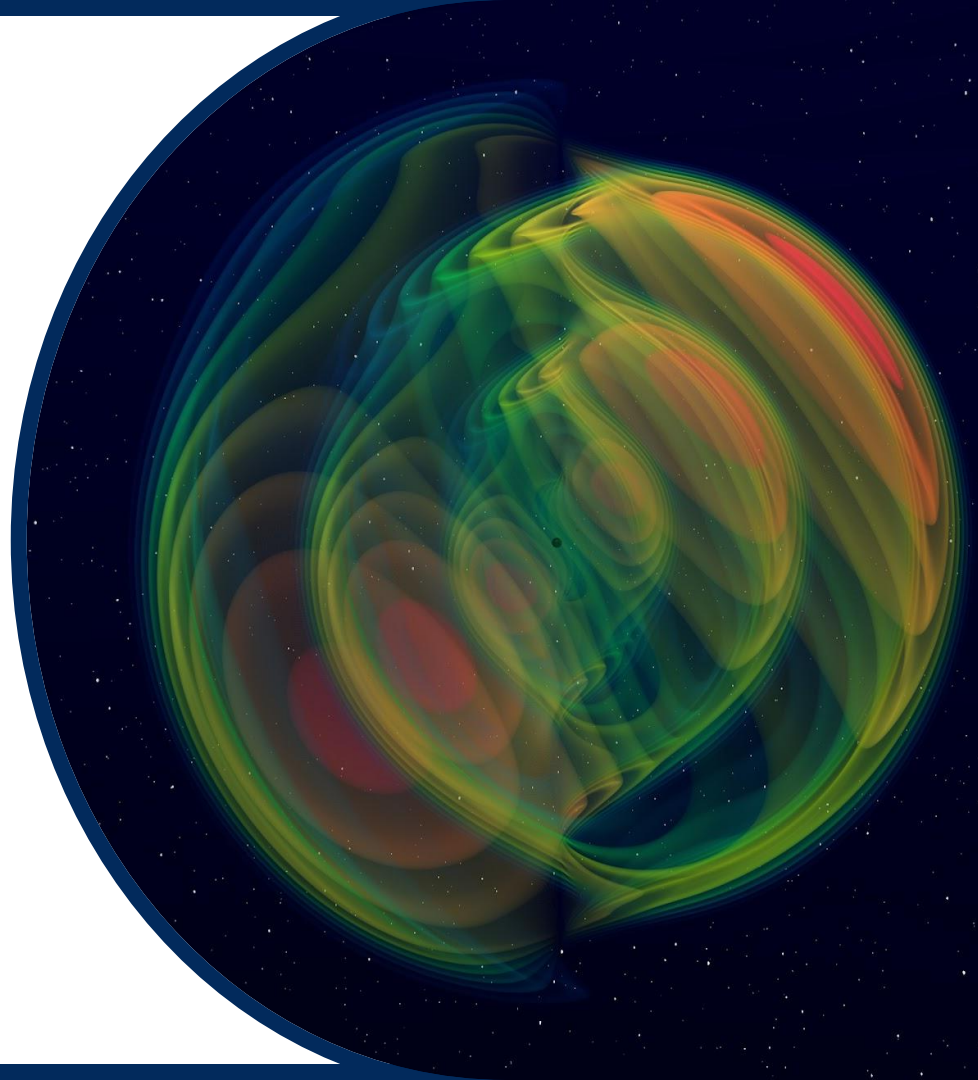
*Stationary axisymmetric black
holes and the breaking of
circularity*

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*SISSA, Trieste
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The talk, in a nutshell

Topic

stationary & axisymmetric
isolated objects
[i.e. rotating BHs]



many well-established properties
are not obvious beyond GR



[Keenan Crane [\[link\]](#)]

CIRCULARITY

'additional'
symmetry of BHs
in (vacuum) GR

- What is circularity?
 - Definition(s) and interpretation
 - Why circularity?
 - Why *not* circularity?
- What do we lose by not having it?



[AI modified]

E. Babichev, **JM**, JCAP 10
(2025) 011 [2505.08880]

F. Del Porro, **JM**,
[2511.02911]

What is circularity?

Def

A stationary & axisymmetric spacetime
is said to be *circular**
if it can be foliated by codimension-2
surfaces orthogonal to the Killing vectors

Killing vectors

ξ^μ stationarity

ψ^μ axisymmetry

by Frobenius' theorem

$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \xi_{\sigma]} = 0$$

$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \psi_{\sigma]} = 0$$

[everywhere]

[*a.k.a. $(t - \varphi)$ symmetric or orthogonally transitive]

General Relativity

Robert M. Wald



Circularity and geometry

Circularity conditions

$$\begin{aligned}\xi_{[\mu}\psi_{\nu}\partial_{\rho}\xi_{\sigma]} &= 0 && \text{geometric statement on} \\ \xi_{[\mu}\psi_{\nu}\partial_{\rho}\psi_{\sigma]} &= 0 && \text{Killing flows}\end{aligned}$$

Practically,
there exist coordinates such that

$$g_{\mu\nu} = \begin{pmatrix} * & 0 & 0 & * \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ * & 0 & 0 & * \end{pmatrix}$$

t
 r
 θ
 φ

$t, \varphi, \dots$ Killing coordinates

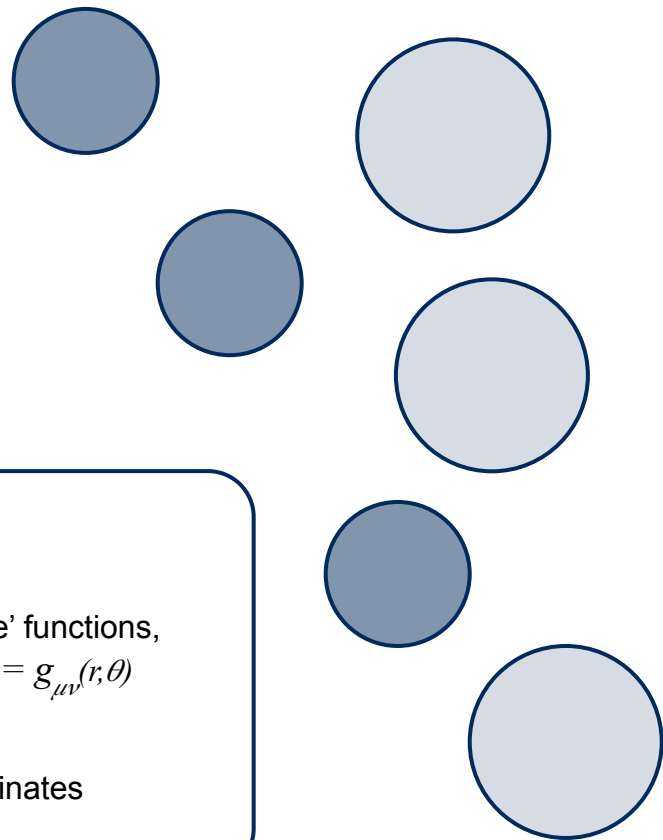
$(t - \varphi)$ symmetry:

$$t \rightarrow -t, \varphi \rightarrow -\varphi$$

i.e. Boyer–Lindquist coordinates

5 ‘free’ functions,

$$g_{\mu\nu} = g_{\mu\nu}(r, \theta)$$



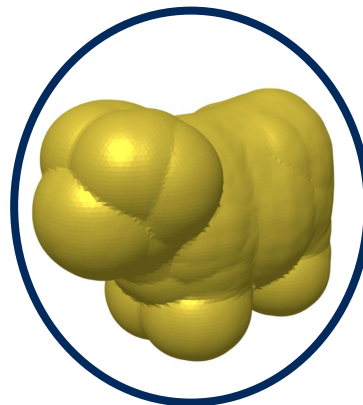
Why circularity?

Circularity is quite ubiquitous, because:

- many GR sol's are circular
- some beyond-GR sol's are circular
- *'square peg in a circular hole'*: beyond-GR sol's that are continuously connected to GR are circular

[Xie, Zhang, Silva, de Rham, Witek, Yunes,
PRL 126 (2021) 241104 [2103.03925]]

- simplicity
 - block-diagonalisation of $g_{\mu\nu}$
 - and more...



[Lehman [2504.00506]]

Why *not* circularity?

Some beyond GR sol's are *not* circular

- Cubic Galileon [numerical]

Grandclement, CQG 41, 025012 (2024)
[2308.11245]

Van Aelst, Gourgoulhon, Grandclément,
Charmousis, CQG 37 035007 (2020)
[1910.08451]

- Semiclassical gravity [numerical]

Fernandes,
PRD 108 (2023) 6, 6 [2305.10382]

- Einstein-æther [numerical]

Adam, Figueras, Jacobson, Wiseman,
Class. Quant. Grav. 39 (2022) 12, 125001
[2108.00005]

- Disformal Kerr [analytical]

Anson, Babichev, Charmousis, Hassaine,
JHEP 2021 (2021) 18 [2006.06461]

Ben Achour, Liu, Motohashi, Mukohyama,
Noui, JCAP 11 (2020) 001 [2006.07245]

- Generalised Proca [analytical]

Fernandes
[2601.21163]

- 'Locality principle' regular BHs
[analytical-ish]

Eichhorn, Held,
JCAP 05 (2021) 073 [2103.13163]

Moving beyond circularity

To explore consequences of
circularity breaking,

need 'new' *framework*

①

'need a metric':

need gauge choice that combines
generality and minimality
[i.e. equivalent of Boyer–Lindquist gauge]

[E. Babichev, **JM**,
JCAP 10 (2025) 011 [2505.08880]]

②

then it depends, we'll see...

e.g. for thermodynamics, maybe
need new ways of computing
surface gravity, etc.

[F. Del Porro, **JM**,
[2511.02911]]

General stationary & axisymmetric metric

What is the *simplest* way of writing the *most general* stationary and axisymmetric metric?

No one really knows...

Kerr-like
gauge

'Kerr' as in 'Kerr ingoing
coordinates'
[not Kerr solution]

$$g_{\mu\nu} = \begin{pmatrix} * & * & 0 & * \\ * & 0 & 0 & * \\ 0 & 0 & * & 0 \\ * & * & 0 & * \end{pmatrix}$$

ν r θ φ

6 'free' functions

$$\begin{cases} g_{vv}(r, \theta) \\ g_{vr}(r, \theta) \\ g_{r\phi}(r, \theta) \\ g_{\phi\phi}(r, \theta) \\ g_{\theta\theta}(r, \theta) \end{cases}$$

existence not obvious,
[see paper]

gauge fixing not
complete

[our answer]

Killing coordinates

non-Killing coordinates

Comments:

1. Important open problem!
2. Beware of claims saying that BL-like metrics are most general: no!
3. Current phenomenological parametrisations all circular by design

'Solving' circularity

Kerr-like gauge describes
circular and *non-circular* spacetimes

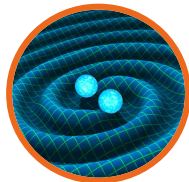
Idea: use it to 'solve' circularity conditions

$$\begin{array}{l} \xi_{[\mu} \psi_{\nu} \partial_{\rho} \xi_{\sigma]} = 0 \\ \xi_{[\mu} \psi_{\nu} \partial_{\rho} \psi_{\sigma]} = 0 \end{array} \longleftrightarrow \begin{array}{l} g^{vr} = f(r) g^{rr} \\ g^{r\phi} = h(r) g^{rr} \end{array}$$

used them to construct two examples,
both simple deformations of Kerr

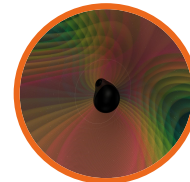
'Minimal' breaking

$$g^{vr} = g_{\text{Kerr}}^{vr} + \delta(r, \theta)$$



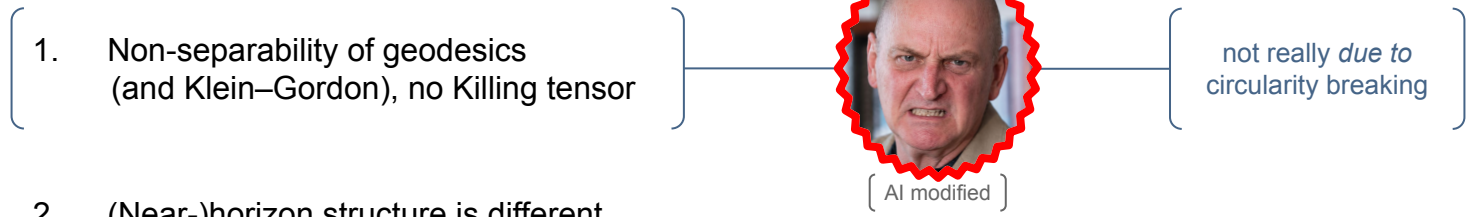
'Not-so-minimal' breaking

$$M \mapsto m(r, \theta)$$

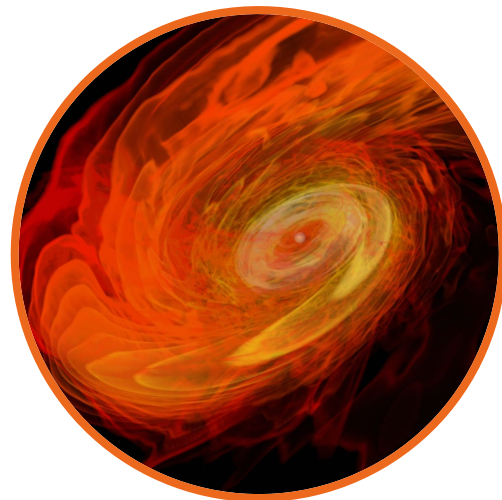
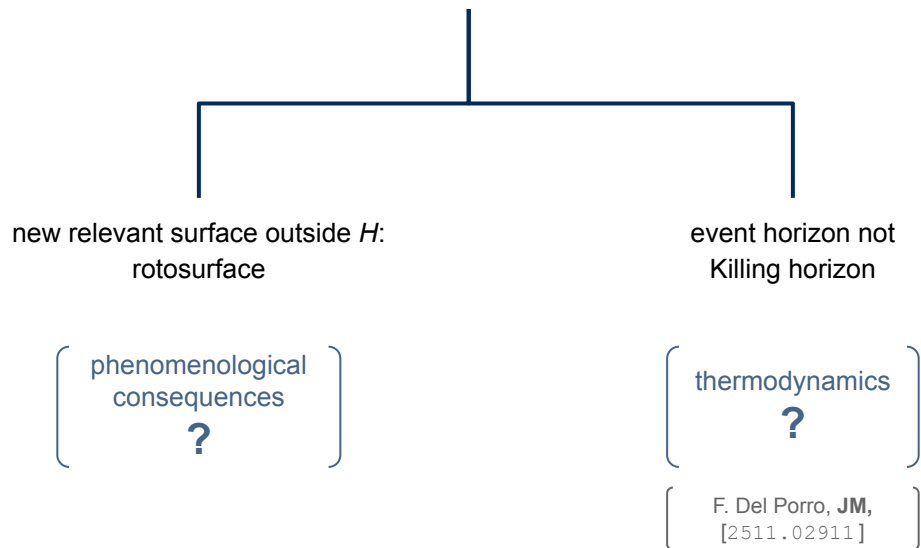


[E. Babichev, **JM**, JCAP 10
(2025) 011 [2505.08880]]

Properties of non-circular BHs



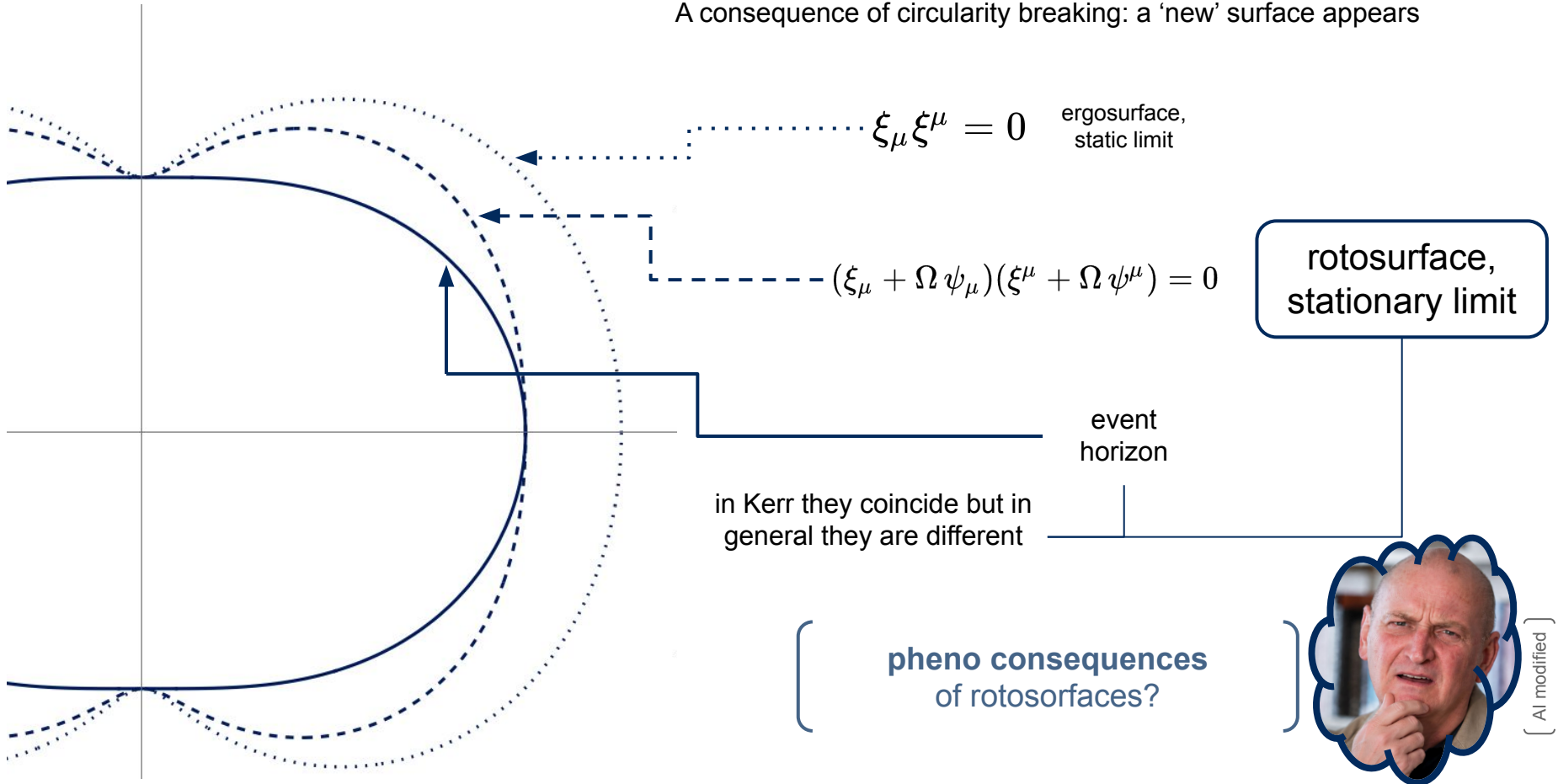
2. (Near-)horizon structure is different



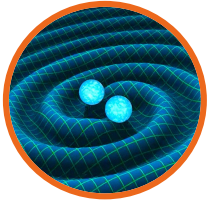
[NASA/AEI/ZIB/M. Koppitz and L. Rezzolla]

The rotosurface

A consequence of circularity breaking: a 'new' surface appears

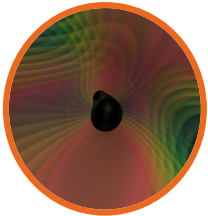


Rotosurface and examples



Minimal

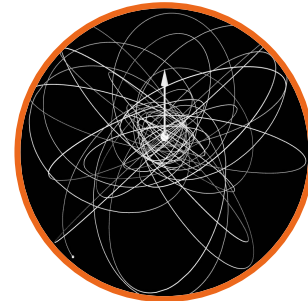
Rotosurface and horizon coincide,
horizon is a 'sphere'



Non-minimal

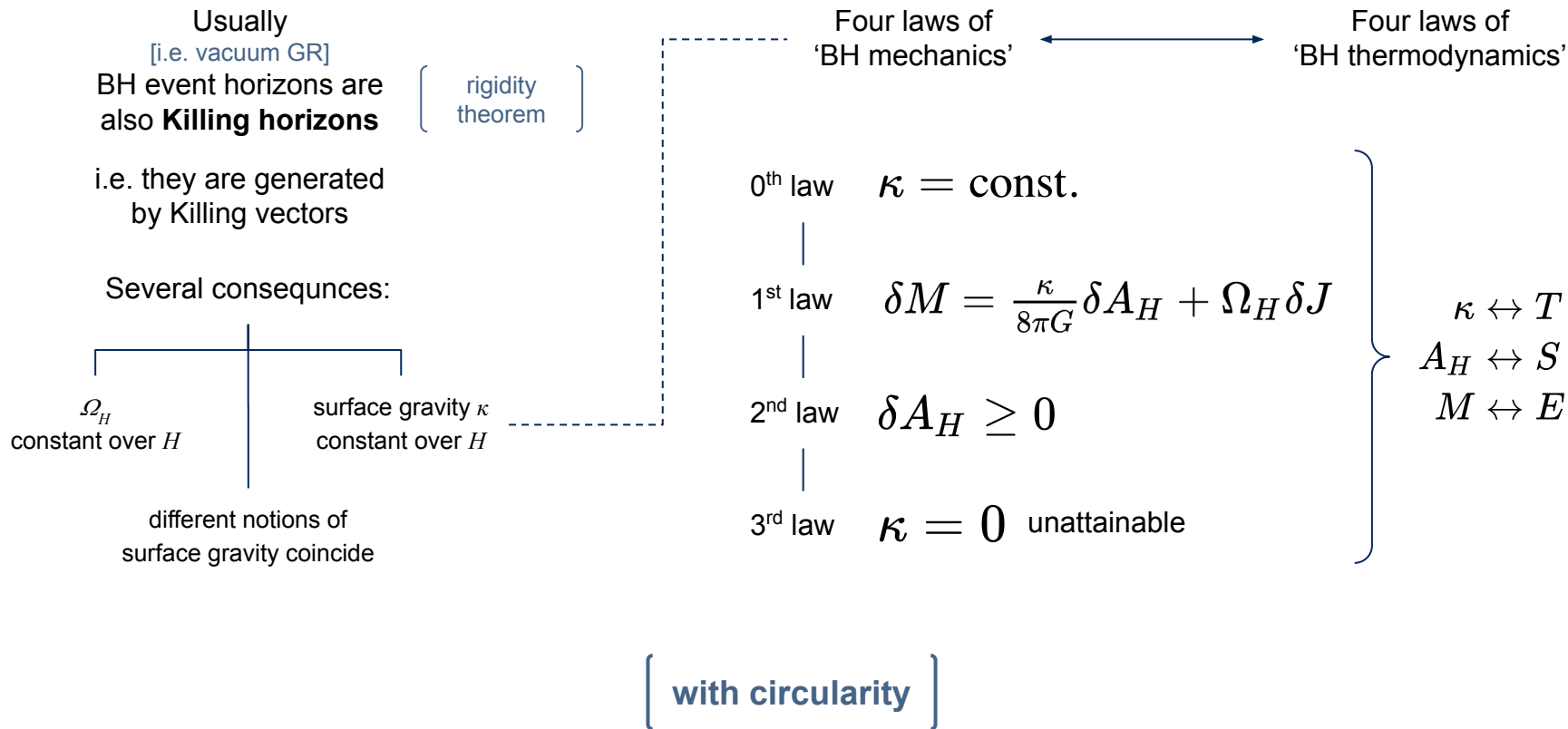
Rotosurface and horizon differ,
Everything known analytically

Have fun!



[Maarten van de Meent [\[link\]](#)]

The horizon



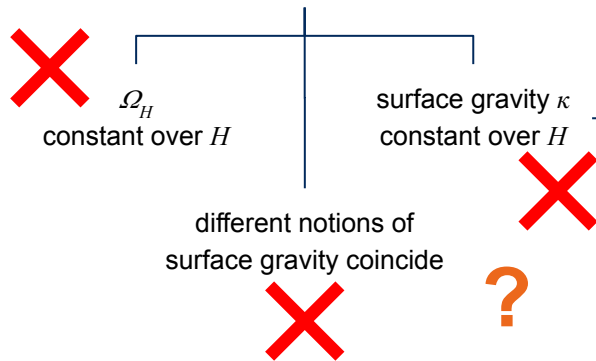
Non-Killing horizon

Horizons
not Killing,
in general

Usually
[i.e. vacuum GR]
BH event horizons are
also **Killing horizons**
i.e. they are generated
by Killing vectors

[rigidity
theorem]

Several consequences:



[without circularity]

Four laws of
'BH mechanics'

Four laws of
'BH thermodynamics'

0th law $\kappa = \text{const.}$

1st law $\delta M = \frac{\kappa}{8\pi G} \delta A_H + \Omega_H \delta J$

2nd law $\delta A_H \geq 0$

3rd law $\kappa = 0$ unattainable

$\kappa \leftrightarrow T$
 $A_H \leftrightarrow S$
 $M \leftrightarrow E$

F. Del Porro, JM,
[2511.02911]

Surface gravities

We computed three notions
of surface gravity $\left[\begin{array}{l} \text{F. Del Porro, JM,} \\ [2511.02911] \end{array} \right]$

κ inaffinity

$$\zeta^\nu \nabla_\nu \zeta^\mu \Big|_{r=H} = \kappa_i \zeta^\mu \Big|_{r=H}$$

κ normal

$$\nabla_\mu (\zeta_\nu \zeta^\nu) \Big|_{r=H} = 2\kappa_n \zeta_\mu \Big|_{r=H}$$

κ peeling

[ask me later...]

on-horizon
surface gravities

near-horizon
surface gravity

assume the horizon is $r = H(\theta)$

then define $\zeta_\mu := \frac{\alpha}{g^{rr}} \partial_\mu [r - H(\theta)]$

$$\kappa_i(\theta) \neq \kappa_n(\theta) = \kappa_p(\theta)$$

non-constant, and different



Hawking radiation

The peeling entails Hawking radiation

Via tunnelling method, we find:
an observer at infinity detects particles with spectral density

$$\langle \hat{\mathcal{N}}_{\omega m} \rangle = \frac{v(\omega, m)}{\exp \left[\frac{2\pi}{\kappa_p} (\omega - m\sigma_H) \right] - 1}$$

Bose–Einstein-ish
distribution w/

$$T_H = \frac{\kappa_p}{2\pi} \quad \text{and} \quad \mu_H = m\sigma_H$$



[AI generated]

Towards thermodynamics

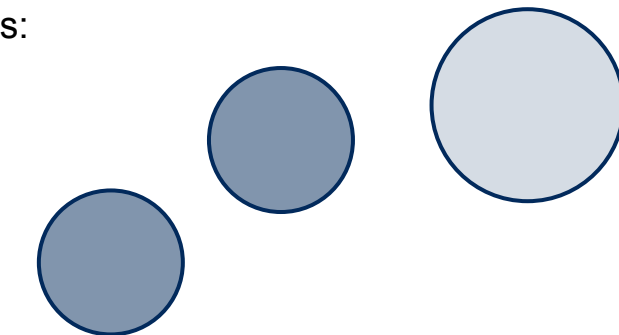
Comments on the four laws of mechanics of non-Killing horizons:

[0th law] constancy of κ : does not hold

[1st law] energy conservation: [w/ caveats]
holds in averaged form

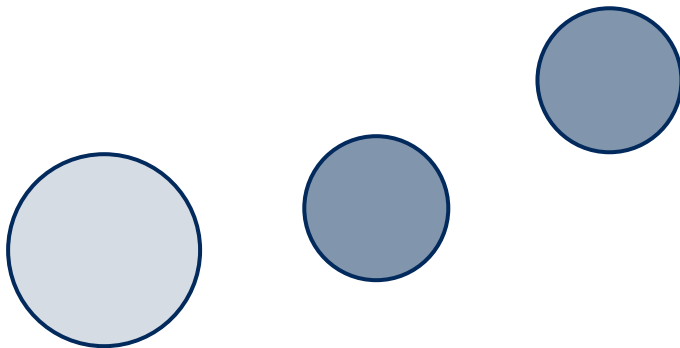
[2nd law] horizon's area cannot decrease: holds
under the same assumptions as usual

[3rd law] unattainability of extremal state: same as
usual



Open questions:

- Thermodynamic interpretation?
- Out of equilibrium / local
(i.e. non global) equilibrium / ?
- [...]



What's next?

What about slow rotation?

Usually, slow rotation means Hartle–Thorne [Lense–Thirring] metric

$$ds^2 = -f(r)dt^2 + \tilde{f}(r)dr^2 + r^2 [d\theta^2 + \sin^2(\theta)d\phi^2] \\ - 2r^2 \sin^2(\theta)\Omega(r, \theta)dtd\phi \\ [\Omega(r, \vartheta) = \mathcal{O}(a)]$$

this is circular

$$ds^2 = -f(r)dv^2 + 2h(r)dvdr + r^2 [d\theta^2 + \sin^2(\theta)d\phi^2] \\ - 2r^2 \sin^2(\theta) [\Omega(r, \theta)dtd\phi + \Lambda(r, \theta)drd\phi] \\ [\Omega(r, \vartheta), \Lambda(r, \vartheta) = \mathcal{O}(a)]$$

e.g. this is not circular

soon on
(non-)circularity TV

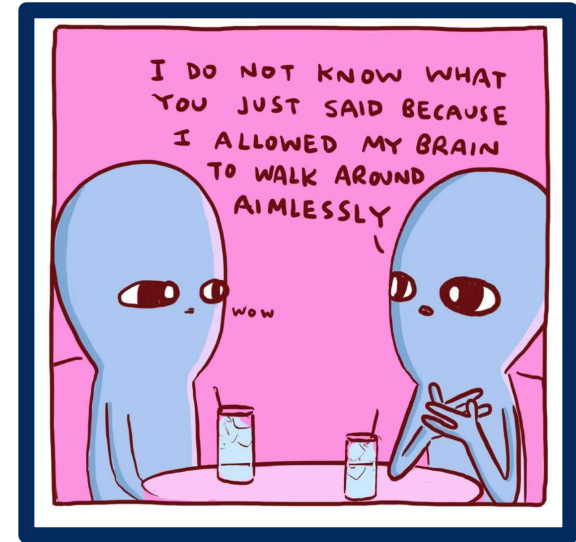
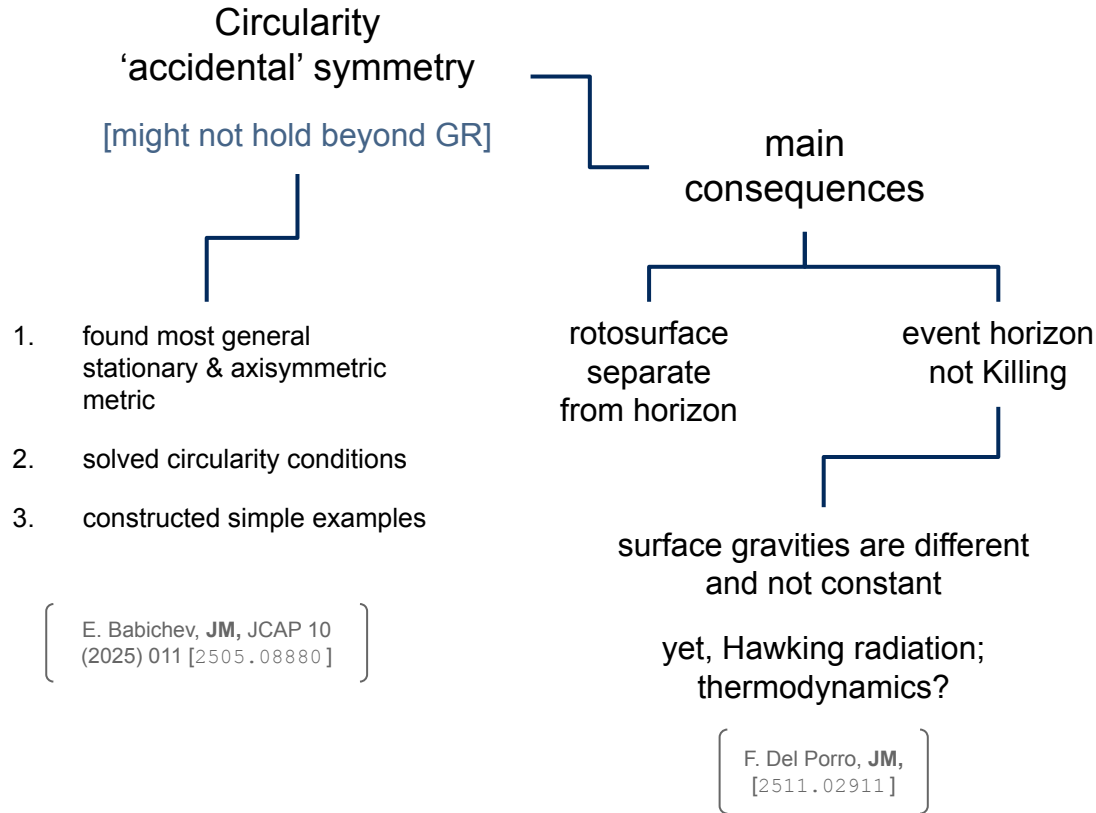
Is there something
'protecting' circularity
at $\mathcal{O}(a)$,

or are we missing some
slowly rotating sol's?

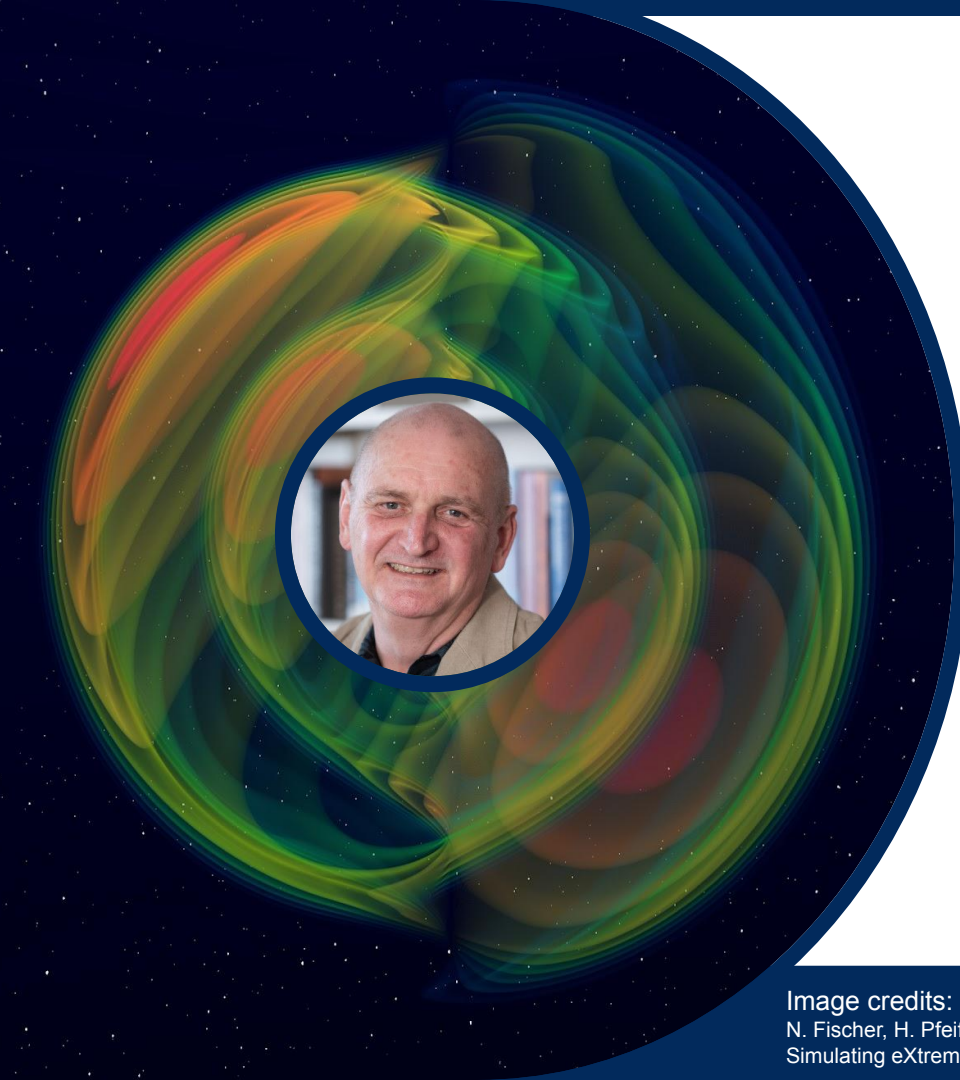


stay tuned...

Recap & Food for thought



[Nathan W. Pyle]



Thanks!

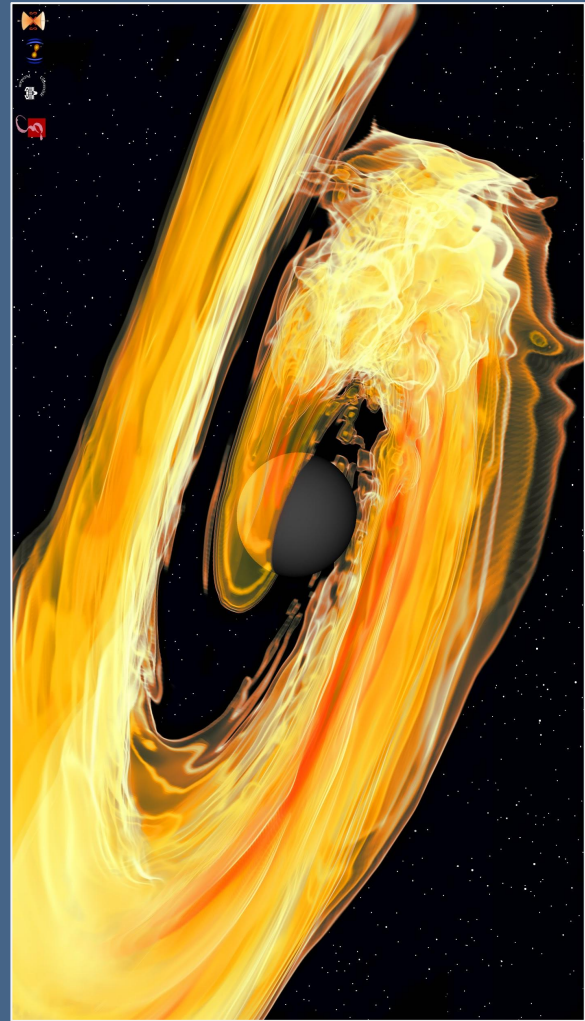
Get in touch

`jacopo.mazza@ijclab.in2p3.fr`

Image credits:

N. Fischer, H. Pfeiffer, A. Buonanno (Max Planck Institute for Gravitational Physics),
Simulating eXtreme Spacetimes (SXS) Collaboration

Backup Slides



[BckUp] Circularity and (*more*) geometry

Since you liked...

$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \xi_{\sigma]} = 0$$

$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \psi_{\sigma]} = 0$$

[everywhere]

... you might also like

$$\xi^{\mu} R_{\mu[\nu} \psi_{\rho} \xi_{\sigma]} = 0$$

$$\psi^{\mu} R_{\mu[\nu} \psi_{\rho} \xi_{\sigma]} = 0$$

[everywhere]



$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \xi_{\sigma]} = 0$$

$$\xi_{[\mu} \psi_{\nu} \partial_{\rho} \psi_{\sigma]} = 0$$

[at least at one point]

equivalent formulations
in 'normal' situations

(use Killing eq.'s and
some assumptions)

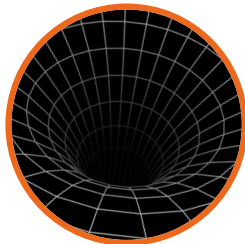
[BckUp] Circularity and matter

$$\left\{ \begin{array}{l} \xi^\mu R_{\mu[\nu} \psi_{\rho} \xi_{\sigma]} = 0 \\ \psi^\mu R_{\mu[\nu} \psi_{\rho} \xi_{\sigma]} = 0 \end{array} \right. \leftarrow \begin{array}{l} \text{statement on} \\ \text{symmetry of sources} \\ \text{via field equations} \end{array}$$

[everywhere]

Einstein spaces
automatically circular

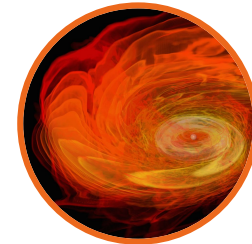
$$\left[\begin{array}{l} \text{Kerr-Newman} \\ \text{(A)dS, etc.} \end{array} \right]$$



[James Zanoni/[Quanta Magazine](#)]

SET of a fluid:
velocity

$$u^\mu = \alpha \xi^\mu + \beta \psi^\nu$$
$$\left[u^{[\mu} \xi^\nu \psi^{\rho]} = 0 \right]$$



[NASA/AEI/ZIB/M. Koppitz and L. Rezzolla]

[BckUp] Square peg in circular hole

$$\left[\begin{array}{l} \text{Y. Xie, J. Zhang, H.O. Silva, C. de} \\ \text{Rham, H. Witek and N. Yunes,} \\ \text{PRL 126 (2021) 241104} \\ \text{[2103.03925]} \end{array} \right] \text{ 'Square peg in a} \\ \text{cirucla hole'}$$

beyond-GR theories like

$$\left[\begin{array}{l} \text{operators up} \\ \text{to dim 4} \end{array} \right] \mathcal{L}_0(g_{\mu\nu}, \varphi) + \alpha \mathcal{L}_M(g_{\mu\nu}, \varphi) \left[\begin{array}{l} \text{higher order} \end{array} \right]$$

theorem

if

then

that allow perturbative treatment

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \sum_n \alpha^n g_{\mu\nu}^{(n)}$$

$$\varphi = \varphi^{(0)} + \sum_n \alpha^n \varphi^{(n)}$$

$$\varphi \text{ scalar } \dots \dots \xi^\mu \partial_\mu \varphi = 0$$

$$\varphi \text{ vector } \dots \dots \varphi \text{ circular}$$

[...]

$$g_{\mu\nu} \text{ circular} \\ \text{order by order in } \alpha$$

[BckUp] Kerr-like gauge

Kerr-like gauge condition

$$\begin{aligned}
 0 &= g_{v\theta} = g_{\tilde{v}\tilde{v}} \frac{\partial V}{\partial \theta} + g_{\tilde{v}\tilde{r}} \frac{\partial R}{\partial \theta} + g_{\tilde{v}\tilde{\theta}} \frac{\partial \Theta}{\partial \theta} + g_{\tilde{v}\tilde{\phi}} \frac{\partial \Phi}{\partial \theta} \\
 0 &= g_{\phi\theta} = g_{\tilde{\phi}\tilde{v}} \frac{\partial V}{\partial \theta} + g_{\tilde{\phi}\tilde{r}} \frac{\partial R}{\partial \theta} + g_{\tilde{\phi}\tilde{\theta}} \frac{\partial \Theta}{\partial \theta} + g_{\tilde{\phi}\tilde{\phi}} \frac{\partial \Phi}{\partial \theta} \\
 0 &= g_{r\theta} = [g_{v\theta}] \frac{\partial V}{\partial r} + [g_{\phi\theta}] \frac{\partial \Phi}{\partial r} \\
 &\quad + \left[g_{\tilde{v}\tilde{r}} \frac{\partial R}{\partial r} + g_{\tilde{v}\tilde{\theta}} \frac{\partial \Theta}{\partial r} \right] \frac{\partial V}{\partial \theta} + \left[g_{\tilde{r}\tilde{\phi}} \frac{\partial R}{\partial r} + g_{\tilde{\theta}\tilde{\phi}} \frac{\partial \Theta}{\partial r} \right] \frac{\partial \Phi}{\partial \theta} \\
 &\quad + \left[g_{\tilde{r}\tilde{r}} \frac{\partial R}{\partial r} + g_{\tilde{r}\tilde{\theta}} \frac{\partial \Theta}{\partial r} \right] \frac{\partial R}{\partial \theta} + \left[g_{\tilde{r}\tilde{\theta}} \frac{\partial R}{\partial r} + g_{\tilde{\theta}\tilde{\theta}} \frac{\partial \Theta}{\partial r} \right] \frac{\partial \Theta}{\partial \theta} \\
 0 &= g_{rr} = g_{\tilde{v}\tilde{v}} \left(\frac{\partial V}{\partial r} \right)^2 + g_{\tilde{r}\tilde{r}} \left(\frac{\partial R}{\partial r} \right)^2 + g_{\tilde{\theta}\tilde{\theta}} \left(\frac{\partial \Theta}{\partial r} \right)^2 + g_{\tilde{\phi}\tilde{\phi}} \left(\frac{\partial \Phi}{\partial r} \right)^2 \\
 &\quad + 2g_{\tilde{v}\tilde{r}} \frac{\partial V}{\partial r} \frac{\partial R}{\partial r} + 2g_{\tilde{v}\tilde{\theta}} \frac{\partial V}{\partial r} \frac{\partial \Theta}{\partial r} + 2g_{\tilde{v}\tilde{\phi}} \frac{\partial V}{\partial r} \frac{\partial \Phi}{\partial r} \\
 &\quad + 2g_{\tilde{r}\tilde{\theta}} \frac{\partial R}{\partial r} \frac{\partial \Theta}{\partial r} + 2g_{\tilde{r}\tilde{\phi}} \frac{\partial R}{\partial r} \frac{\partial \Phi}{\partial r} + 2g_{\tilde{\theta}\tilde{\phi}} \frac{\partial \Theta}{\partial r} \frac{\partial \Phi}{\partial r}
 \end{aligned}$$

perform coordinate change

[tilt Cauchy surface]

$$r \rightarrow r, \quad \theta \rightarrow \theta + r$$

$$\frac{\partial U_i}{\partial \theta} \rightarrow \frac{\partial U_i}{\partial \theta}, \quad \frac{\partial U_i}{\partial r} \rightarrow \frac{\partial U_i}{\partial r} + \frac{\partial U_i}{\partial \theta}$$

Cauchy–Kovalvskaya form

$$\frac{\partial U_i}{\partial \theta} = F \left(g_{\tilde{\mu}\tilde{\nu}}, \frac{\partial U_i}{\partial r}, \partial U_j \right)$$

[BckUp] Remarks

Indices up

$$\frac{g^{vr}}{g^{rr}} = f(r)$$

$$\frac{g^{r\phi}}{g^{rr}} = h(r)$$

nice

Indices down

$$\frac{g_{vr}g_{\phi\phi} - g_{r\phi}g_{v\phi}}{g_{v\phi}^2 - g_{vv}g_{\phi\phi}} = f(r)$$

$$\frac{g_{r\phi}g_{vv} - g_{vr}g_{v\phi}}{g_{v\phi}^2 - g_{vv}g_{\phi\phi}} = h(r)$$

not (so) nice

Solved conditions are
existence conditions for
coordinate change to
'Boyer–Lindquist form'

$$g_{\mu\nu} = \begin{pmatrix} g_{vv} & 0 & 0 & g_{v\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ * & 0 & 0 & g_{\phi\phi} \end{pmatrix}$$



[BckUp] 'Minimal' deformation

Solved circularity conditions

$$g^{vr} = f(r)g^{rr}$$

$$g^{r\phi} = h(r)g^{rr}$$

'soft' breaking

$$g^{vr} = f_{\text{Kerr}} g_{\text{Kerr}}^{rr} + f(r, \theta)_{\text{deform}}$$

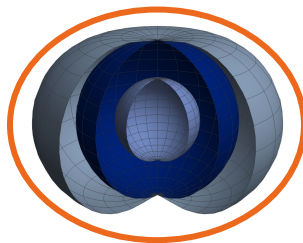
$$g^{r\phi} = h_{\text{Kerr}} g_{\text{Kerr}}^{rr}$$

$$\left| \begin{aligned} g^{\mu\nu} &= g_{\text{Kerr}}^{\mu\nu} + \frac{\delta(r, \theta)}{\Sigma} \delta^\mu_v \delta^\mu_\phi \\ g_{\mu\nu} &= (\text{Quite-A-Mess})_{\mu\nu} \end{aligned} \right|$$

surfaces @ same location

$$r_{\text{erg}} = M + \sqrt{M^2 - a^2 \cos^2 \theta}$$

$$r_{\text{H}} = r_{\text{rot}} = M + \sqrt{M^2 - a^2}$$



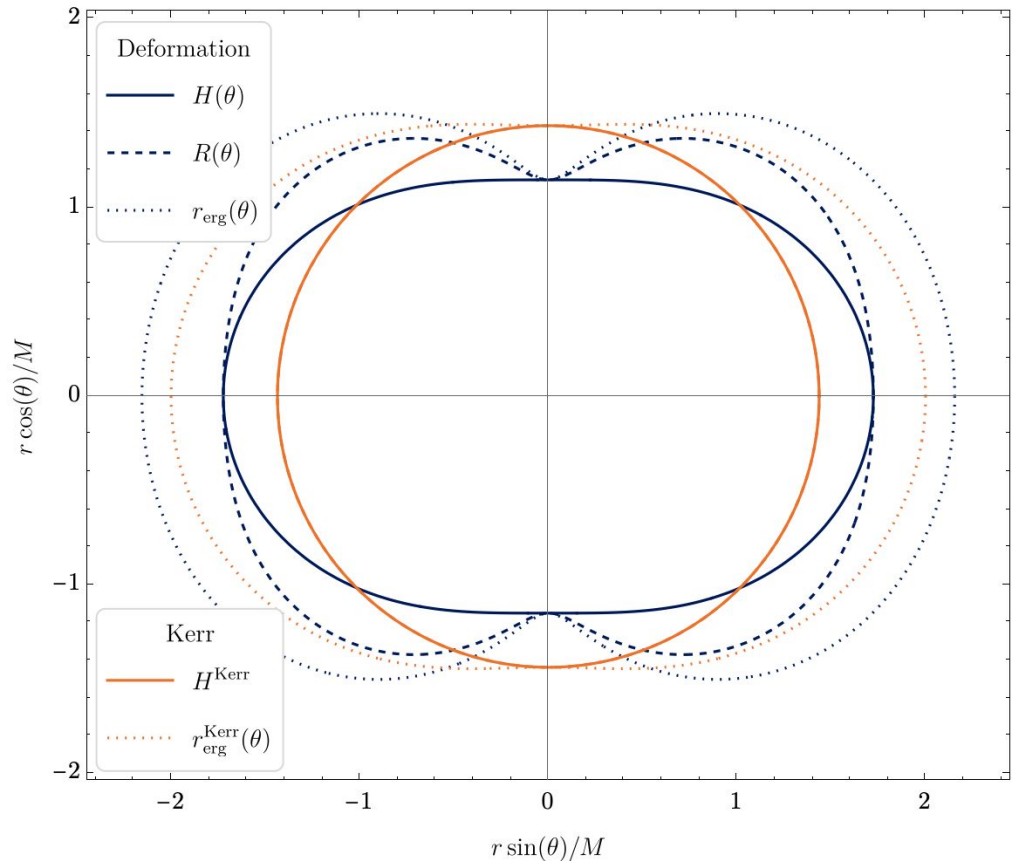
but H not Killing, and

$$\kappa = \left. \frac{r-M}{2rM + \delta(r, \theta)} \right|_{r_H}$$

[BckUp] 'Not-so-minimal' deformation

Choose horizon profile $H(\theta)$,
then reverse-engineer the metric

horizon, rotosurface, ergosphere
known analytically



[BckUp] Horizon generators

Horizon

surface	$r = H(\theta)$
normal	$\zeta_\mu := \frac{\alpha}{g^{vr}} \partial_\mu [r - H(\theta)]$
generator	$\zeta_\mu \zeta^\mu \Big _{r=H} = 0$

$$\zeta^\mu \Big|_{r=H} = \alpha \left[\underbrace{\Xi^\mu}_{\text{would-be Killing vector}} - \frac{H' g^{\theta\theta}}{g^{vr}} \underbrace{\tau^\mu}_{\text{tangent to H}} \right] \Big|_{r=H}$$



[BckUp] Surface gravities

Assume the horizon is $r = H(\theta)$ then define $\zeta_\mu := \frac{\alpha}{g^{vr}} \partial_\mu [r - H(\theta)]$

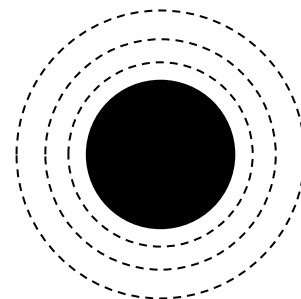
inaffinity surface gravity	$\zeta^\nu \nabla_\nu \zeta^\mu \Big _{r=H} = \kappa_i \zeta^\mu \Big _{r=H}$	}	on-horizon surface gravities
normal surface gravity	$\nabla_\mu (\zeta_\nu \zeta^\nu) \Big _{r=H} = 2\kappa_n \zeta_\mu \Big _{r=H}$		

compute them both

$$\kappa_n = \frac{\alpha}{2g^{vr}} \left[\partial_r g^{rr} + (H')^2 \partial_r g^{\theta\theta} \right] \Big|_{r=H}$$

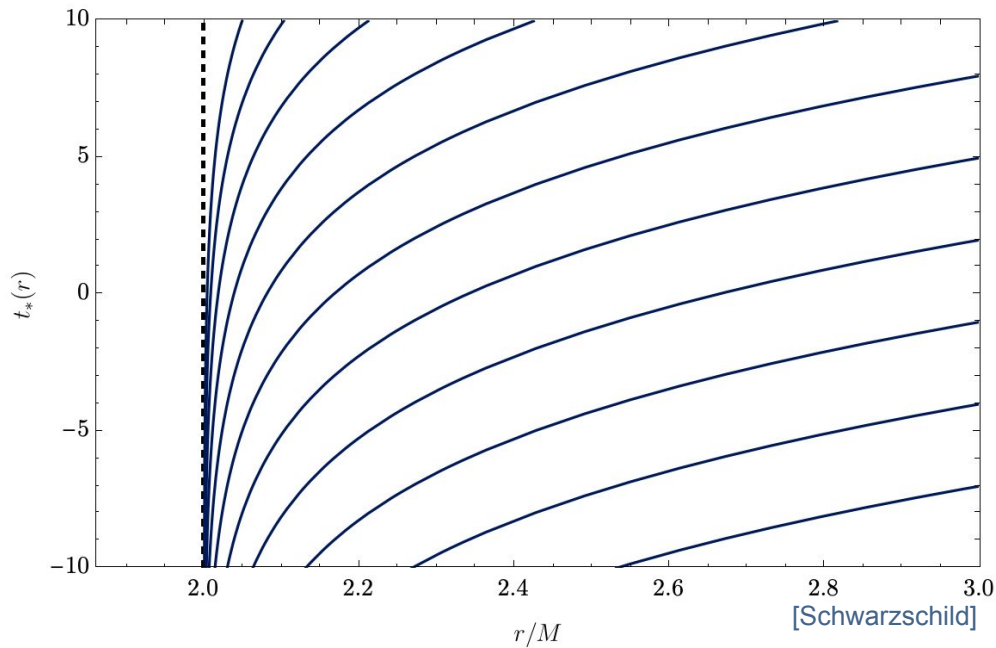
$$\kappa_i = \kappa_n + \zeta^\mu \partial_\mu \log\left(\frac{\alpha}{g^{vr}}\right) \Big|_{r=H}$$

depend on
position on H



[BckUp] Peeling surface gravity

Outgoing causal geodesics “peel off” of the horizon



$$t_* \sim \frac{1}{\kappa_p} \log(r)$$



peeling surface gravity

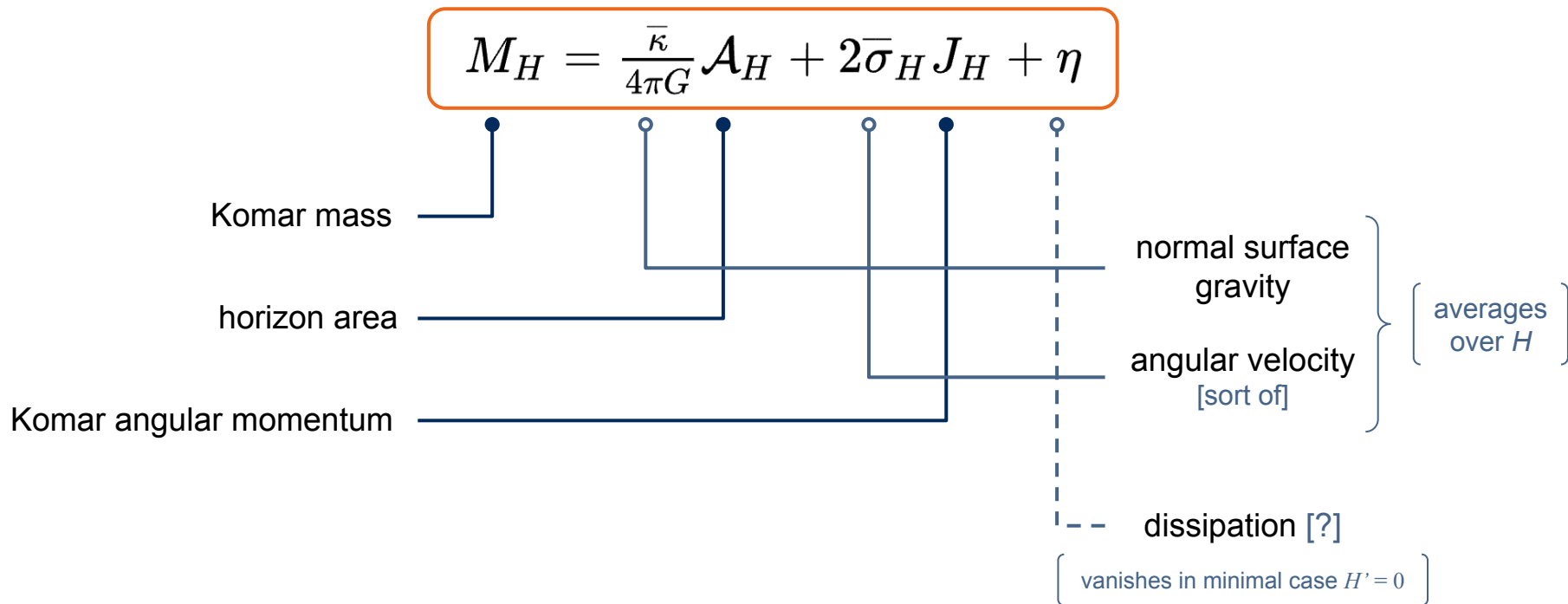
despite the lack of geodesic separability, we can compute:

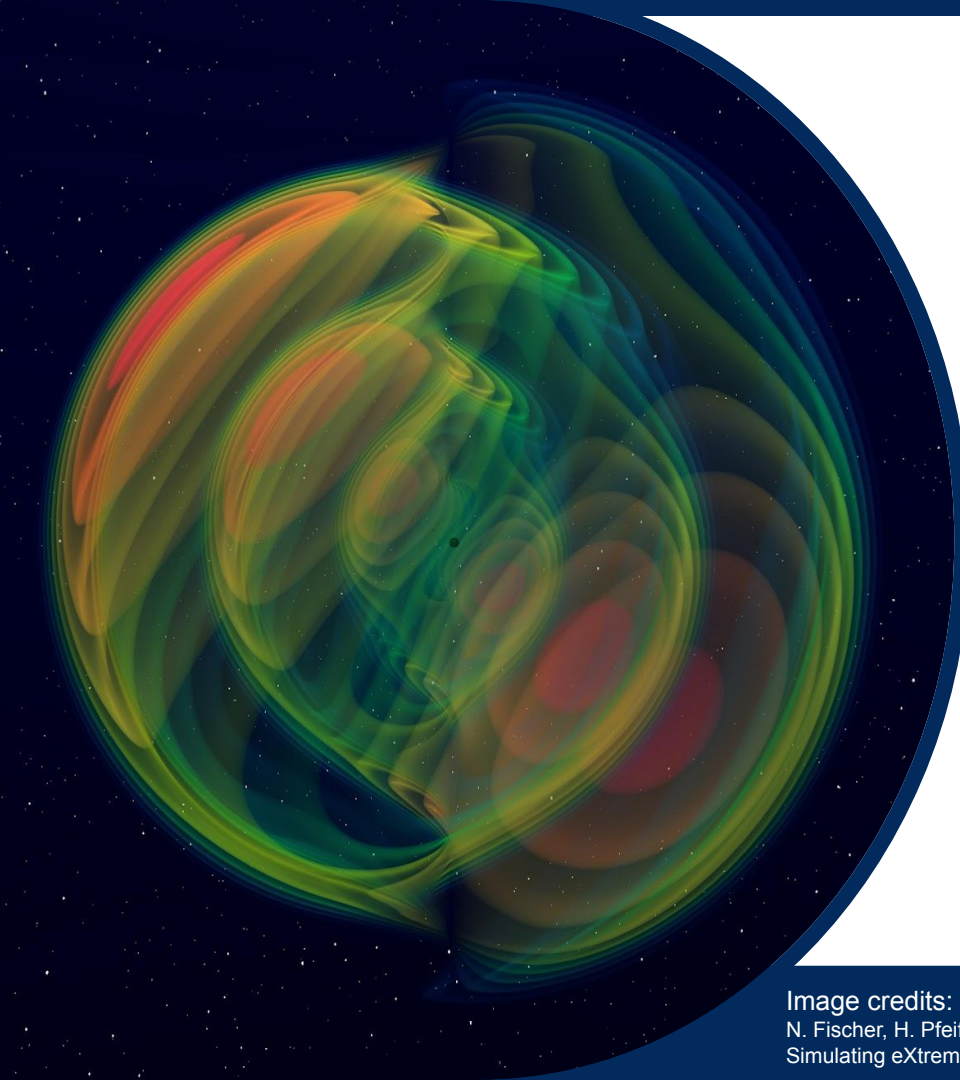
$$\kappa_p = \kappa_n$$

(w/ good choice of
normalisation)

[BckUp] Smarr's formula

We can prove this geometric identity, no field equations





Thanks!

Get in touch

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Image credits:

N. Fischer, H. Pfeiffer, A. Buonanno (Max Planck Institute for Gravitational Physics),
Simulating eXtreme Spacetimes (SXS) Collaboration

Tuesday - 25 August 2026 - <https://indico.sissa.it/event/175/overview>

Timing: 20 min

Beyond circles: stationary axisymmetric black holes and the breaking of circularity

Circularity is an accidental symmetry of the Kerr metric, one that is widely assumed when searching for rotating black hole solutions in modified gravity as well as when constructing models of Kerr mimickers. Though extremely enticing, circularity is often an excessively restrictive assumption, and understanding the consequences of its loss is thus crucially relevant. In this seminar, I wish to present some recent results on the subject: after describing in detail what this symmetry entails, I will show how to construct stationary and axisymmetric spacetimes exhibiting a controlled breaking of circularity; then, I will describe the impact of circularity breaking on the hole's horizon, focusing in particular on the laws of black hole mechanics. This discussion is thus going to be pertinent for anyone with an interest in compact astrophysical objects and their phenomenology, in general relativity and beyond.



50 Shades of Matt

