New perspectives on primordial black hole formation

A geometrical route towards PBH abundance and scalar-induced gravitational waves

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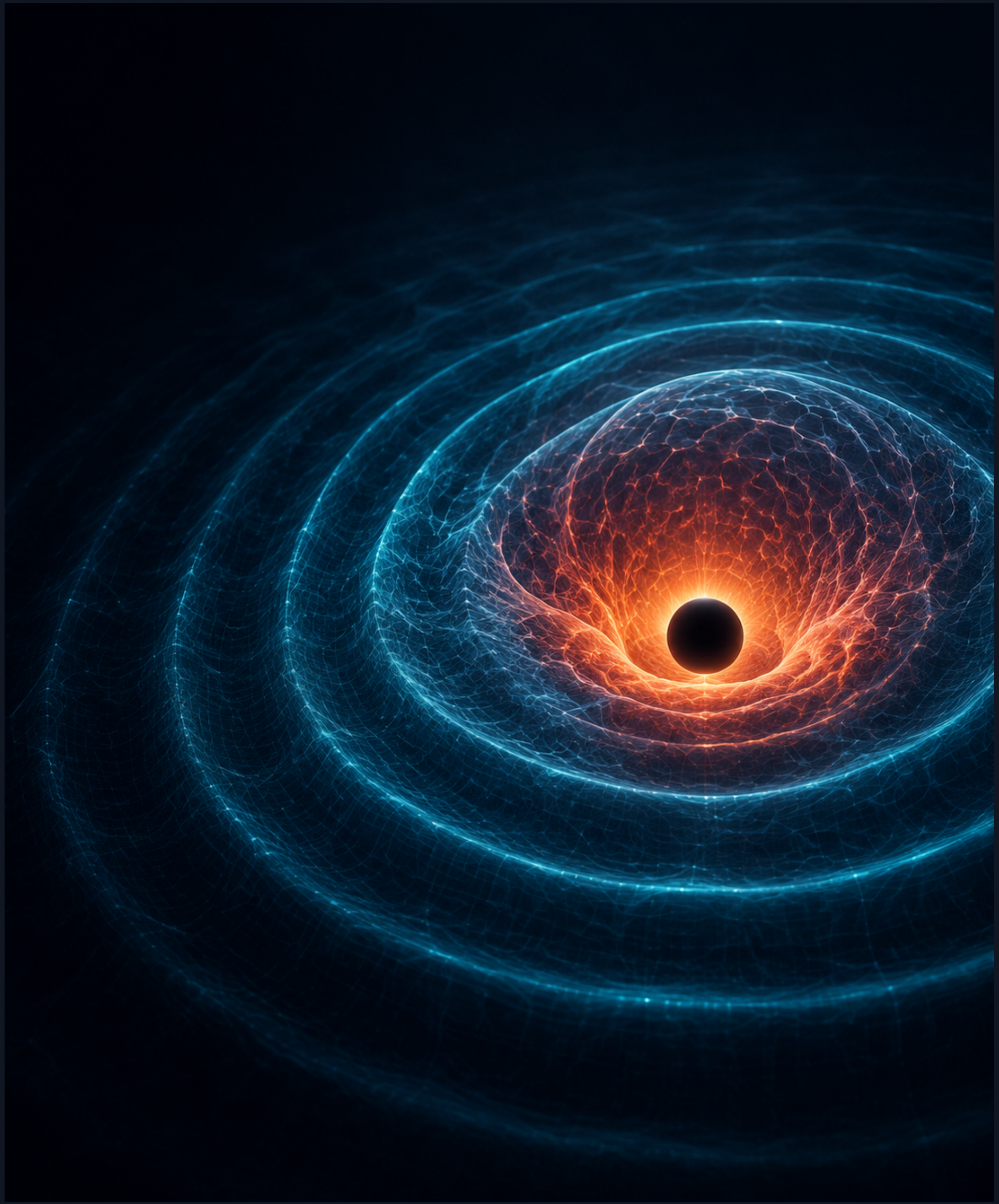
From Black Holes to the Cosmos

SISSA, Miramare Campus | 27 August 2026

SMASH

UNG



Summary

1 PBH formation across cosmological eras

- Horizon formation and critical collapse behaviour
- Equation-of-state transitions and characteristic PBH masses

2 Initial conditions and critical dynamics

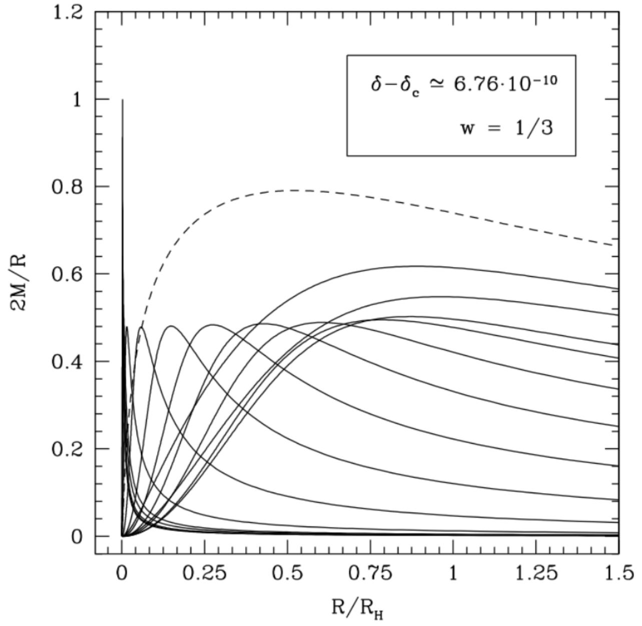
- Gradient-expansion initial data and curvature profile shape
- Power-spectrum prescription and PBH threshold

3 A geometrical potential for nonlinear evolution

- Misner–Sharp potential and Bardeen matching
- Nonlinear scalar response for PBH abundance and SIGWs
- Critical potential and PBH abundance

PBH formation / bounce

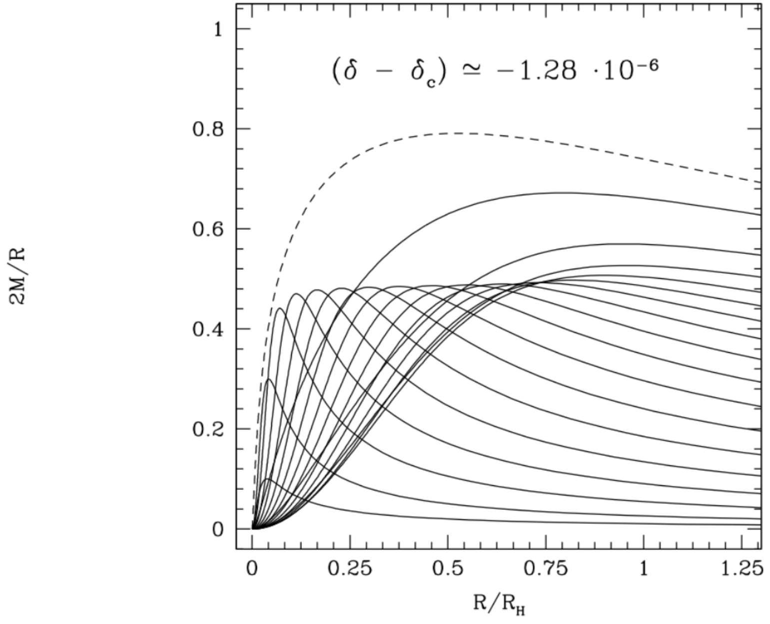
$$\delta > \delta_c$$



$$R(r,t) = 2M(r,t)$$

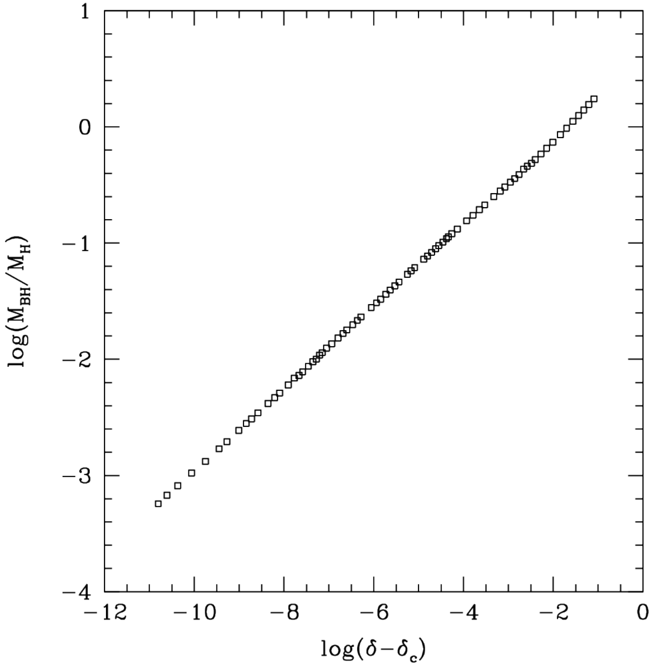
Collapse and apparent-horizon formation

$$\delta < \delta_c$$



Pressure gradients drive a hydrodynamical bounce

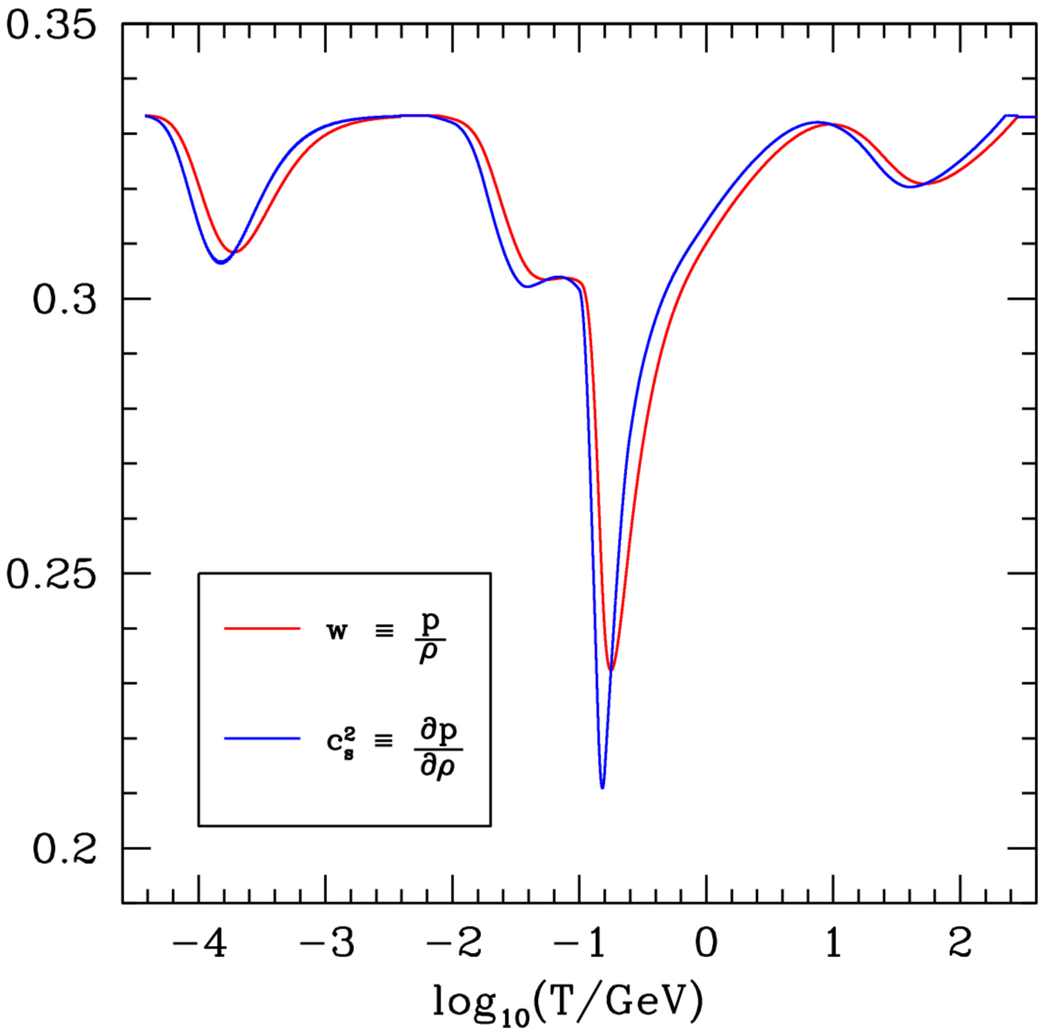
$$M_{\mathrm{PBH}} = \mathcal{K}\,(\delta - \delta_c)^{\gamma} M_H$$



Scaling law for critical collapse

Musco, Miller & Rezzolla, CQG 22, 1405 (2005)
Musco, Miller & Polnarev, CQG 26, 235001 (2009)

The early-Universe equation of state



Electroweak transition

$$M_{\mathrm{PBH}} \sim 5\times10^{-7} - 10^{-3} M_\odot$$

Planetary-mass scale; possible LISA signals

QCD transition

$$M_{\mathrm{PBH}} \sim 10^{-1} - 10^{2} M_\odot$$

Solar to intermediate masses; possible LVK signals

Primordial nucleosynthesis

$$M_{\mathrm{PBH}} \sim 10^{4} - 10^{6} M_\odot$$

SMBH seeds; open problem: e^+e^- production, radiation transport

Franciolini, Musco, Pani & Urbano, PRD 106, 123526 (2022)
Musco, Jedamzik & Young, PRD 109, 083506 (2024)
Blas et al. PLB 879, 140635 (2026)

Gradient expansion / nonlinear initial data

Long-wavelength expansion, nonlinear in amplitude

$$\epsilon(t) \equiv \frac{1}{a(t)H(t)r_m} \ll 1$$

$$ds^2 = -dt^2 + a^2(t)e^{2\zeta(\hat{r})}\left[d\hat{r}^2 + \hat{r}^2 d\Omega^2\right] + \mathcal{O}(\epsilon^2)$$

Equivalent curvature-coordinate description

$$r = \hat{r}\, e^{\zeta(\hat{r})}, \qquad K(r)r^2 = -2\hat{r}\zeta'(\hat{r}) - [\hat{r}\zeta'(\hat{r})]^2$$

Compaction function: definition and long-wavelength limit

$$\mathcal{C}_{\rm com}(r,t) \equiv \frac{2[M(r,t) - M_b(R,t)]}{R(r,t)}$$

$$\mathcal{C}_{\rm com}(r) = f(w)K(r)r^2 + \mathcal{O}(\epsilon^2), \qquad f(w) = \frac{3(1+w)}{5+3w}$$

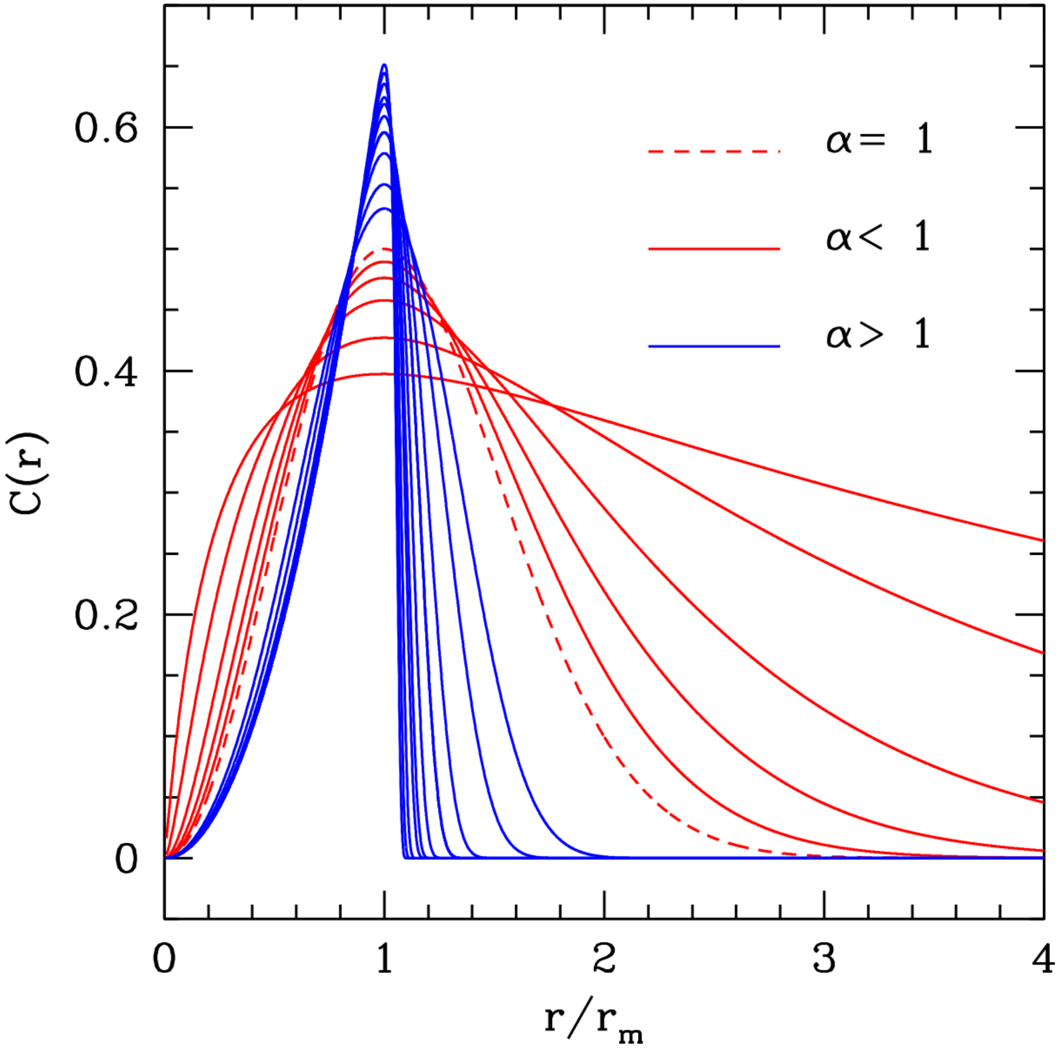
$$\mathcal{C}'_{\rm com}(r_m) = 0$$

The maximum fixes the characteristic scale; K(r) carries the nonlinear profile shape.

Only spatial gradients are expanded: the amplitude of K(r) can be nonlinear.

Musco, PRD 100, 123524 (2019)

Profile shape controls the initial threshold for PBHs



$$\mathcal{P}_\zeta(k) \longrightarrow \{\zeta(\hat{r}),\, K(r)\} \longrightarrow \mathcal{C}_{\rm com}(r) \longrightarrow r_m, \alpha$$

$$\alpha \equiv -\frac{r_m^2 \mathcal{C}''_{\rm com}(r_m)}{4\mathcal{C}_{\rm com}(r_m)}, \qquad \delta_c = \delta_c(w, \alpha)$$

- The maximum fixes the characteristic scale r_m.
- Local curvature at the maximum distinguishes broad and narrow profiles.
- The power spectrum therefore determines the relevant family of collapse thresholds.

Musco, PRD 100, 123524 (2019)
Musco et al., PRD 103, 063538 (2021)

The geometrical potential removes the w-dependence

Misner–Sharp kinematics and constraint

$$U \equiv D_t R, \qquad \Gamma \equiv D_r R, \qquad \Gamma^2 = 1 + U^2 - \frac{2M}{R}$$

Geometrical potential

$$\Phi_{\rm MS} \equiv \frac{1}{2}(1-\Gamma^2) = \frac{M}{R} - \frac{U^2}{2}$$

Gradient-expansion contributions

$$\frac{\delta M}{R} = \frac{3(1+w)}{2(5+3w)} K(r) r^2, \qquad -\frac{1}{2}(U^2 - H^2R^2) = \frac{1}{5+3w} K(r) r^2$$

Mass excess and kinetic correction depend separately on the equation of state.

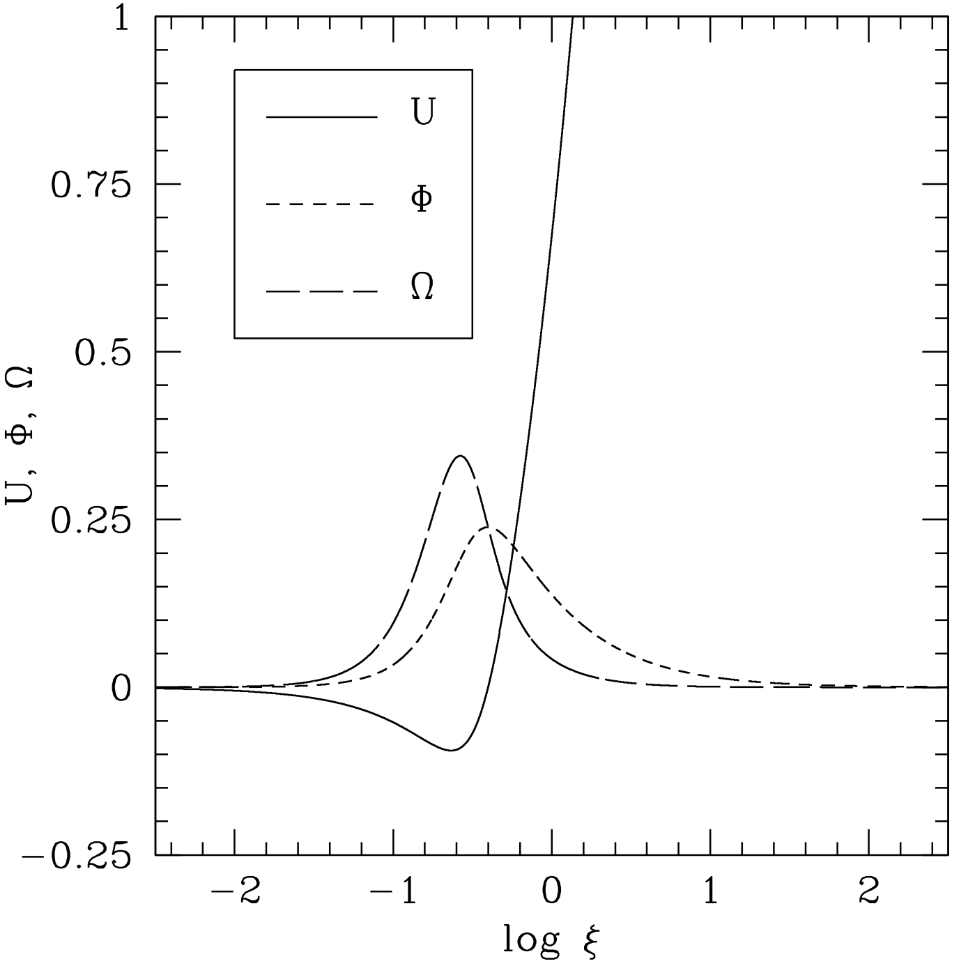
Coefficient cancellation

$$\frac{3(1+w)}{2(5+3w)} + \frac{1}{5+3w} = \frac{1}{2}$$

$$\Phi_{\rm MS} = \frac{1}{2} K(r) r^2$$

The normalization 1/2 is fixed geometrically by the constraint equation.

Self-similarity and the critical potential



One self-similar solution for each equation of state

$$\xi \equiv -\frac{R}{u}, \qquad \Omega \equiv 4\pi R^2 e, \qquad \Phi \equiv \frac{M}{R}, \qquad p = we$$

$$\delta_c = \delta_c(w, \alpha), \qquad \Phi_c = \Phi_c(w)$$

The initial threshold retains the profile dependence through the shape parameter.

$$\Phi \equiv \frac{M}{R}, \qquad U = 0 \implies \Phi_{\mathrm{MS}} = \Phi, \qquad \Phi_c = \Phi(\xi_{\max}; w)$$

- At the maximum of the compactness, the velocity vanishes: the geometrical potential coincides with the self-similar mass potential.
- This defines a shape-independent critical threshold depending only on the equation of state.

Musco & Miller, CQG 30, 145009 (2013)

Connection to the Bardeen potential

Nonlinear long-wavelength geometry

$$\Phi_{\mathrm{MS}} = -\hat{r}\,\zeta'(\hat{r})\left[1 + \frac{1}{2}\hat{r}\,\zeta'(\hat{r})\right]$$

Linearize in amplitude

$$\Phi^{(1)}_{\mathrm{MS}}(\hat{r}) = -\hat{r}\,\zeta'(\hat{r}), \qquad \Phi_{\mathrm{B}} = -f(w)\zeta, \quad f(w) = \frac{3(1+w)}{5+3w}$$

Match the same scalar geometry

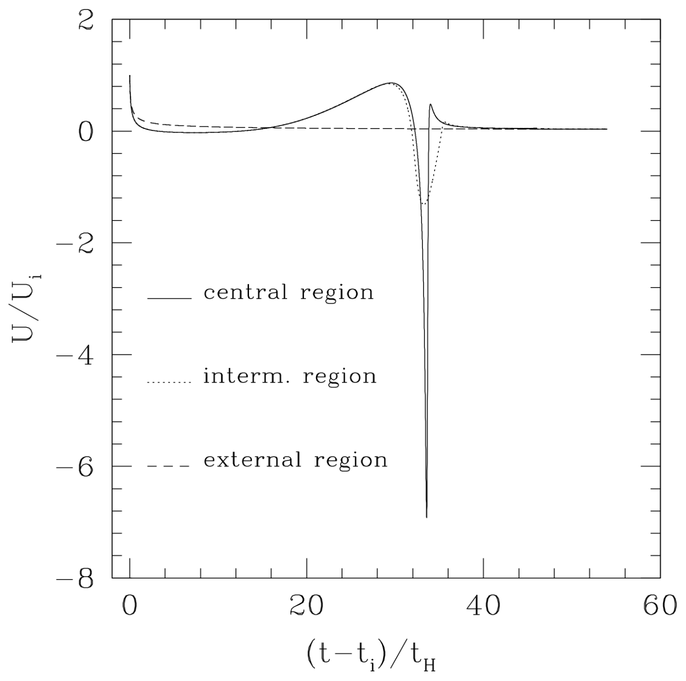
$$\Phi^{(1)}_{\mathrm{MS}}(\hat{r}) = \frac{1}{f(w)}\hat{r}\,\Phi'_{\mathrm{B}}(\hat{r}), \qquad \Phi^{(1)}_{\mathrm{MS}}(k) = -\frac{1}{f(w)}\left(3 + k\partial_k\right)\Phi_{\mathrm{B}}(k)$$

- The differential operator above is a geometrical mapping, not a transfer function.
- This perturbative matching provides the reference needed to reconstruct a Bardeen-like response from the nonlinear simulation.

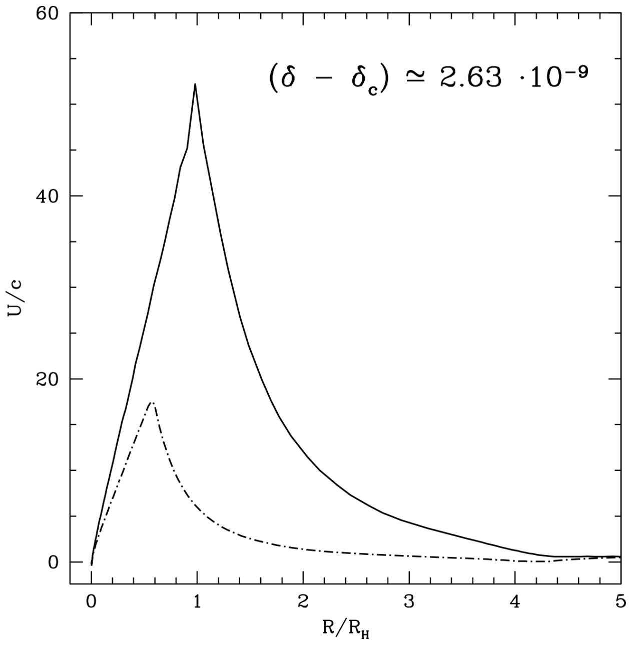
Bardeen, PRD 22, 1882 (1980)
Musco et al. (in progress)

Near-threshold dynamics is intrinsically nonlinear

Subcritical bounce



Relativistic wind



<u>Strong pressure gradients generate large velocities and a deeply nonlinear scalar evolution around horizon entry.</u>

Musco, Miller & Rezzolla, CQG 22, 1405 (2005)
Musco, Miller & Polnarev, CQG 26, 235001 (2009)

Linear SIGW source

$$y \equiv \frac{\delta}{\delta_c(w,\alpha)}, \qquad \Phi_{\rm B}^{(1)}(k,\eta) = T_{\rm L}(k,\eta)\,\Phi_{\rm B}^{(1)}(k,\eta_{\rm ini})$$

$$y \ll 1, \quad w = \frac{1}{3}: \qquad T_{\rm L}(k,\eta) = 3\,\frac{\sin x - x\cos x}{x^3}, \qquad x \equiv \frac{k\eta}{\sqrt{3}}$$

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 4S^{\rm TT}_{ij}, \qquad \mathcal{H} \equiv \frac{a'}{a} = aH$$

- The equation is linear in the tensor modes.
- The TT-projected source is quadratic in the scalar perturbations and inherits their amplitude and phase.

What replaces $T_{\rm L}$
when the scalar evolution is nonlinear?

Kohri & Terada, PRD 97, 123532 (2018)

The nonlinear response is conditioned by amplitude and shape

$$\Phi_{\mathrm{MS}}(r,t;y,\alpha) \longrightarrow \Phi_{\mathrm{B}}^{\mathrm{rec}} \longrightarrow T_{\mathrm{eff}}(k,\eta;y,\alpha)$$

Define the nonlinear correction through the reconstructed scalar field

$$\Phi_{\mathrm{B}}^{\mathrm{rec}}(\mathbf{k},\eta) = T_{\mathrm{L}}(k,\eta)\Phi_{\mathrm{B}}^{(1)}(\mathbf{k}) + \int \frac{d^3q}{(2\pi)^3}\,\mathcal{K}_{\mathrm{NL}}(\mathbf{k},\mathbf{q};\eta,y,\alpha)\Phi_{\mathrm{B}}^{(1)}(\mathbf{q})\Phi_{\mathrm{B}}^{(1)}(\mathbf{k}-\mathbf{q}) + \cdots$$

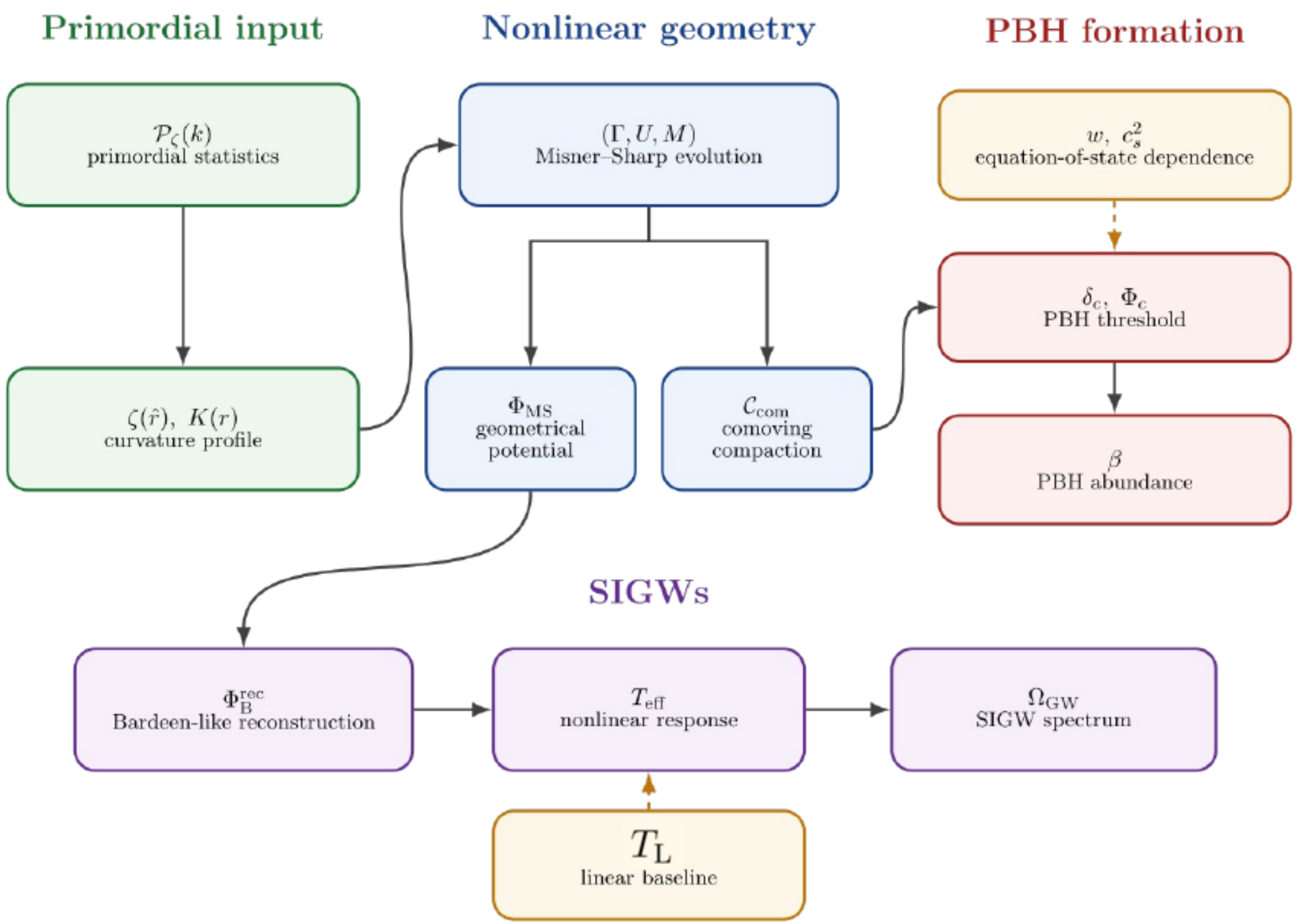
Operationally, the kernel is obtained by comparing simulations at fixed profile shape and normalized amplitude.

Exact output	Reconstruction	Conditional kernel
$\Phi_{\mathrm{MS}}[\Gamma, U, M]$	$\Phi_{\mathrm{B}}^{\mathrm{rec}}$	$\mathcal{K}_{\mathrm{NL}}(\mathbf{k},\mathbf{q};\eta,y,\alpha)$

- Near-threshold bounce and relativistic wind modify the scalar source around horizon entry.
- The same nonlinear response affects both the SIGW kernel and the PBH abundance.

Musco et al. (in progress)

A geometrical route to PBH abundance and SIGWs



$$\mathcal{P}_\zeta(k) \longrightarrow \delta_c(w, \alpha) \qquad \text{critical dynamics} \longrightarrow \Phi_c(w) \qquad \Phi_{\rm MS} \longrightarrow \{\beta_{\rm PBH}, \Omega_{\rm GW}\}$$

Backup — Self-similar system in cosmic/null time

Musco & Miller, CQG 30, 145009 (2013)

$$\xi \equiv -\frac{R}{u}, \qquad \Omega \equiv 4\pi R^2 e, \qquad \Phi \equiv \frac{M}{R}, \qquad p = we$$

$$\frac{d\ln U}{d\ln\xi} = \frac{(\Phi + w\Omega)^2 - 2w\Gamma^2\Phi}{U^2(\Phi+w\Omega)^2 - w\Gamma^2(\Omega-\Phi)^2}\left[(\Omega-\Phi) - \frac{(1+w)\Omega U}{\Gamma+U}\right],$$

$$\frac{d\ln\Omega}{d\ln\xi} = \frac{(1+w)(\Omega-\Phi)}{\Phi+w\Omega}\frac{d\ln U}{d\ln\xi} + \frac{2w}{\Phi+w\Omega}\left[(\Omega-\Phi) - \frac{(1+w)\Omega U}{\Gamma+U}\right],$$

$$\frac{d\ln\Phi}{d\ln\xi} = \frac{1}{\Phi}\left[(\Omega-\Phi) - \frac{(1+w)\Omega U}{\Gamma+U}\right], \qquad \Gamma^2 = 1 + U^2 - 2\Phi.$$

$$U(\xi) = -\frac{2}{3(1+w)}\xi, \qquad \Phi(\xi) = k\xi^2, \qquad \Omega(\xi) = 3k\xi^2 \qquad (\xi \to 0)$$

Regularity at the sonic point selects k and therefore the critical solution for each w.

Backup — Misner–Sharp metric and evolution system

Metric and derivative operators

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + R^2 d\Omega^2, \qquad D_t \equiv A^{-1}\partial_t, \qquad D_r \equiv B^{-1}\partial_r$$

Evolution equations

$$D_t R = U, \qquad D_t M = -4\pi p R^2 U,$$

$$D_t U = -\frac{\Gamma}{\rho + p} D_r p - \frac{M}{R^2} - 4\pi p R, \qquad D_t \rho = -(\rho + p)\Theta,$$

$$D_t \Gamma = -\frac{U}{\rho + p} D_r p, \qquad D_t \Theta = -\frac{1}{3}\Theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - 4\pi(\rho + 3p) + \nabla_\mu a^\mu,$$

$$\Theta \equiv -K = \frac{2U}{R} + \frac{D_r U}{\Gamma}, \qquad a_\mu = -\frac{h_\mu{}^\nu \nabla_\nu p}{\rho + p}.$$

The expansion scalar $\Theta = -K$ makes the continuity equation manifestly geometrical; $K(r)$ remains reserved for the initial curvature profile.

Backup — Constraints and Φ_{MS} dynamics

Radial constraints

$$D_r R = \Gamma, \qquad D_r M = 4\pi\rho R^2\Gamma,$$

$$\Gamma^2 = 1 + U^2 - \frac{2M}{R}, \qquad \frac{D_r A}{A} = -\frac{D_r p}{\rho + p}.$$

Geometrical potential

$$\Phi_{\mathrm{MS}} \equiv \frac{1}{2}(1 - \Gamma^2) = \frac{M}{R} - \frac{U^2}{2}$$

$$D_t \Phi_{\mathrm{MS}} = \frac{\Gamma U}{\rho + p} D_r p,$$

$$D_r \Phi_{\mathrm{MS}} = 4\pi\rho R\Gamma - \frac{M\Gamma}{R^2} - U D_r U.$$

The temporal equation evolves the geometrical potential directly; the radial equation provides an independent consistency relation.

Backup — Near-threshold SIGW kernel

Numerical scalar response

$$T_{\mathrm{num}}(k,\eta;y) \equiv \frac{\Phi_{\mathrm{B}}^{\mathrm{rec}}(k,\eta;y)}{\Phi_{\mathrm{B}}^{\mathrm{rec}}(k,\eta_{\mathrm{ini}};y)}, \qquad \Delta T = T_{\mathrm{num}} - T_{\mathrm{L}}$$

Corrected time kernel

$$I_{\mathrm{corr}}(u,v,x;y) = \int_0^x d\bar{x}\, G_k(x,\bar{x})\, f_{\mathrm{num}}(u\bar{x}, v\bar{x}; y)$$

Tensor equation and scalar source

$$h_{ij}'' + 2\mathcal{H}h_{ij}' - \nabla^2 h_{ij} = 4S_{ij}^{\mathrm{TT}}, \qquad \mathcal{H} \equiv \frac{a'}{a} = aH$$

$$S_{ij}^{\mathrm{TT}} = \Lambda_{ij}{}^{lm} S_{lm}$$

TT projection of the quadratic scalar source

$$S_\lambda(\mathbf{k},\eta) = \int \frac{d^3q}{(2\pi)^3}\, e_\lambda^{ij}(\mathbf{k}) q_i q_j\, F[\Phi_{\mathrm{B}}(\mathbf{q},\eta), \Phi_{\mathrm{B}}(\mathbf{k}-\mathbf{q},\eta)]$$

The kernel F is quadratic in the scalar potential and its time derivative; it therefore retains amplitude and phase information.

Linear, interference and nonlinear terms

$$I_{\mathrm{corr}}^2 = I_{\mathrm{L}}^2 + 2I_{\mathrm{L}}I_{\mathrm{cross}} + I_{\mathrm{cross}}^2$$

- The cross term is sensitive to the amplitude and phase of the near-threshold response.
- The quadratic term is positive and need not be small close to the collapse threshold.

A physical prediction finally requires weighting over the distribution of subcritical amplitudes.

Backup — Peak theory and abundance

Spectral moments and peak parameters

$$\sigma_j^2(R) = \int_0^\infty \frac{dk}{k}\, k^{2j} W^2(kR)\mathcal{P}_\delta(k), \qquad \gamma_{\rm pk} = \frac{\sigma_1^2}{\sigma_0\sigma_2}, \qquad R_* = \sqrt{3}\,\frac{\sigma_1}{\sigma_2}$$

From peaks above threshold to the mass fraction

$$\frac{dn_{\rm pk}}{d\nu} = \frac{e^{-\nu^2/2}}{(2\pi)^2 R_*^3}\, G(\gamma_{\rm pk}, \gamma_{\rm pk}\nu), \qquad \nu_c = \frac{\delta_c(w,\alpha)}{\sigma_0},$$

$$\beta(M_H) = V_H \int_{\nu_c}^\infty d\nu\, \frac{M_{\rm PBH}(\nu)}{M_H}\frac{dn_{\rm pk}}{d\nu}, \qquad M_{\rm PBH} = \mathcal{K}[\delta(\nu) - \delta_c]^{\gamma_c} M_H.$$

The power spectrum fixes the peak statistics; nonlinear collapse supplies the shape-dependent threshold and the PBH mass assigned to each peak.

BBKS, ApJ 304, 15 (1986); Young, Musco & Byrnes, JCAP 11, 012 (2019); Young & Musso, JCAP 11, 022 (2020)
Musco, PRD 100, 123524 (2019); Musco et al., PRD 103, 063538 (2021)