Supermassive Black Hole seeds from Direct Collapse of CDM-Curvature Peaks in ΛCDM

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From BHs to the Cosmos

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InDark INFN QUAGRAP

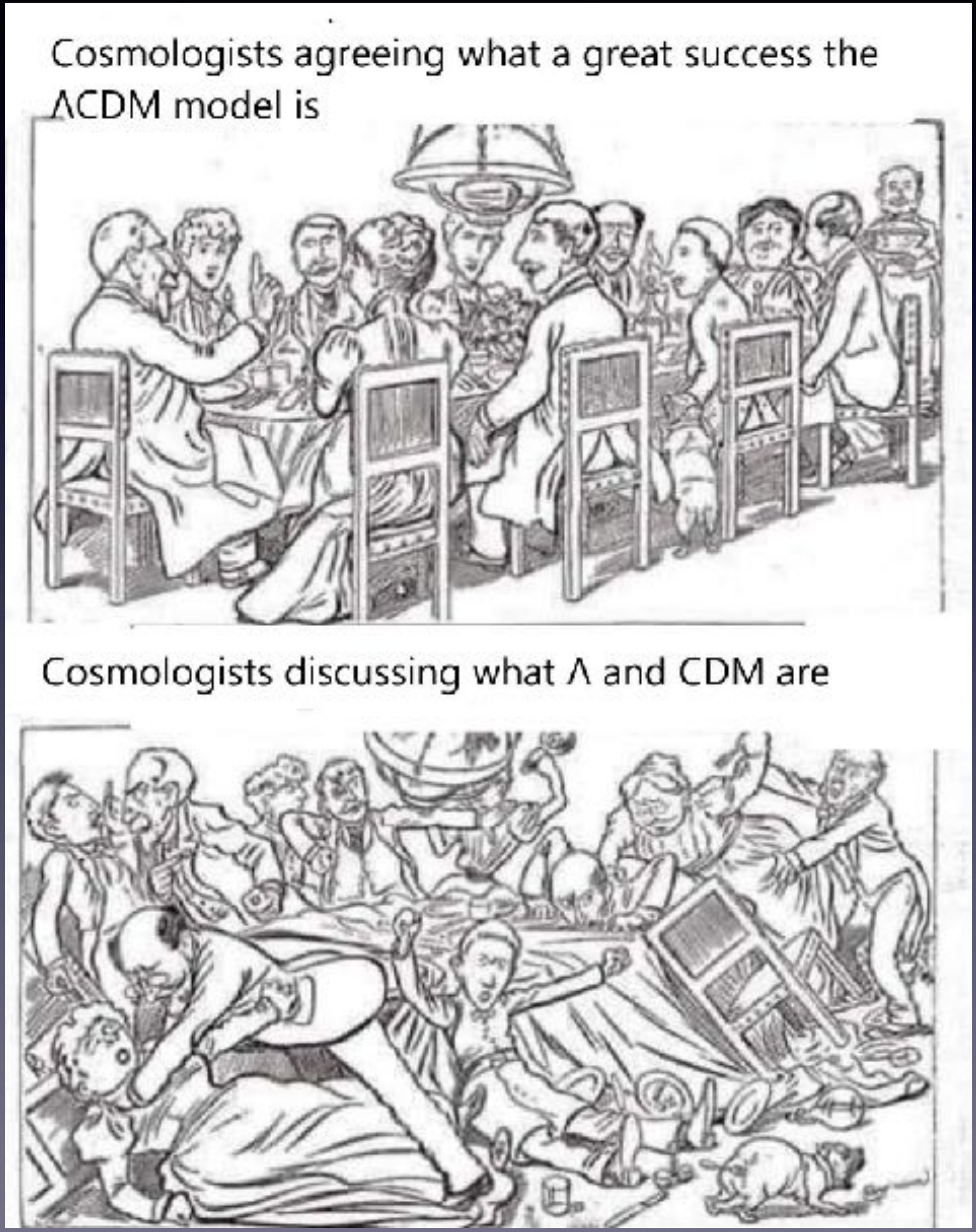
Take Home Message

- In standard ΛCDM, supermassive Black Hole seeds of the order of 10^3-10^6 solar masses can form at redshift z~5 - or even earlier -from the direct collapse of peaks in the perturbative initial spatial distribution of curvature and CDM at z~300
- Some simplifications will need extensions:
  - More general matter description than dust fluid
  - Go beyond spherical symmetry

thanks to all students and collaborators, Cristian Barrera, Robyn Munoz, William Beordo, Marco Galoppo, + Baojiu Li, Kazuya Koyama, Ian Hawke, T. Harada

Concordance Model: ΛCDM

Context of this work and where are we coming from



Questions on ΛCDM

- Recipe for modelling based on 3 main ingredients:
  1. Homogeneous isotropic background, FLRW models
  2. Relativistic Perturbations (e.g. CMB; linear, nonlinear)
  3. Newtonian study of non-linear structure formation (numerical simulations or approx. techniques)
- Is 3 good enough? (more data, precision cosmology @ ~1%, observations and simulations covering large fraction of H^{-1})

▸ It is timely to bridge the gap between 2 and 3, using:

- Full GR simulations with Numerical Relativity or
- other quasi-GR codes (GRAMSES), or at least
- a GR interpretation of Newtonian N-body simulations

Cosmological Numerical Relativity with Einstein Toolkit: quasi-spherical collapse

Robyn Munoz & MB, Phys. Rev. D 107, 123536 (2023), arXiv:2302.09033, based on the Einstein Toolkit fluid code, plus the EBWeyl code presented in Robyn Munoz & MB, Class. Quantum Grav. 40 135010 (2023) arXiv:2211.08133

initial conditions based on
MB, J.C. Hidalgo, N. Meures & D. Wands,
ApJ 785:2, (2014)
arXiv:1307.1478

Basic goal: check the Top-hat collapse model of peaks

- the so-called Top-hat collapse model describes the detachment of an overdensity from the Hubble expansion, and it is based on a spherically symmetric closed model (negative energy, or positive curvature in GR) that expands, reaches a Turn Around, then recollapses.

- Because it is spherical dust, it can be derived in Newtonian theory

- From a GR perspective, it is a closed FLRW model matched to a Schwarzschild hole, surrounded by a flat FLRW background

- It is exactly the same as in the Oppenheimer-Sneyder collapse, with the first part of expansion before turn-around

full GR with ET: Reductionism

- Cosmic web: a network of peaks connected by filaments and separated by voids, see J. R. Bond, L. Kofman, and D. Pogosyan, Nature (London) 380, 603 (1996).
- For our purposes, i.e. understand GR effects in non-linear structure formation with Einstein Toolkit, a reductionist approach is useful.

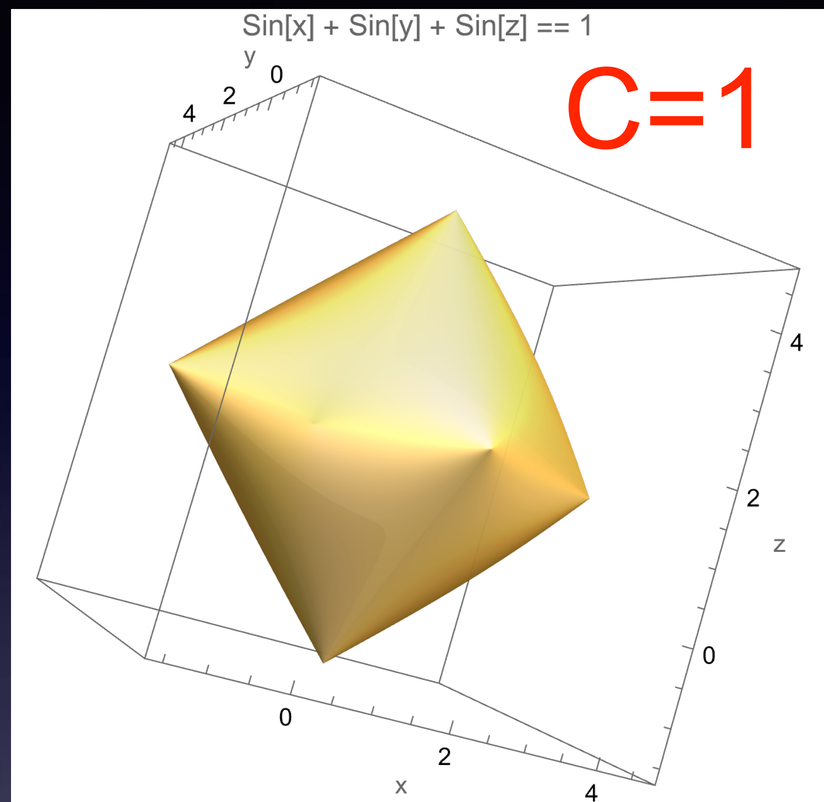
$$ds^2 = -d\tau^2 + \gamma_{ij}dx^i dx^j,$$

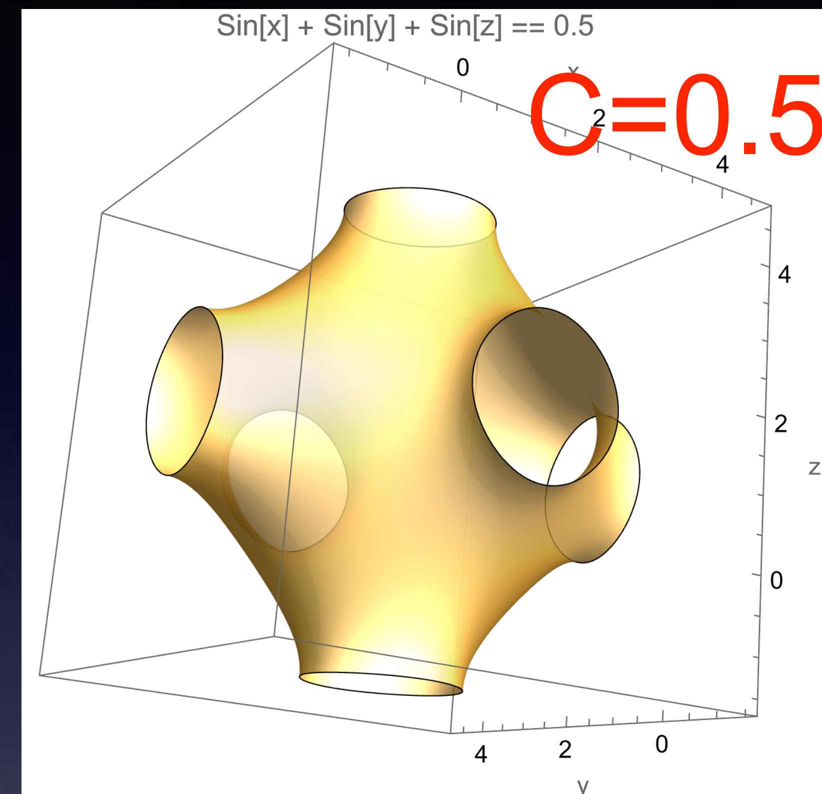
- CDM as dust: fluid simulations in comoving-synchronous gauge, valid up to first "shell-crossing", with initial conditions at z~300
- initial conditions based on curvature perturbation variable $\mathcal{R}_c$ (typically used in inflation) with simple spatial distribution

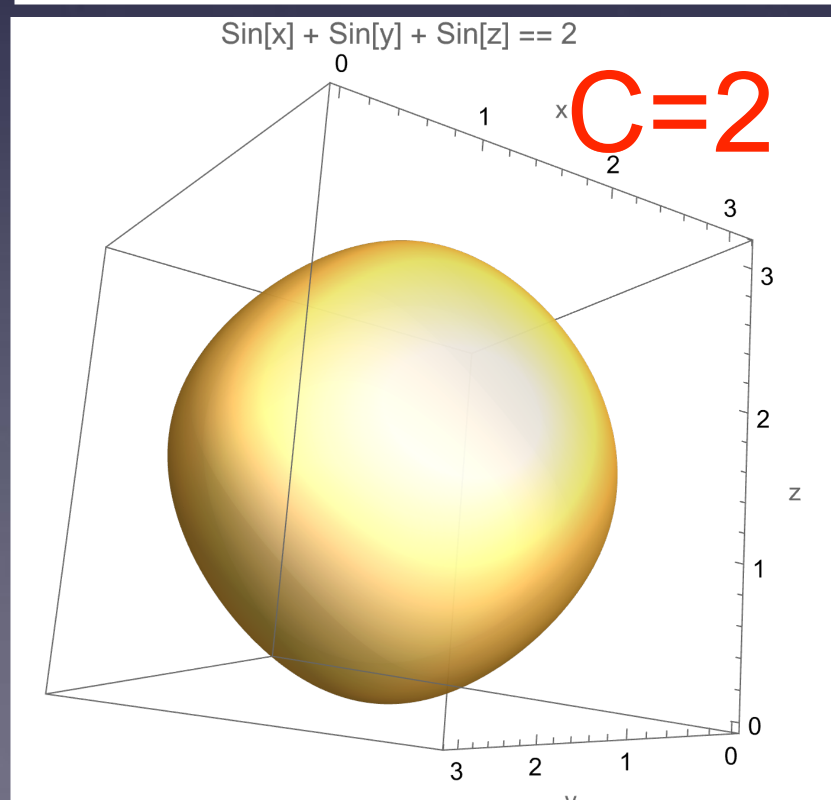
$$\mathcal{R}_c = A_{\text{pert}}(\sin(xk_{\text{pert}}) + \sin(yk_{\text{pert}}) + \sin(zk_{\text{pert}})),$$

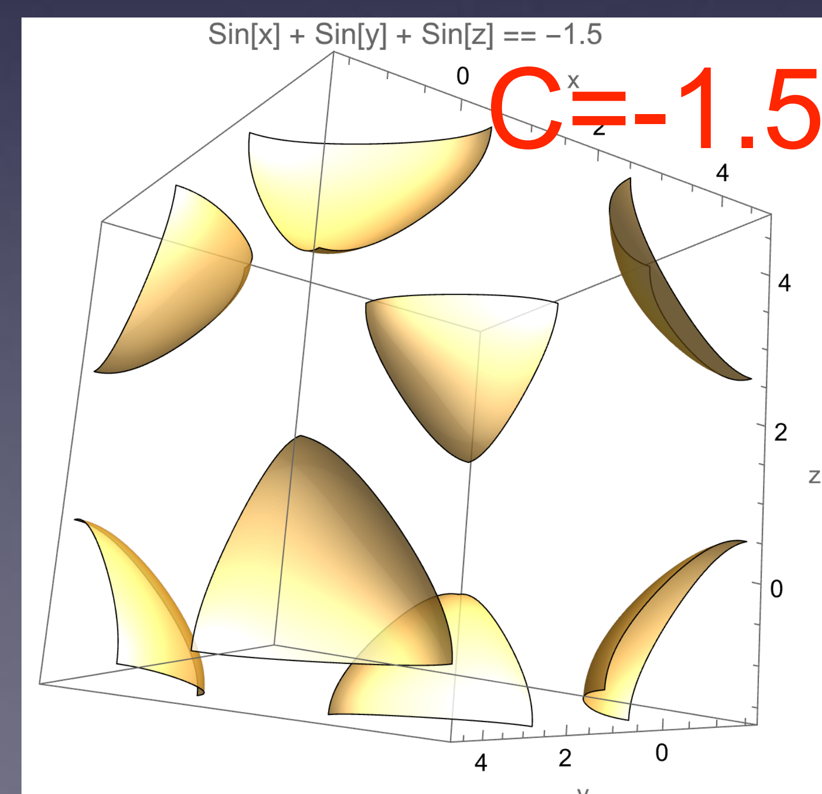
- quasi-spherical around peak, but with filaments and voids

$$\sin(x) + \sin(y) + \sin(z) = C$$

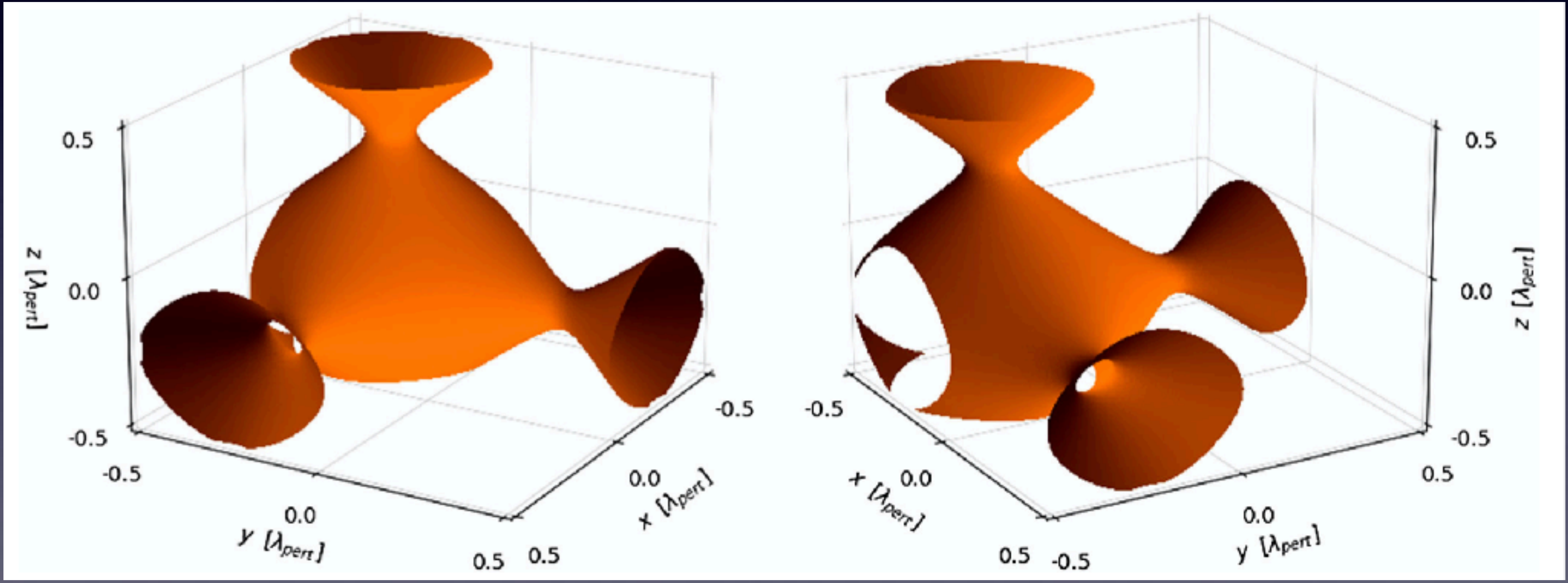






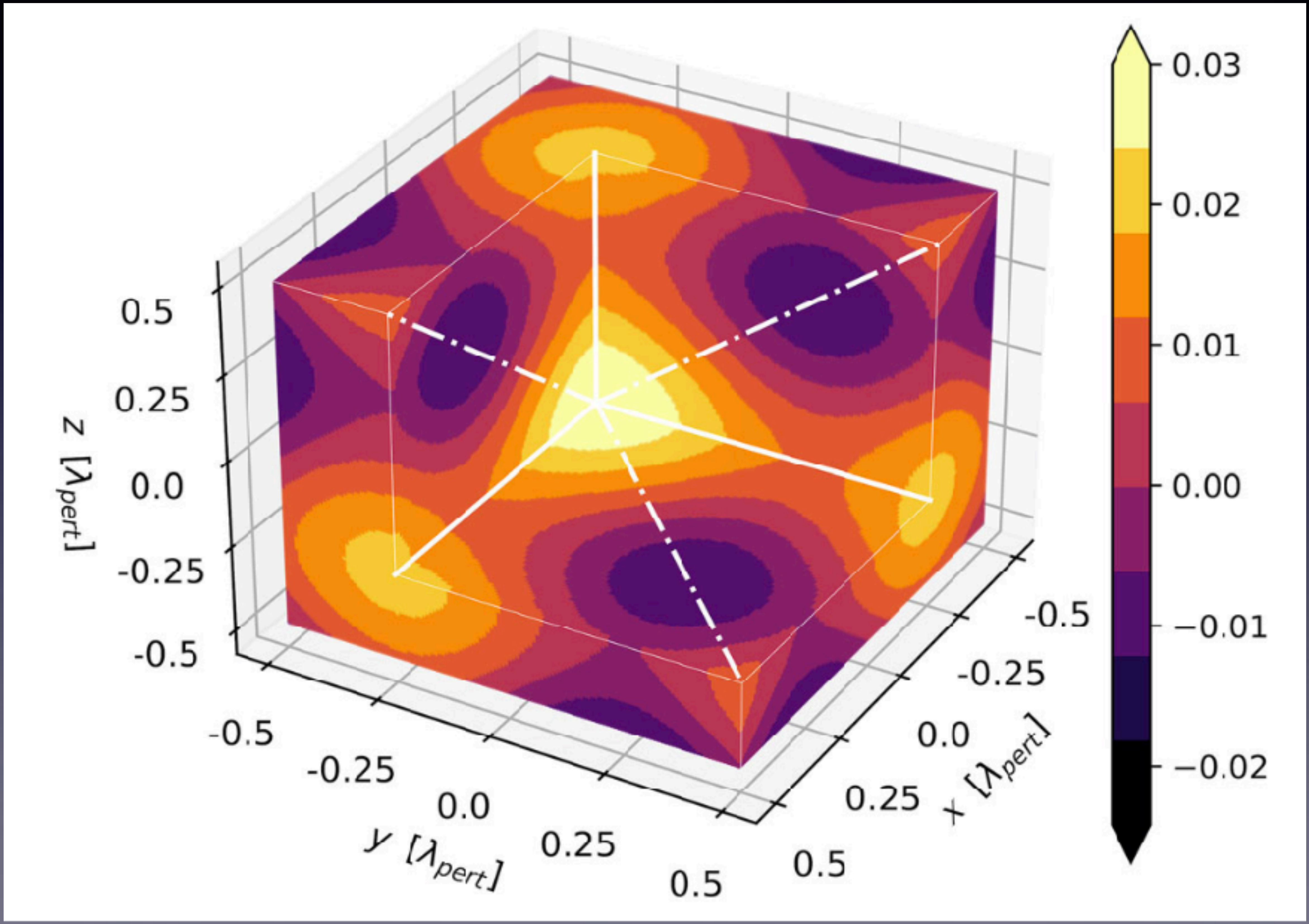


toy-model of a cosmic web of over-densities, filaments and voids



iso-density surface with over-density δ=0.01 at z_{in}=302, two different points of view

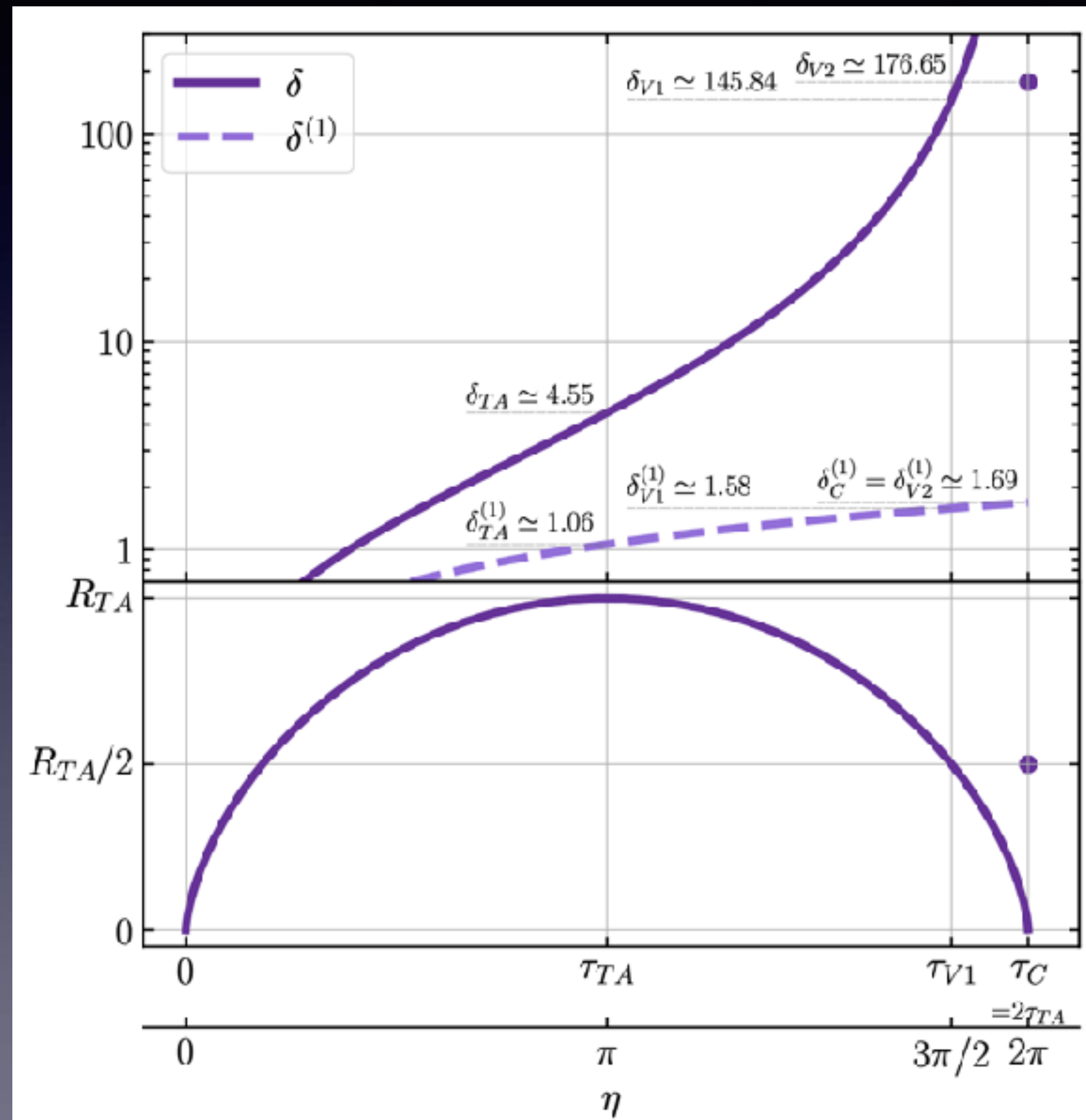
δ: density distribution



Top-hat collapse model

- It is at the base of the mass function theory of Press-Schechter mass function and the Sheth-Tormen extension

- linear value of $\delta^{(1)}$ used as benchmark, conventional values of nonlinear δ used to flag virialization, collapse time predict first shell-crossing when $\delta^{(1)}=1.69$



validity of Top-hat collapse

		Top-Hat, $\Lambda = 0$	Here, $\Lambda = 0$	Here, $\Lambda \neq 0$
Initially	z_{IN}		205.4	302.5
	a/a_{IN}	35.4137	$35.24467 \pm$ 7e-5	$35.195 \pm$ 3e-3
	z		$4.85620 \pm$ 1e-5	$7.6234 \pm$ 7e-4
Turn Around (TA)	$\gamma_{\mathrm{OD}}^{1/6}/\gamma_{\mathrm{IN,OD}}^{1/6}$		$20.10169 \pm$ 3e-5	$20.0600 \pm$ 1e-4
$K_{\mathrm{OD}} = 0$	$\langle\gamma^{1/6}\rangle_{\mathcal{D}}/\langle\gamma^{1/6}\rangle_{\mathcal{D},\mathrm{IN}}$		$35.2064 \pm$ 1e-4	$35.154 \pm$ 3e-3
	$\delta_{\mathrm{OD}}^{(1)}$	1.06241	$1.05734 \pm$ 2e-6	$1.05584 \pm$ 8e-5
	δ_{OD}	4.55165	$4.55164 \pm$ 1e-5	$4.5626 \pm$ 5e-4
	a/a_{IN}	56.22	$55.9 \pm$ 1e-1	$55.87 \pm$ 8e-2
	z		$2.692 \pm$ 7e-3	$4.432 \pm$ 8e-3
Collapse	$\gamma_{\mathrm{OD}}^{1/6}/\gamma_{\mathrm{IN,OD}}^{1/6}$		$0.4 \pm$ 6e-1	$0.8 \pm$ 2e-1
/Crash	$\langle\gamma^{1/6}\rangle_{\mathcal{D}}/\langle\gamma^{1/6}\rangle_{\mathcal{D},\mathrm{IN}}$		$55.8 \pm$ 1e-1	$55.77 \pm$ 2e-2
	$\delta_{\mathrm{OD}}^{(1)}$	1.686	$1.678 \pm$ 3e-3	$1.676 \pm$ 2e-3
	δ_{OD}	$+\infty$	$2e+6 \pm 2e+6$	$4e+5 \pm 4e+5$
Virialization	a/a_{IN}	52.64	$52.5055 \pm$ 9e-4	$52.469 \pm$ 2e-3
$R = R_{\mathrm{TA}}/2$	δ_{OD}	145.84	145.84	145.84
Virialization	a/a_{IN}	56.22	$52.83625 \pm$ 7e-5	$52.801 \pm$ 2e-3
$R = R_{\mathrm{TA}}/2$ & $\tau = \tau_C$	δ_{OD}	176.65	176.65	176.65

why the top-hat model works so well?

- It is all down to the Raychaudhuri equation!

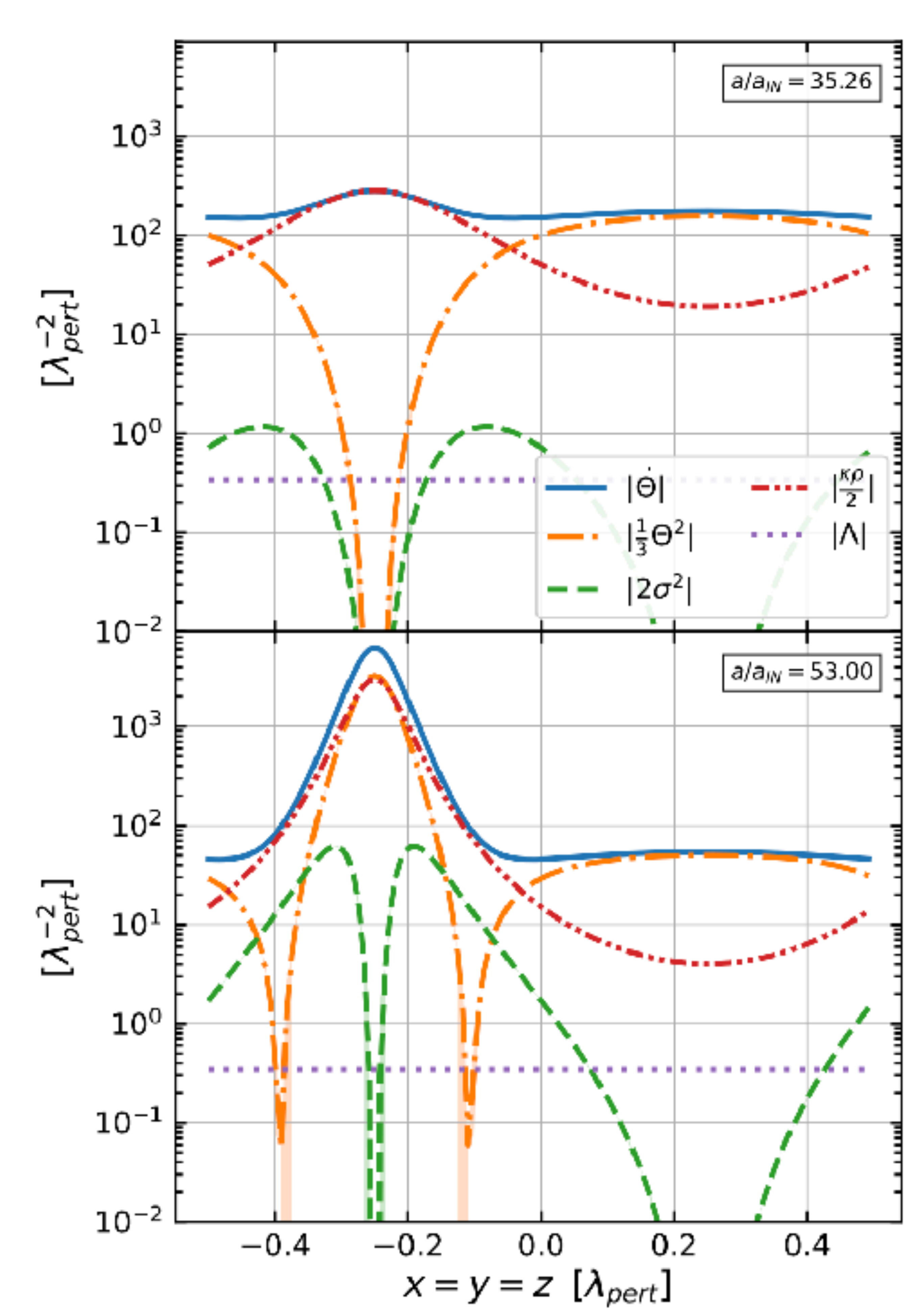
$$\dot{\Theta} = -\frac{\Theta^2}{3} - 4\pi G\rho_M - 2\sigma^2 + 2\omega^2 + \Lambda$$

$\Theta \to 3H = 3\frac{\dot{a}}{a}$, $\sigma = \omega = 0$ for homogeneous and isotropic case

reduces to Friedmann equation $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho_M + \frac{\Lambda}{3}$

contributions to Raychaudhuri

- in our quasi-spherical collapse the shear σ is negligible at the peak initially
- at Turn-Around (Top panel) Θ=0 and the shear remains negligible at the peak, and subdominant in general
- at collapse (Bottom panel) the shear remains negligible at the peak
- 3-Ricci curvature (not shown, from Hamiltonian constraint) is positive and very important at the peak during evolution
- Net result: Raychaudhuri is very well approximated by the Friedmann equation for closed models!!!
- Top-Hat works very well to predict first "shell-crossing", even if overall the collapse is non-spherical



Questions

zooming-in on the collapsing over-density

Given that the collapse of the peak is the same as Oppenheimer-Sneyder, it was natural to ask ourselves some questions:

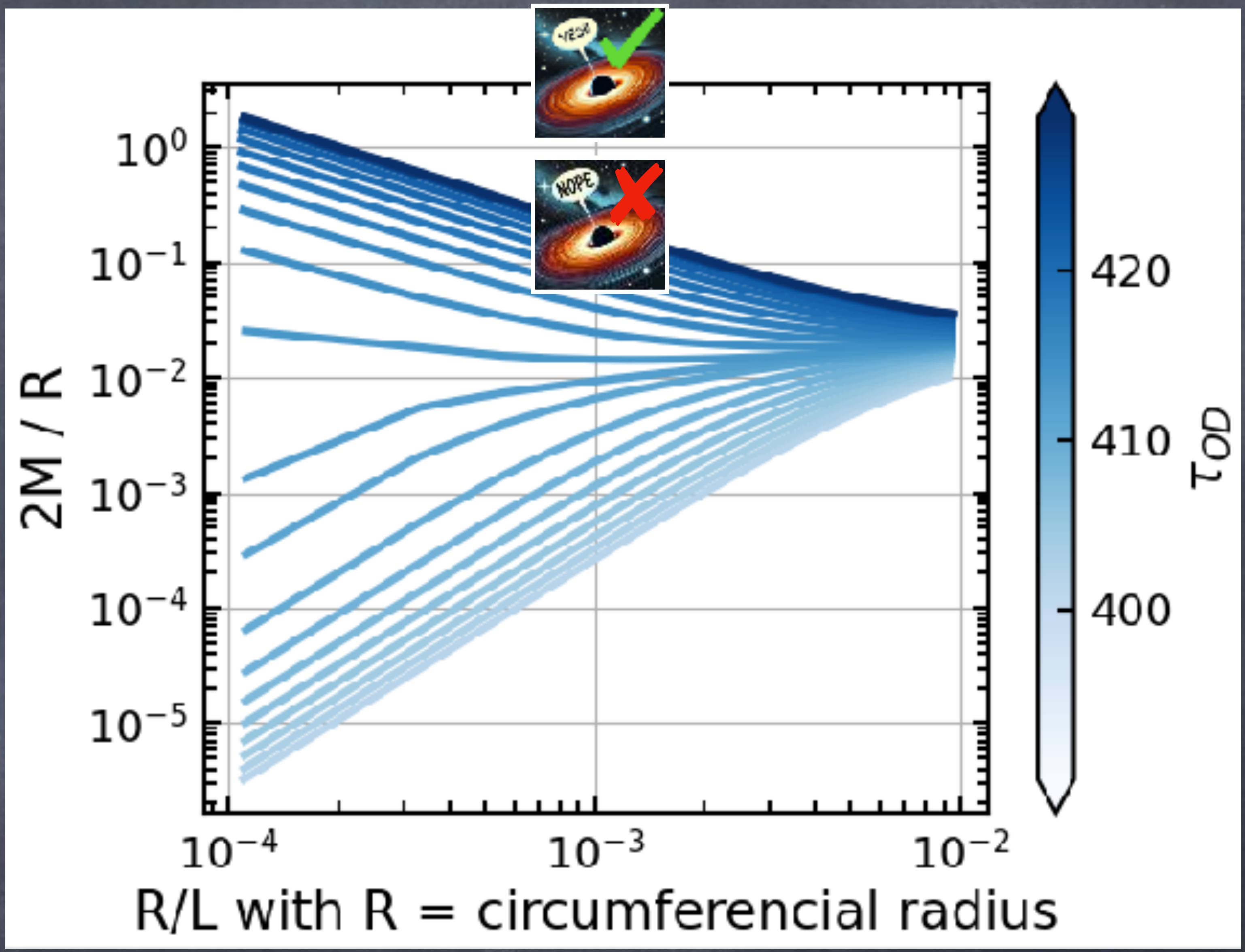
1. Can we form a black hole? (cosmic censorship conjecture by Penrose 1969). Requires finding an apparent horizon (AH) in Numerical Relativity from our cosmological initial data
   - Note: horizon forms in LTB Harada+ arXiv:1512.08639 and in Einstein-Vlasov Andréasson+ arXiv:2410.06701
2. Effect of the anisotropy (shear) at the peak (hoop conjecture by Thorne 1972); "spindle" rather than pancake start to form?
3. Ultimate question: can we form seeds of SMBHs by direct collapse of the core of CDM halos?
   - JWST see galaxies and SMBH at z>10
   - Newtonian simulation of SIDM claim BH formation

Feng, Yu & Zhong ApJL 2021

Massive BH formation?

Preliminary!!!!

- rule-of-thumb criteria
- when 2M/R>1 a BH should form
- which M?
- we still need to show that an apparent horizon forms, using a proper criterion based on the expansion of the outgoing null geodesics



Work in Progress:
Munoz, Byrnes, Bruni and Hawke

Massive BHs formation

- With these full GR (numerical relativity) simulations, we still need to show that an apparent horizon forms
- Need to use a proper criterion based on the expansion of the outgoing null geodesics
- So far, this has been too hard a problem

SMBH seeds from direct collapse of CDM-Curvature peaks

Marco Galoppo, MB & T. Harada, 2605.30145

- We go back to spherical LTB and Szekeres (dipole distribution) exact models, again starting from initial conditions in terms of the gauge-invariant curvature perturbation $\mathcal{R}_c$ at z=300
- Focus on the possible formation of SMBH seeds of CDM, represented by pressures fluid (dust), in the range 10^3 - 10^6 $M_\odot$, by $z \approx 10$ or even earlier.

LTB and Szekeres models

- Standard metric approach for LTB and Szekeres
  - Hellaby, Goode-Weinwrigth, Harada+, etc…

1. Each shell evolves as a closed RW model (top-hat) with its own spatial curvature
2. Simple and useful to predict time of apparent horizon and time of collapse, but
   - hides details of how each shall evolves, and the type of singularity
   - Non-local: each shall evolution dictated by mass inside

Spherical and Szekeres local dynamics

- Physical, covariantly defined local variables
- dynamics obeys "silent" evolution, i.e. nonlinear set of ODEs (2605.30145, based on MB, Matarrese & Pantano, astro-ph/9406068)

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 2\sigma^2 - 4\pi G\rho\,;$$

$$\dot{\rho} = -\Theta\rho\,;$$

$$\dot{\sigma}_{\langle\mu\nu\rangle} = -\frac{2}{3}\Theta\sigma_{\mu\nu} - \sigma_{\alpha\langle\mu}\sigma_{\nu\rangle}{}^{\alpha} - E_{\mu\nu}\,;$$

$$\dot{E}_{\langle\mu\nu\rangle} = -\Theta E_{\mu\nu} - 3\sigma_{\alpha\langle\mu}E_{\nu\rangle}{}^{\alpha} - 4\pi G\rho\sigma_{\mu\nu}$$

+constrains

In the local tetrad

$$\sigma_+ := \frac{1}{2}(\sigma_{22} + \sigma_{33}) = -\frac{1}{2}\sigma_{11}$$

Same for E

4-d Dyn. System

$$\dot{\rho} = -\Theta\rho\,,$$

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - 6\sigma_+^2 - 4\pi G\rho\,,$$

$$\dot{\sigma}_+ = -\frac{2}{3}\Theta\sigma_+ + \sigma_+^2 - E_+\,,$$

$$\dot{E}_+ = -\Theta E_+ - 3E_+\sigma_+ - 4\pi G\rho\sigma_+$$

Spherical and Szekeres local dynamics

- Dimensionless variables, asymptotic to finite values, and time

$$\tau = \pm \int \Theta \mathrm{d}t$$

$$\Omega := \frac{3\rho}{\Theta^2}, \quad \Sigma_+ := \frac{\sigma_+}{\Theta}, \quad \mathcal{E}_+ := \frac{E_+}{\Theta^2}$$

✓ τ with negative sign during collapse

✓ 3-d Dyn. System + Raychaudhuri eq.

✓ Fixed points and their stability type determine the type of singularity

$$\partial_\tau \Theta = \Theta \left[\frac{1}{3} + 6\Sigma_+^2 + \frac{\Omega}{6}\right],$$

$\Theta < 0, \dot{\Theta} < 0,$ Collapse is irreversible

$$\partial_\tau \Omega = -\frac{\Omega}{3}\left[36\Sigma_+^2 - 1 + \Omega\right],$$

$\Omega = 0$ is a stable sub manifold

$$\partial_\tau \Sigma_+ = \Sigma_+ \left[\frac{1}{3} - 6\Sigma_+^2 - \Sigma_+ - \frac{\Omega}{6}\right] + \mathcal{E}_+,$$

$$\partial_\tau \mathcal{E}_+ = \mathcal{E}_+ \left[\frac{1}{3} - 12\Sigma_+^2 + 3\Sigma_+ - \frac{\Omega}{3}\right] + \frac{1}{6}\Sigma_+\Omega$$

Singularity types

- Local kinematics of each fluid element
- Singularities are "velocity dominated"

$$\frac{\dot{\ell}_i}{\ell_i} = \Theta_{(i)} = \frac{1}{3}\Theta + \sigma_{ii}$$

Singularity type	Behaviour of ℓ_i	Asymptotic behaviour of $\Theta_{(i)}$
Point-like	$\ell_1, \ell_2, \ell_3 \to 0$	$\Theta_{(1)}, \Theta_{(2)}, \Theta_{(3)} \to -\infty$
Cigar-like	$\ell_1, \ell_2 \to 0$, $\ell_3 \to +\infty$	$\Theta_{(1)}, \Theta_{(2)} \to -\infty$, $\Theta_{(3)} \to \text{const} > 0$
Filament-like	$\ell_1, \ell_2 \to 0$, $\ell_3 \to \text{const} > 0$	$\Theta_{(1)}, \Theta_{(2)} \to -\infty$, $\Theta_{(3)} \to 0$
Panca		

See the book: Wainwright & Ellis, Dynamical Systems in Cosmology (1997)

Matter becomes negligible in collapse, except for the set of measure zero point-like case

Singularity type	Ω	Σ_+	$\mathcal{E}_+$	Type of stationary point	Collapse asymptotic behaviour
Point-like	1	0	0	DI (repeller)	Flat FLRW
Cigar-like	0	1/3	2/9	DV (attractor)	Kasner-like
Pancake-like	0	$-1/3$	0	DIV (attractor)	Minkowski-like

TABLE II. Classification of the stationary points of the autonomous Szekeres dynamical system for quasispherical models with positive curvature parameter.

SMBH seeds from direct collapse of CDM-Curvature peaks

Marco Galoppo, MB & T. Harada, 2605.30145

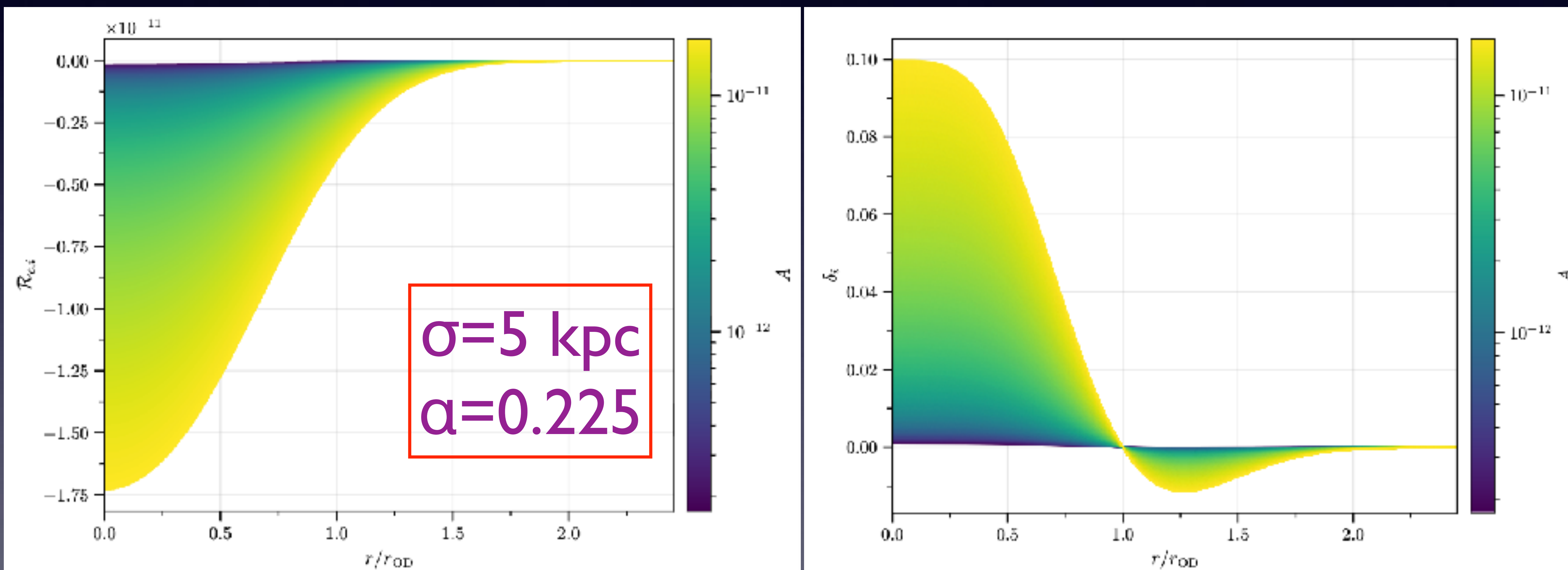
- Initial conditions in terms of $\mathcal{R}_c$ at z=300
  - Based on MB, Hidalgo, Meures & Wands, arXiv:1307.1478
- We tried various profiles for $\mathcal{R}_c$, focusing on a typical from BBKS peak theory (smoothed Gaussian)
- $\delta^{(1)} = \dfrac{\nabla^2 \mathcal{R}_c}{\mathcal{F} H^2}$, hence the over-density is surrounded by under-density, matched to a flat ΛCDM at $r = r_*$

$$\mathcal{R}^{[m]}_{c,i}(r) = \begin{cases} -A\left[e^{-x^2/2}\left(1+\alpha x^2+\beta x^4\right) - e^{-x_\star^2/2}\left(1+\alpha x_\star^2+\beta x_\star^4\right)\right] & r < r_\star, \\ 0 & r \geq r_\star, \end{cases}$$

$$\boxed{x \equiv r/\sigma,} \qquad \boxed{\alpha < \frac{1}{2} \;\; ; \;\; \beta = \frac{\alpha}{2} - \frac{1}{8}}$$

Curvature and density profiles

- Over-density surrounded by under-density, matched to a flat ΛCDM at $r = r_*$



$$\mathcal{R}_{c,i}^{[m]}(r) = \begin{cases} -A\left[e^{-x^2/2}\left(1+\alpha x^2+\beta x^4\right) - e^{-x_\star^2/2}\left(1+\alpha x_\star^2+\beta x_\star^4\right)\right] & r < r_\star, \\ 0 & r \geq r_\star, \end{cases}$$

$$\boxed{x \equiv r/\sigma,}$$

Characteristic times

- Using exact expressions (Harada+, 1512.08639), we can find turnaround (TA), horizon formation (HF) and collapse times t_c of each individual shell at coordinate r in terms of $\mathcal{R}_c$

$$\boxed{\epsilon = r\mathcal{R}'_{c,i}}$$

$$t_{\rm ta}(r) = \frac{\pi}{2^{5/2}} H_i^2 a_i^3 r^{3/2} \left[\mathcal{R}'^{-3/2}_{c,i} + 3\left(r^2 + \frac{2}{5a_i^2 H_i^2}\right)\frac{\mathcal{R}'^{-1/2}_{c,i}}{r} + \mathcal{O}\left(\epsilon^{1/2}\right)\right],$$

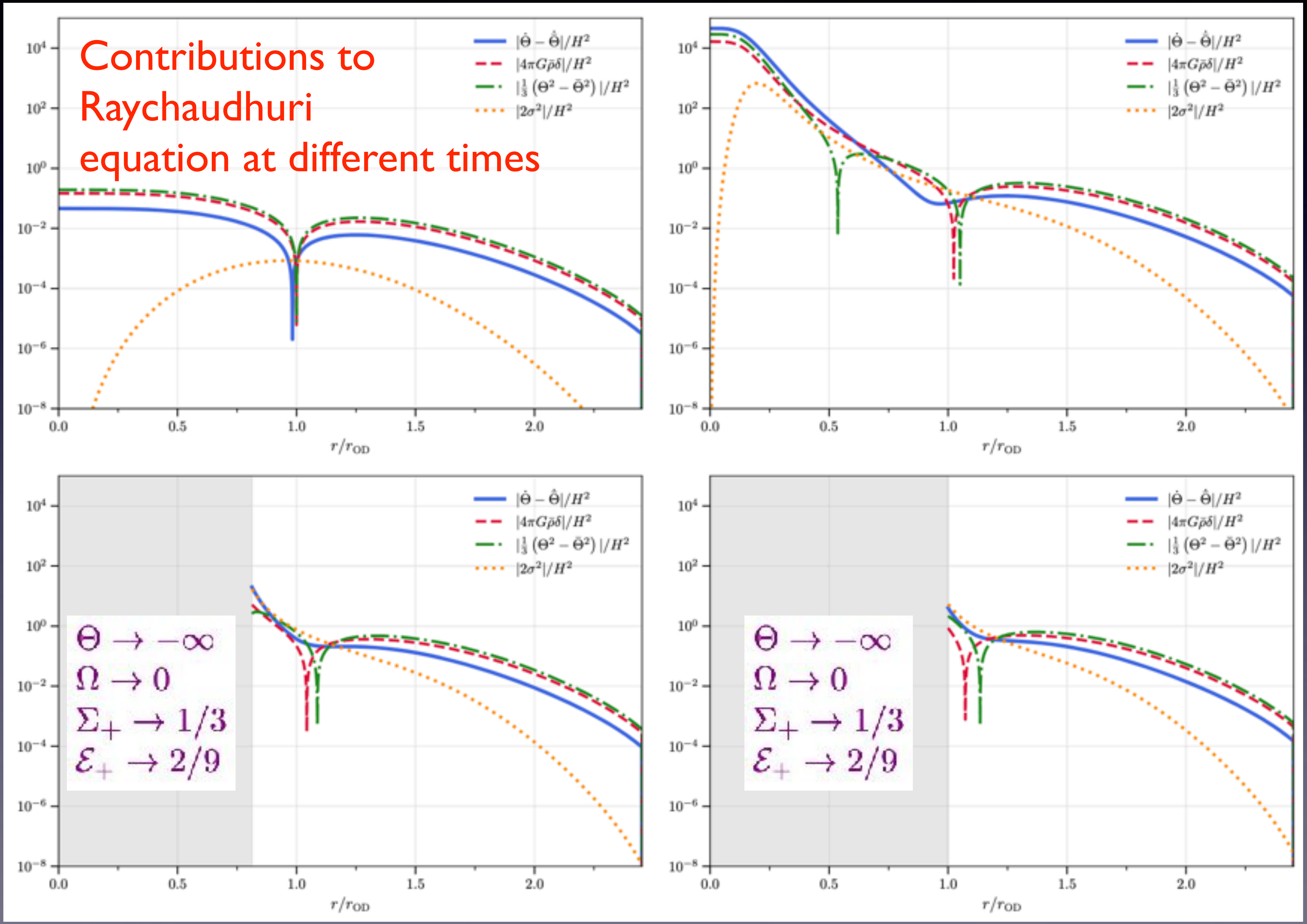
$$t_{\rm col}(r) = \frac{\pi}{2^{3/2}} H_i^2 a_i^3 r^{3/2} \left[\mathcal{R}'^{-3/2}_{c,i} + 3\left(r^2 + \frac{2}{5a_i^2 H_i^2}\right)\frac{\mathcal{R}'^{-1/2}_{c,i}}{r} + \mathcal{O}\left(\epsilon^{1/2}\right)\right],$$

$$t_{\rm FH}(r) = \frac{\pi}{2^{3/2}} H_i^2 a_i^3 r^{3/2} \left[\mathcal{R}'^{-3/2}_{c,i} + 3\left(r^2 + \frac{2}{5a_i^2 H_i^2}\right)\frac{\mathcal{R}'^{-1/2}_{c,i}}{r} + \frac{2^{3/2} r^{3/2}}{3\pi} + \mathcal{O}\left(\epsilon^{1/2}\right)\right]$$

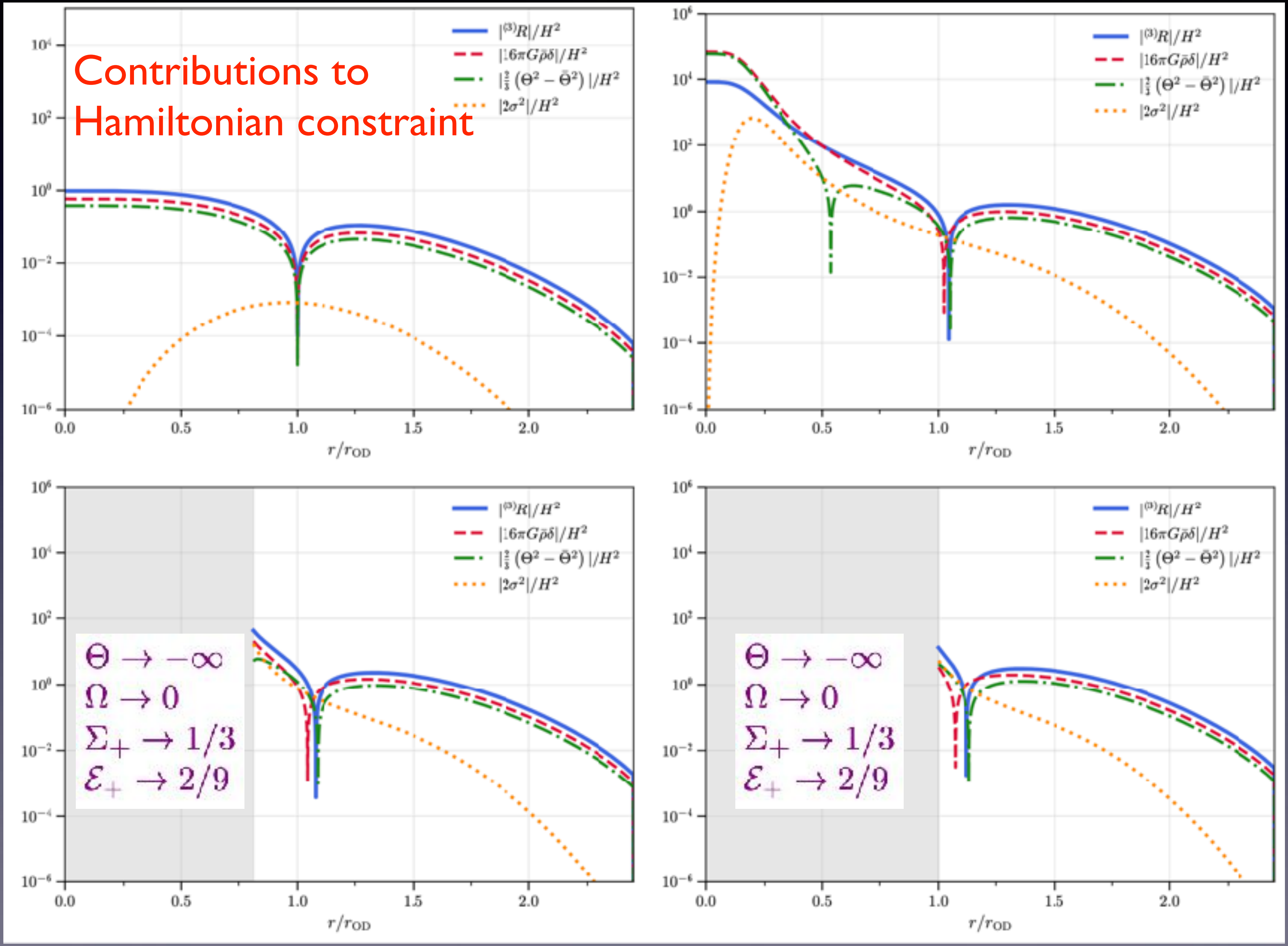
$$\mathcal{R}^{[m]}_{c,i}(r) = \begin{cases} -A\left[e^{-x^2/2}\left(1+\alpha x^2+\beta x^4\right) - e^{-x_\star^2/2}\left(1+\alpha x_\star^2+\beta x_\star^4\right)\right] & r < r_\star, \\ 0 & r \geq r_\star, \end{cases}$$

$$\boxed{x \equiv r/\sigma,}$$

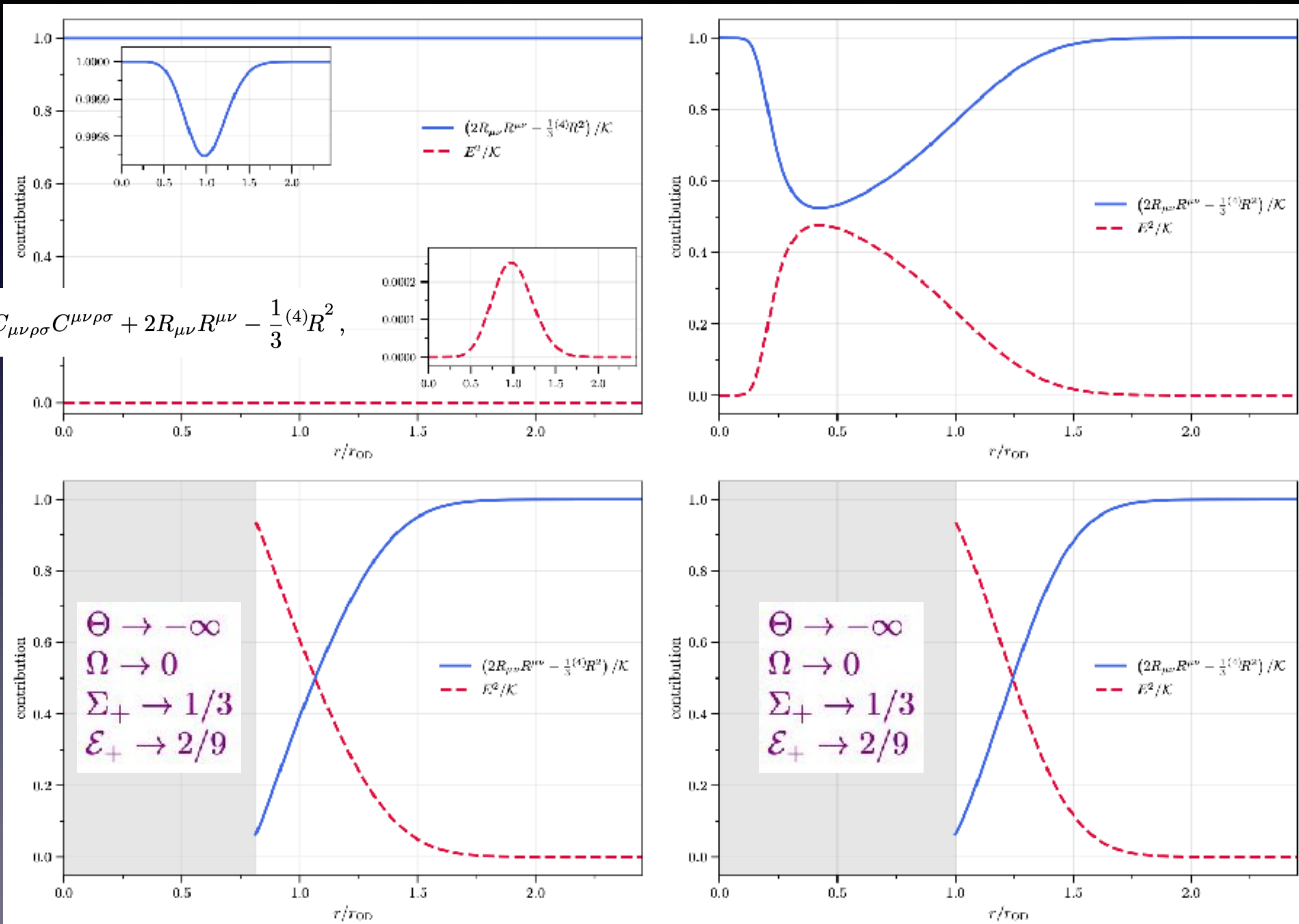
Profiles of contributions



Profiles of contributions



Ricci, Weyl & Kretschmann



$$\mathcal{K} = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}{}^{(4)}R^2 ,$$

SMBH seeds from direct collapse of CDM-Curvature peaks

- Take home message: both spherical and dipole-like initial perturbation at z=300 can form black holes in an interesting range of masses, in an interesting range of redshifts, in a flat ΛCDM background. Starting at $z \lesssim 500$ we have the same masses fully in the AH at z~10-15

A	σ [kpc]	$\delta_i^{\rm peak}$	$M_{\rm BH}$ [$M_\odot$]	$z_{\rm col}(r=0)$	$z_{\rm FH}(r=r_{\rm OD})$
$[10^{-13}, 7\times10^{-13}]$	1	$[10^{-3}, 10^{-1}]$	$\sim 10^3$	[1.5, 16]	[−0.3, 6]
$[1.7\times10^{-13}, 1.7\times10^{-11}]$	5	$[10^{-3}, 10^{-1}]$	$\sim 10^5$	[−0.3, 16]	[−1, 6]
$[7\times10^{-13}, 7\times10^{-11}]$	10	$[10^{-3}, 10^{-1}]$	$\sim 10^6$	[−0.3, 16]	[−1, 7]

TABLE III. Summary of BH collapse outcomes for initial perturbations set at $z = 300$ within a representative subset of the allowed parameter space shown in Fig. 2.

$$\mathcal{R}^{[m]}_{c,i}(r) = \begin{cases} -A\left[e^{-x^2/2}\left(1+\alpha x^2+\beta x^4\right) - e^{-x_\star^2/2}\left(1+\alpha x_\star^2+\beta x_\star^4\right)\right] & r<r_\star, \\ 0 \qquad\qquad x\equiv r/\sigma, & r\geq r_\star, \end{cases}$$

Summary and Outlook

- Quasi-spherical collapse follows the Top-Hat prediction, but finding an apparent horizon has proved challenging
- Reductionisms pays off: back to exact solutions:
  - effect of shear in LTB and Szekeres models: in GR spaghetti-fication (spindle/cigar, Kasner-like Weyl-dominated collapse) can win over pancakes, unlike in the Newtonian case (Zeldovich-approx.): SMBHs can be seeded by CDM-curvature perturbation peaks
  - SMBH seeds in the range 10^3 - 10^6 $M_{\odot}$ can form by $z \approx 5$-10, or earlier
  - ultimate goal: SMBHs seeds from direct collapse of the core of CDM halos (more work with Einstein Toolkit and other GR codes needed!)
- Work in progress and for the future:
  - general idea: extend quasi-spherical collapse, including anisotropy at the peak, with fluid (ET) and with particles (GRAMSES), looking for AH (ET) and virializasion (GRAMSES), developing old ideas (*)

(*) MB, Matarrese & Pantano, astro-ph/9406068, astro-ph/9407054, plus following works by Maartens+, Heinesen & Macpherson arXiv:2111.14423

do not underestimate the power of the force!

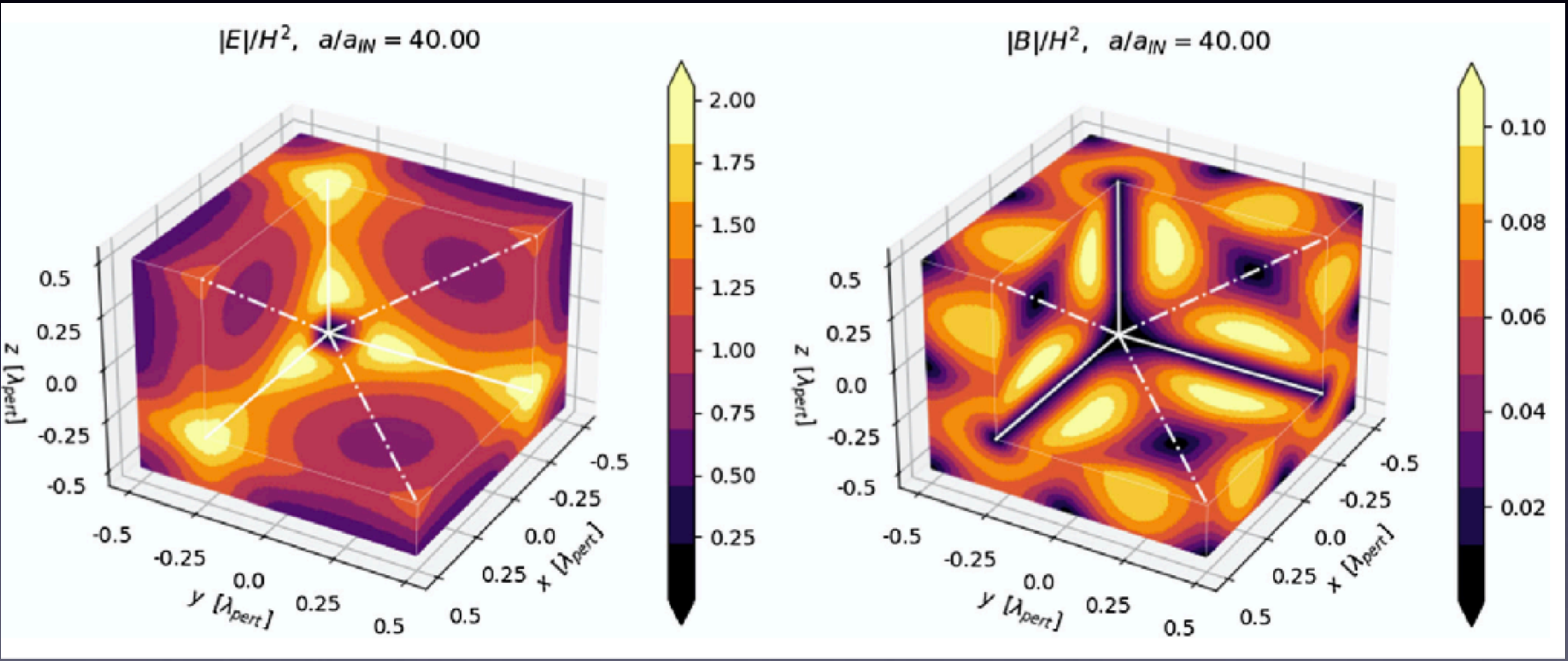
Initial Conditions

- initial conditions based on gauge-invariant curvature perturbation variable $\mathcal{R}_c$ with simple spatial distribution
- directly related to the 3-Ricci scalar $^{(3)}R^{(1)} = 4\nabla^2\mathcal{R}_c,$
- is conserved (constant in time) at first-order
- it sources the growing mode in first-order perturbation theory $4H\dot{\delta}^{(1)} + 6H^2\Omega_m\delta^{(1)} = {}^{(3)}R^{(1)},$
- Use first-order metric and extrinsic curvature constructed from $\mathcal{R}_c$ to compute exact 3-Ricci, plug this in Hamiltonian constrain and define fully-nonlinear matter density from it $$\rho = \frac{1}{2\kappa}\left({}^{(3)}R + K^2 - K^{ij}K_{ji} - 2\Lambda\right)$$

A couple of objections

- How about anisotropy at the peak?
  - Peak theory predicts rare peaks to be very spherical
  - For large masses the horizon can form in any case
- Is the p=0 (dust) assumption for the description of CDM good enough when the horizon starts to form?
  - Densities are low for large masses, and p=0 is at least as good as when we treat DM as Cold at very high z, when we run a Boltzmann code

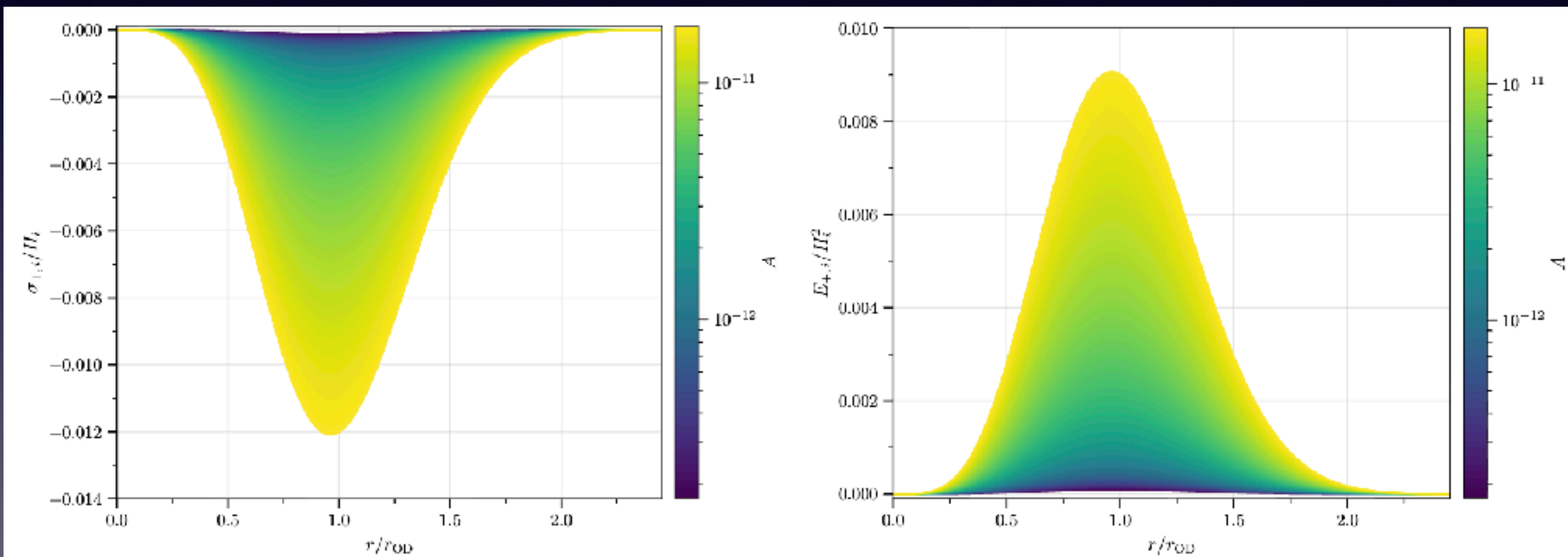
E and B parts of the Weyl curvature



simple structure makes evident that E is stronger around the peaks and *along* filaments, while B is stronger *around* filaments

Shear and E-Weyl profiles

- Over-density surrounded by under-density, matched to a flat ΛCDM at $r = r_*$



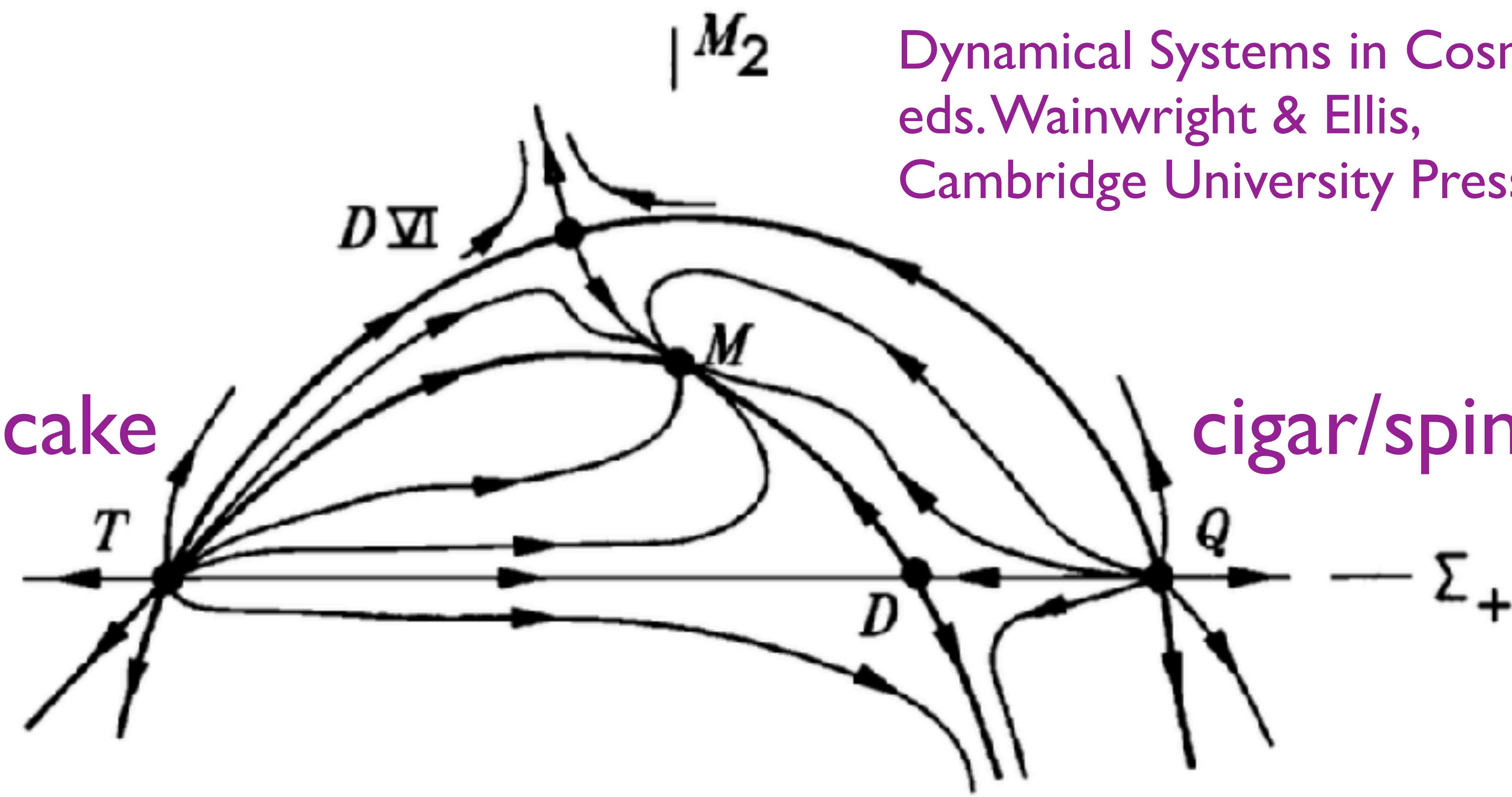
$$\mathcal{R}^{[m]}_{c,i}(r) = \begin{cases} -A\left[e^{-x^2/2}\left(1+\alpha x^2+\beta x^4\right) - e^{-x_\star^2/2}\left(1+\alpha x_\star^2+\beta x_\star^4\right)\right] & r < r_\star, \\ 0 & r \geq r_\star, \end{cases}$$

$$\boxed{x \equiv r/\sigma,}$$

Ω=0 Phase Space (assuming expansion)

Dynamical Systems in Cosmology,
eds. Wainwright & Ellis,
Cambridge University Press, 1997

pancake



cigar/spindle

Fig. 13.2. Phase portrait for the invariant set $\Omega_m = 0$, showing the evolution of the vacuum Szekeres solutions. We have projected this invariant set, which is the surface $\Sigma_+^2 + M_1 + 2M_2 = 1$ in (Σ_+, M_1, M_2)–space into the $\Sigma_+ M_2$–plane.

Work in progress: GRAMSES

with Cristian Barrera, Baojiu Li, Kazuya Koyama

- use relativistic N-body code to:
  - Start from initial conditions similar to work with Munoz
  - Evolve beyond $\delta^{(1)}=1.69$ (first shell-crossing) and look at virialization
  - Isotropic and anisotropic overdensity peaks

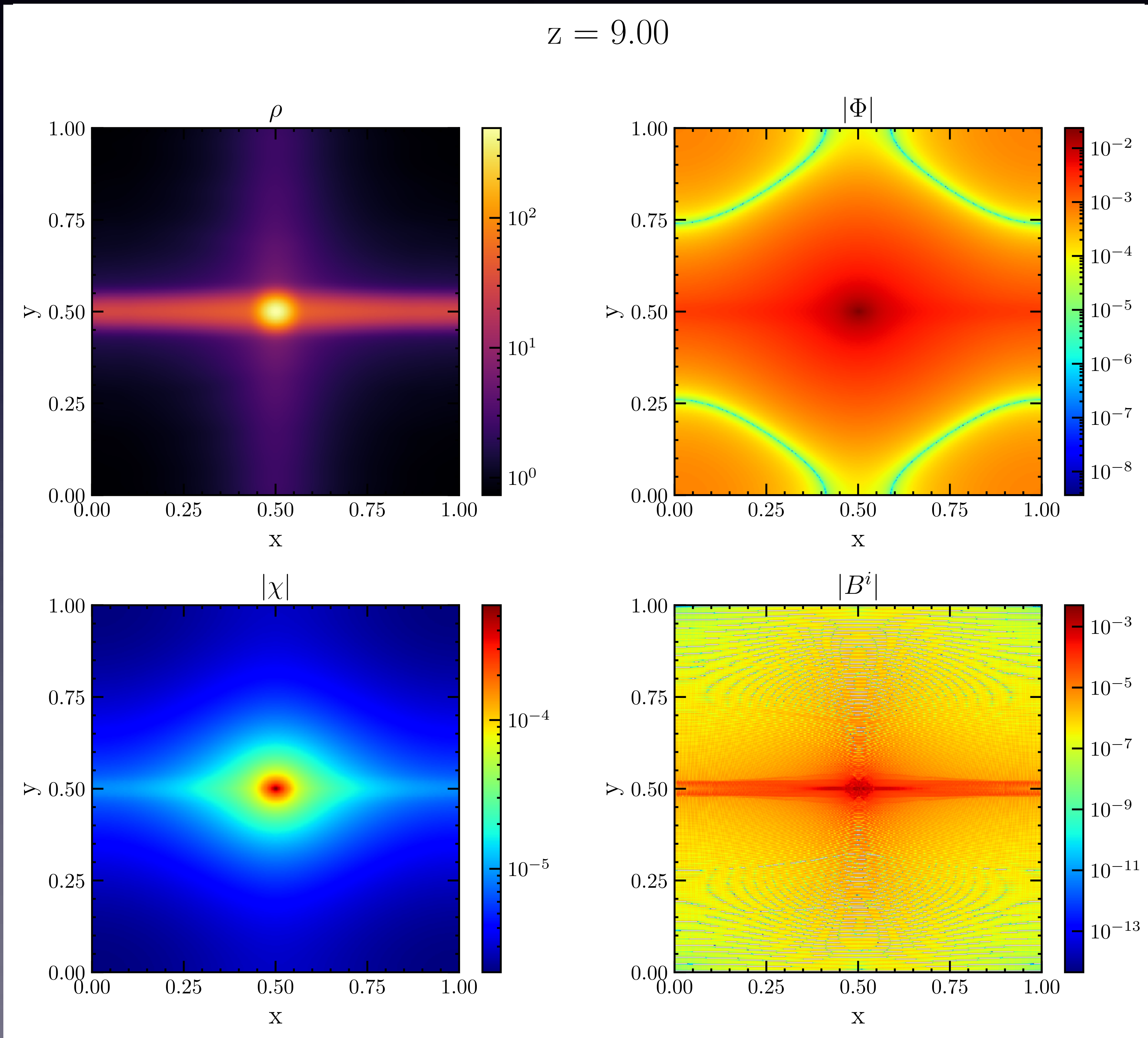
GRAMSES: beyond shell-crossing and virialization

extend work with Munoz beyond first shell-crossing

Evolution beyond δ_L=1.69 and effect of tidal fields

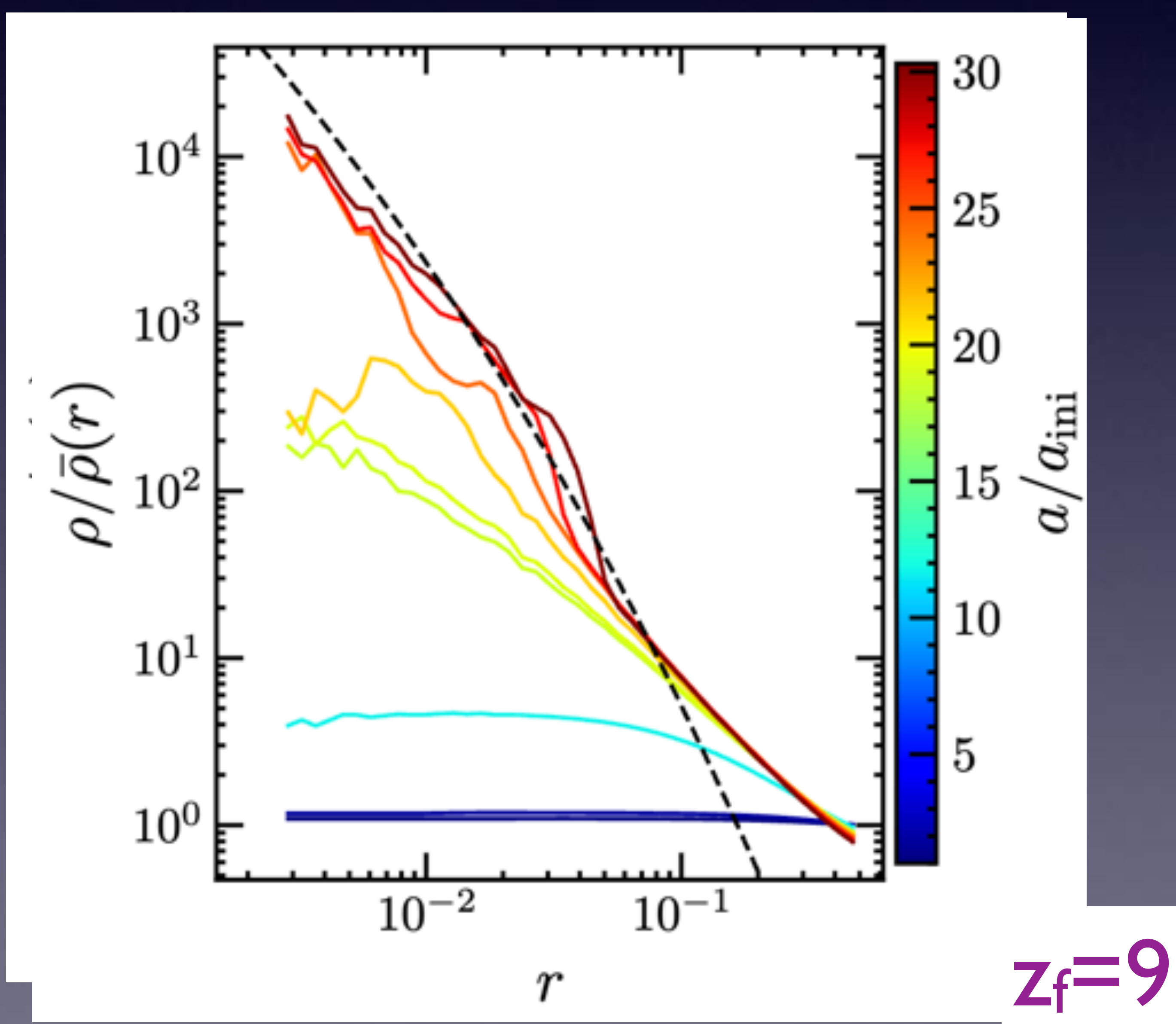
δ_{in}=9x10^{-3}

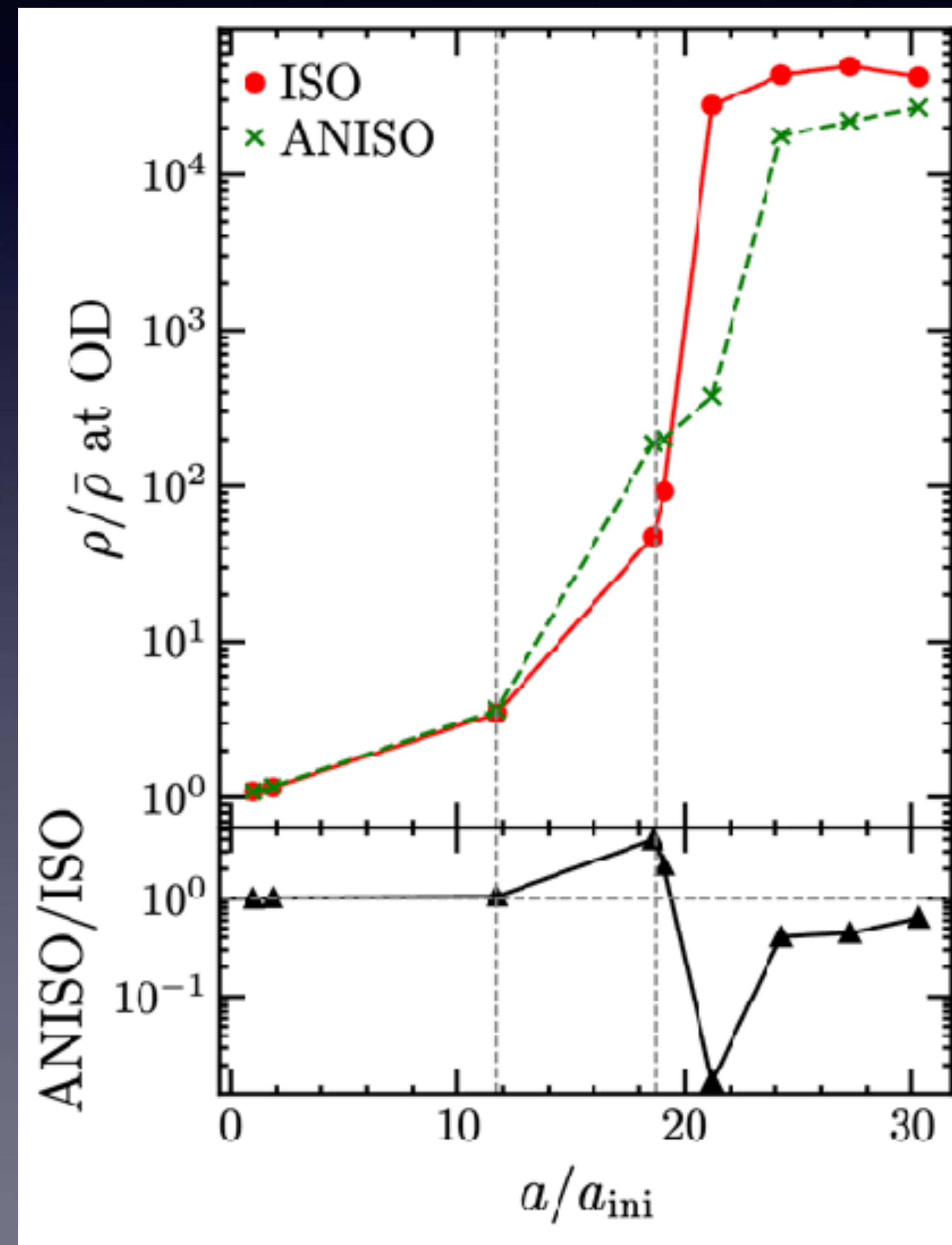
Add shear at the peak



Average and OD Evolution

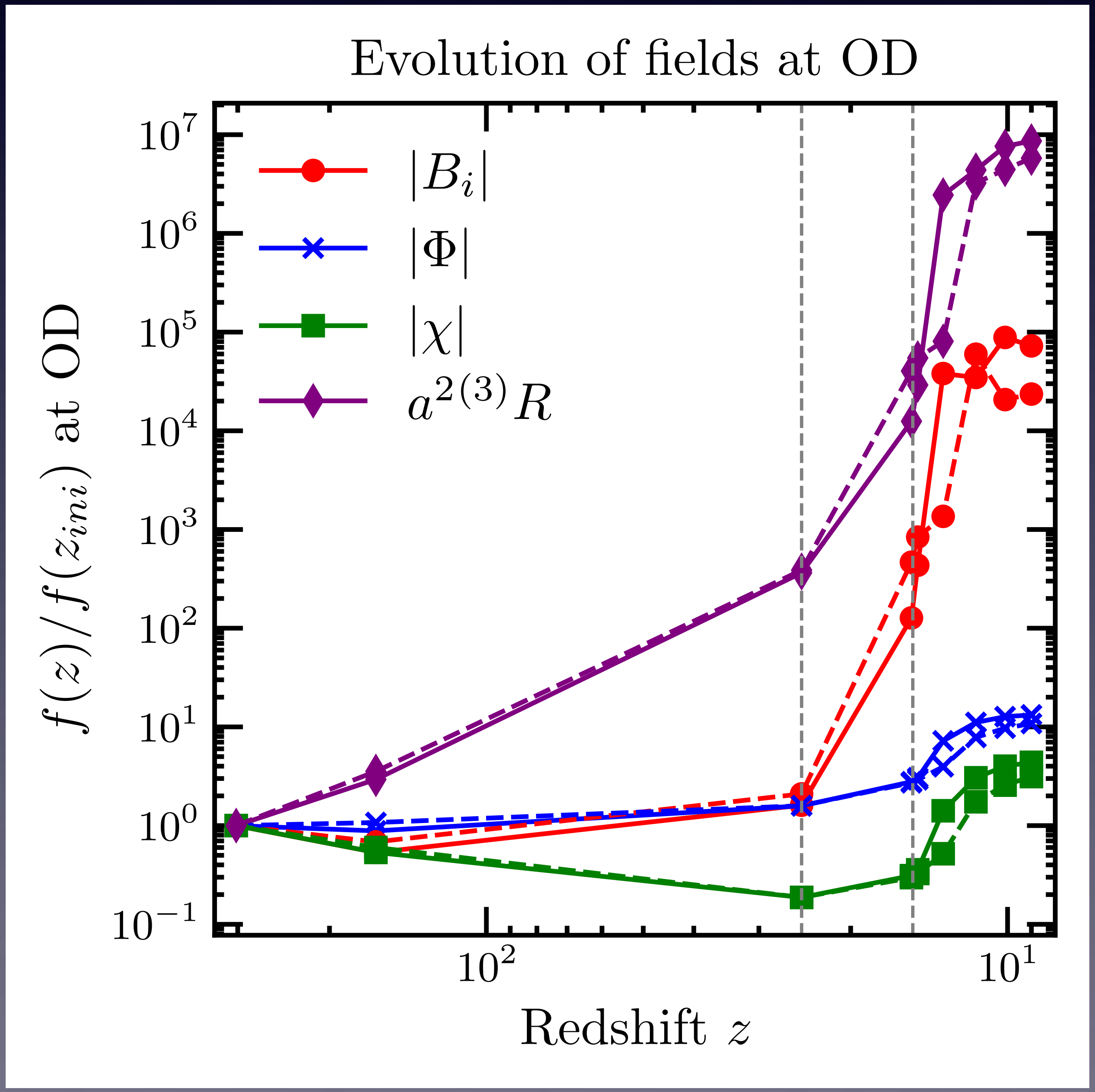
Solid and dashed lines represent the isotropic and anisotropic case, respectively. Vertical lines correspond, from left to right, to the time of TA and time of collapse, as predicted by the top-hat model.

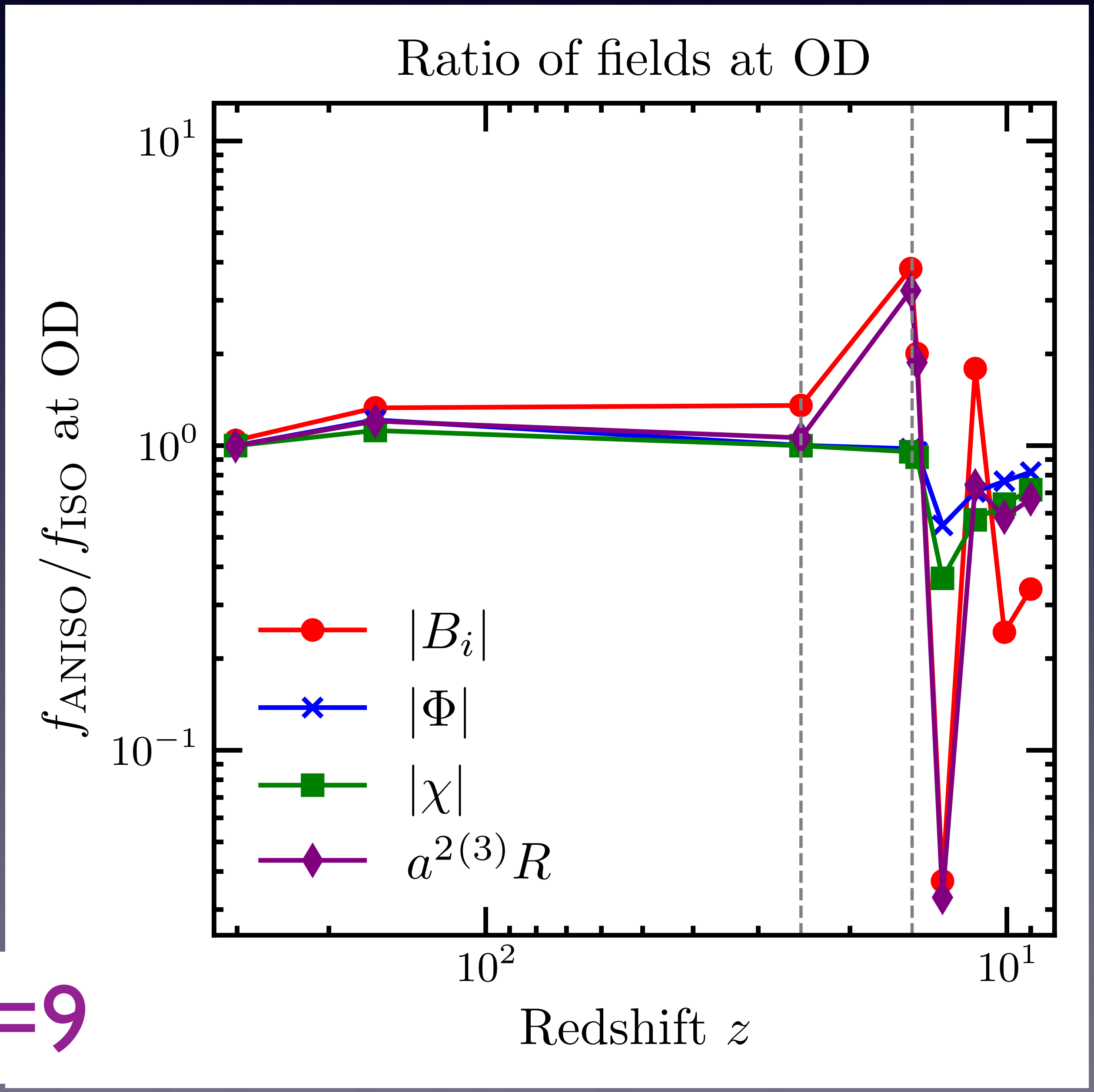




Evolution at the OD

Solid and dashed lines represent the isotropic and anisotropic case, respectively. Vertical lines correspond, from left to right, to the time of TA and time of collapse, as predicted by the top-hat model.

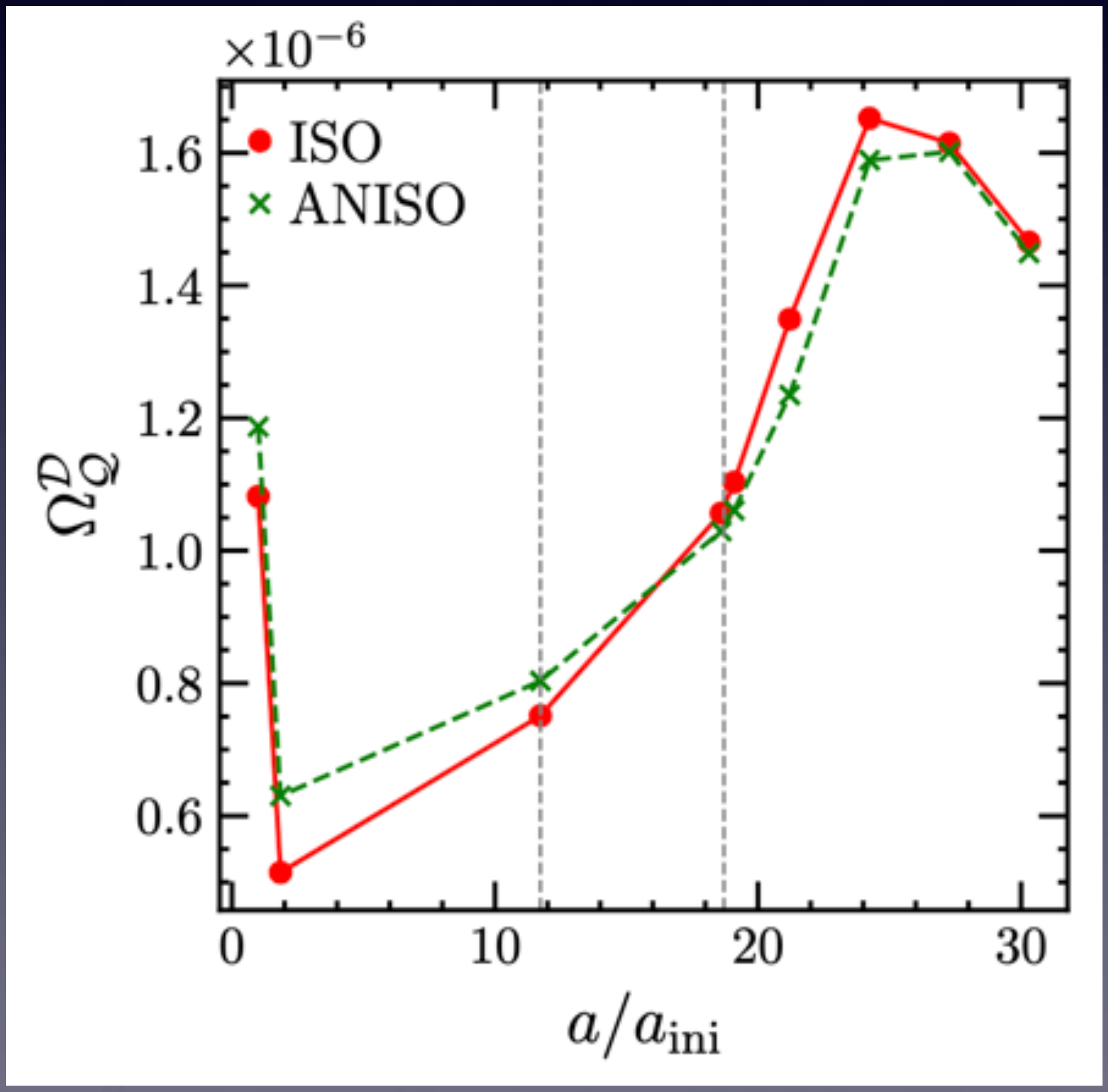


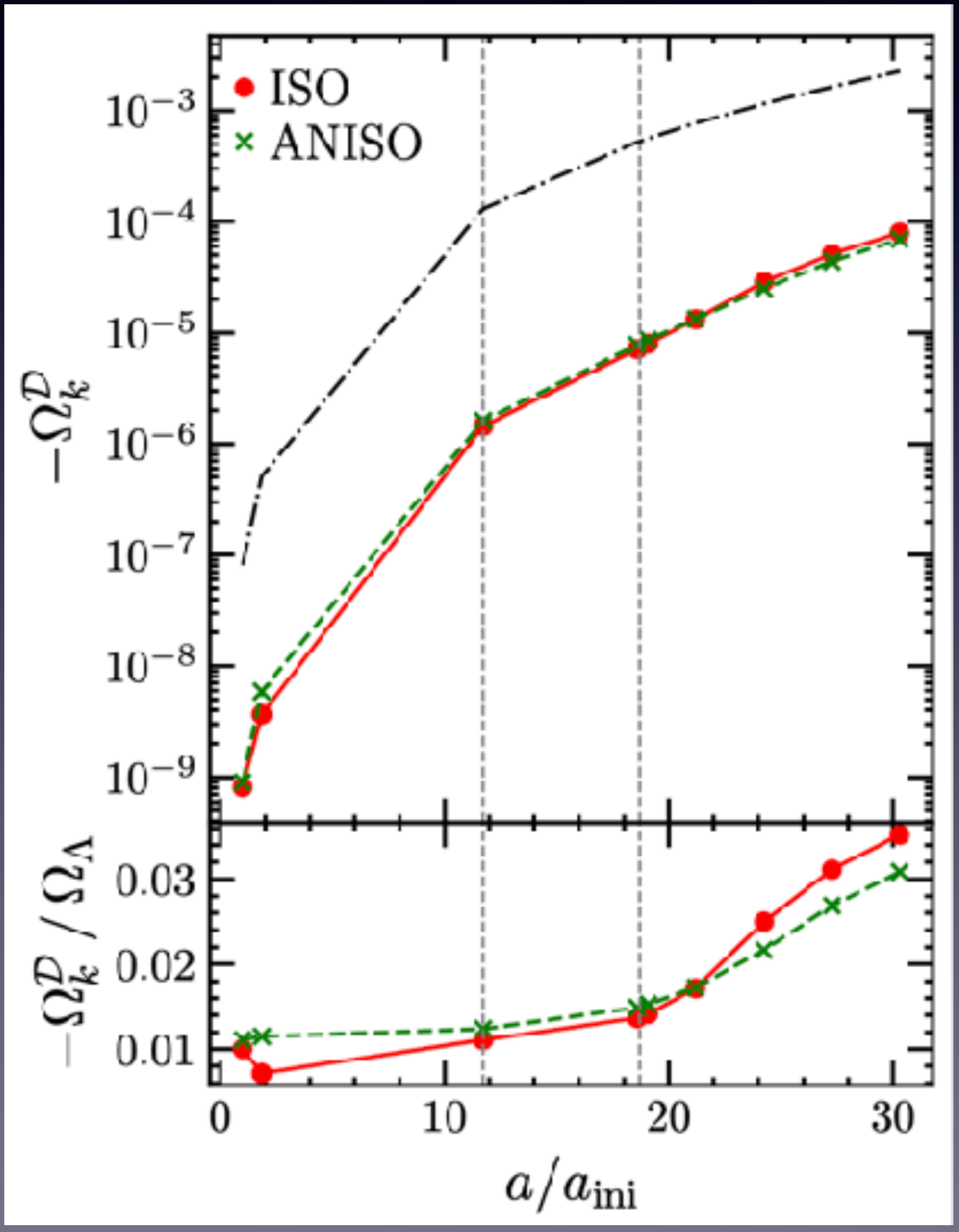


z_f=9

Curvature Evolution

Solid and dashed lines represent the isotropic and anisotropic case, respectively. Vertical lines correspond, from left to right, to the time of TA and time of collapse, as predicted by the top-hat model.





$z_f=9$

Curvature Evolution

Solid and dashed lines represent the isotropic and anisotropic case, respectively. Vertical lines correspond, from left to right, to the time of TA and time of collapse, as predicted by the top-hat model.

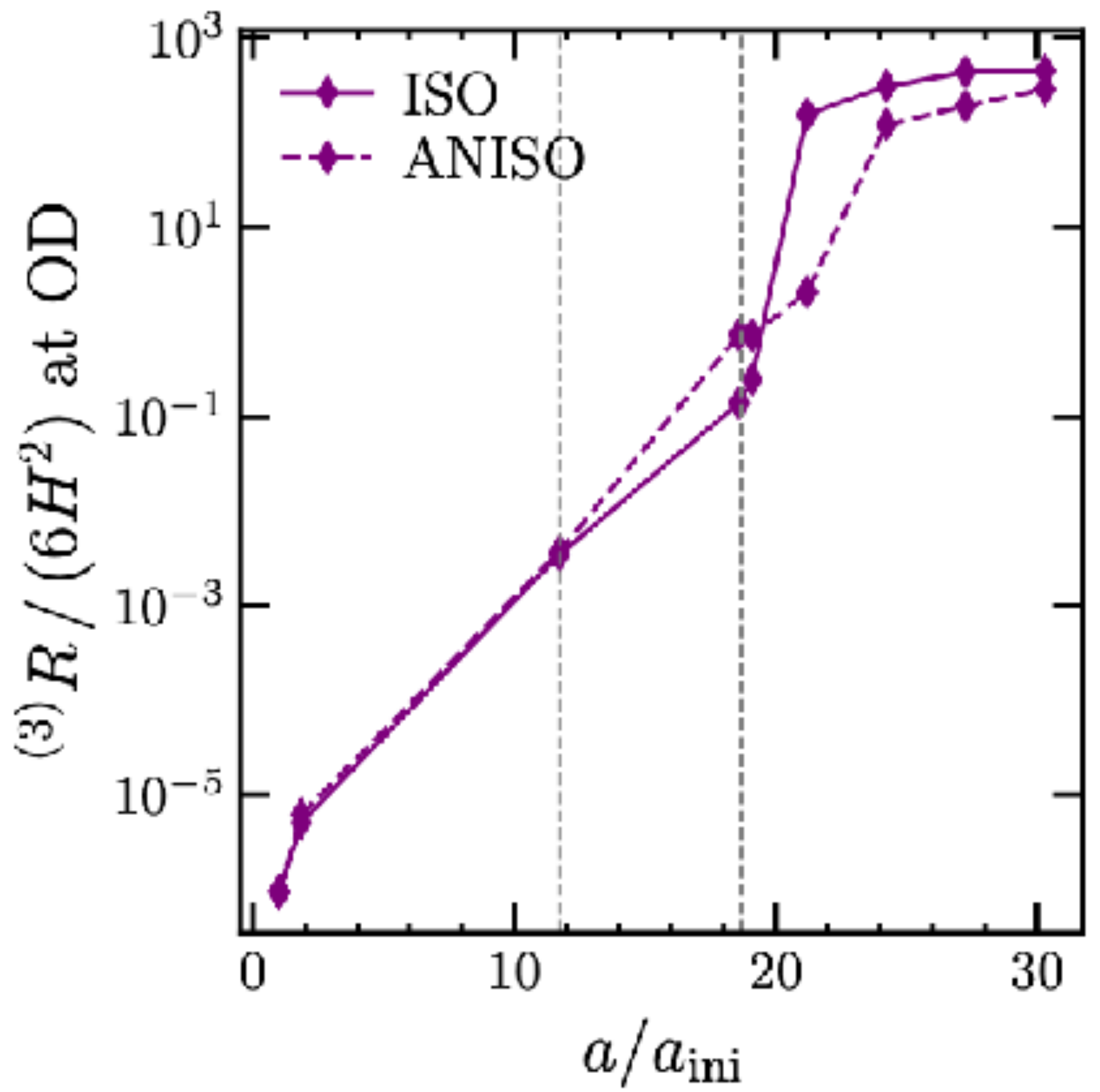


FIG. 11. Evolution of Ricci at OD.

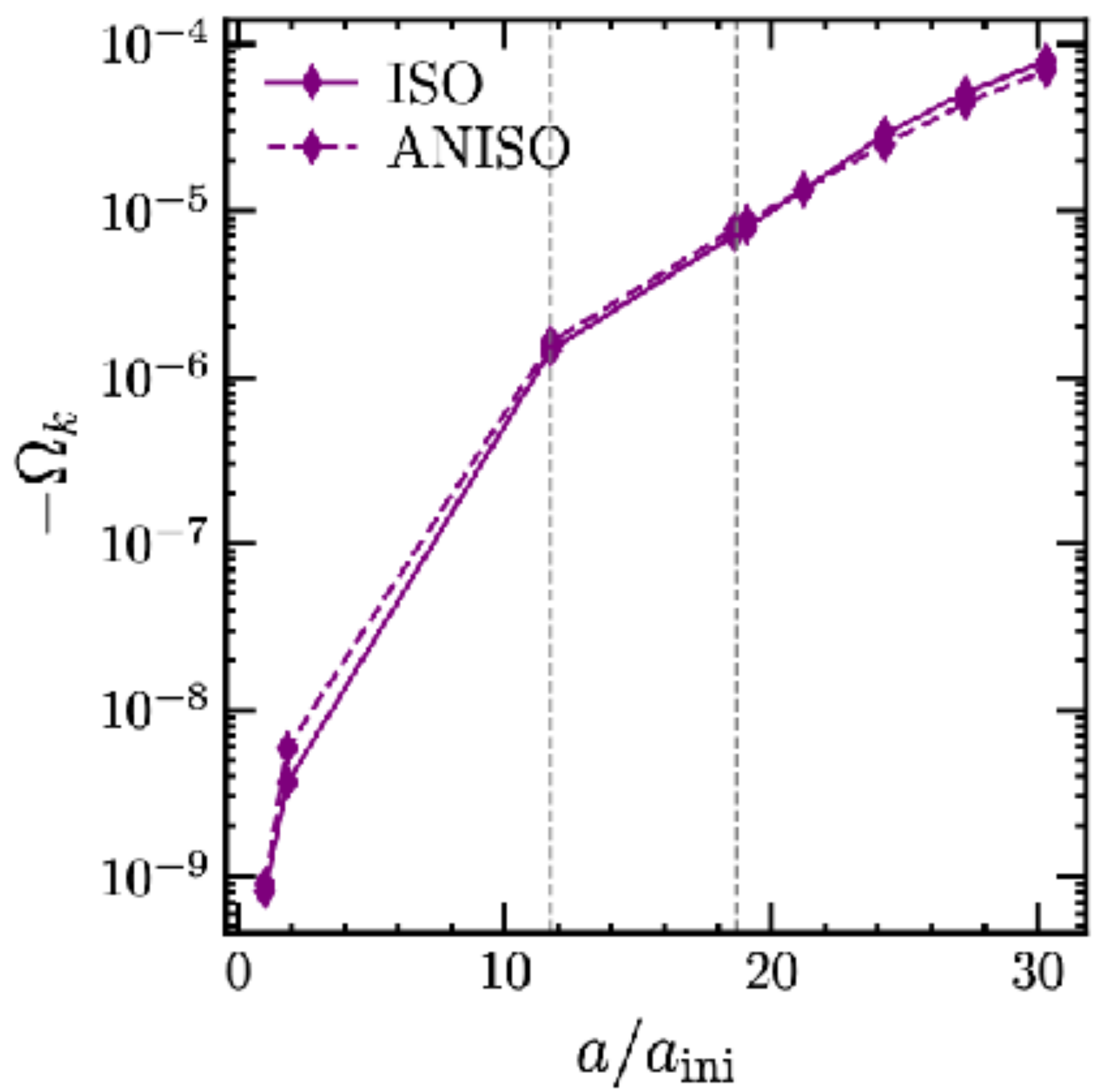


FIG. 13. Evolution of Ω_k.

$z_f=9$