Gravitational Lensing of Gravitational Waves in the Wave-Optics Regime

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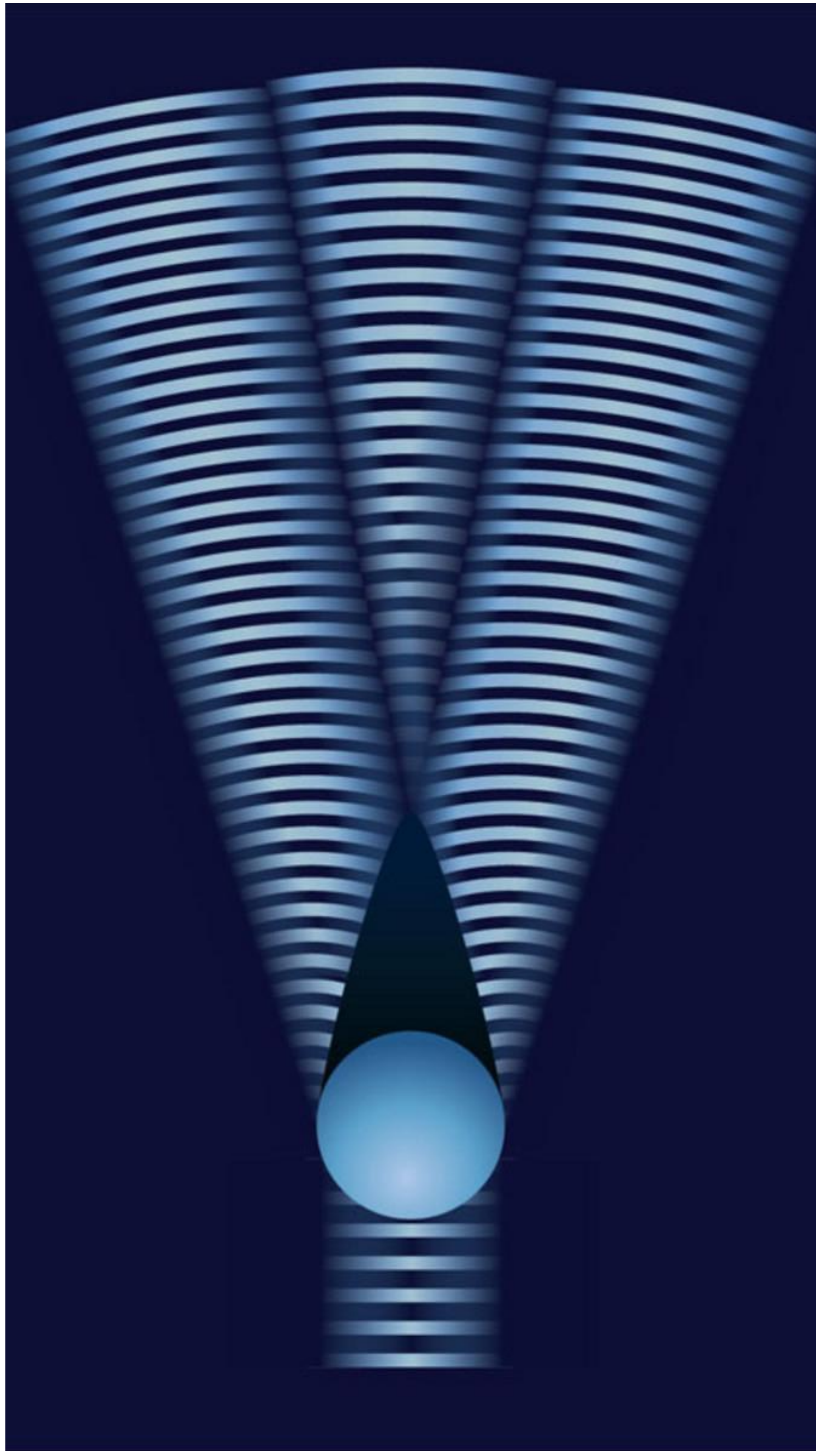
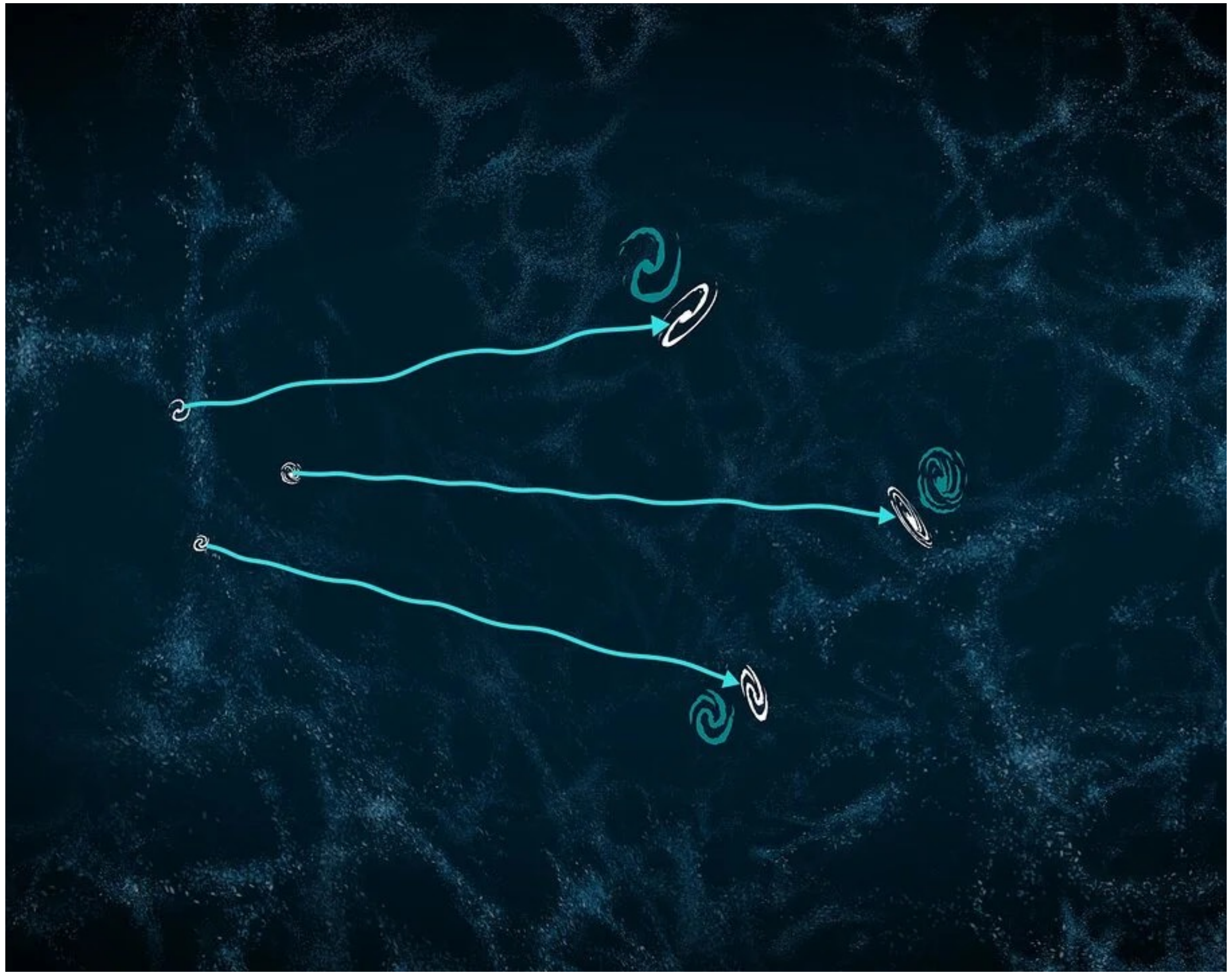
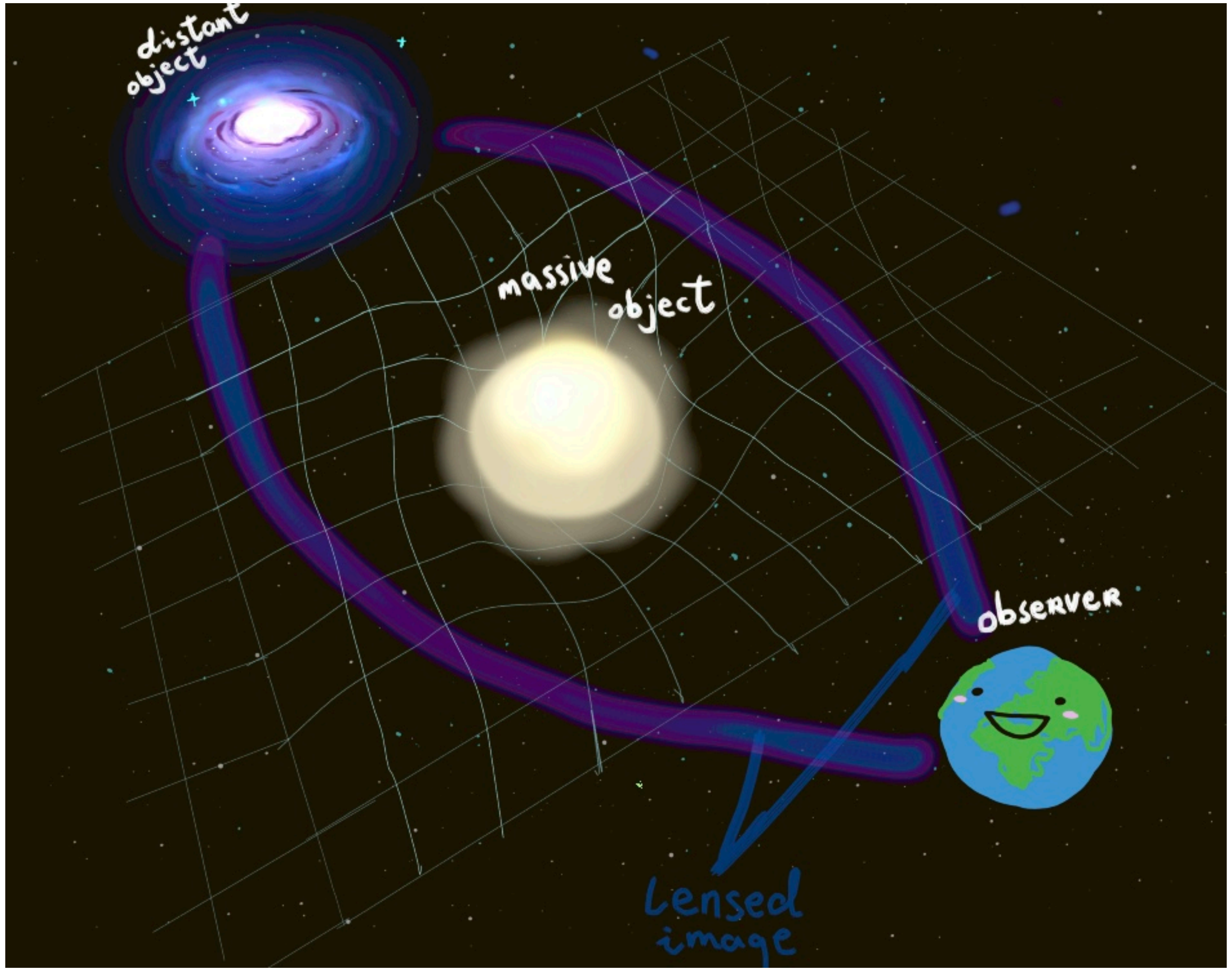


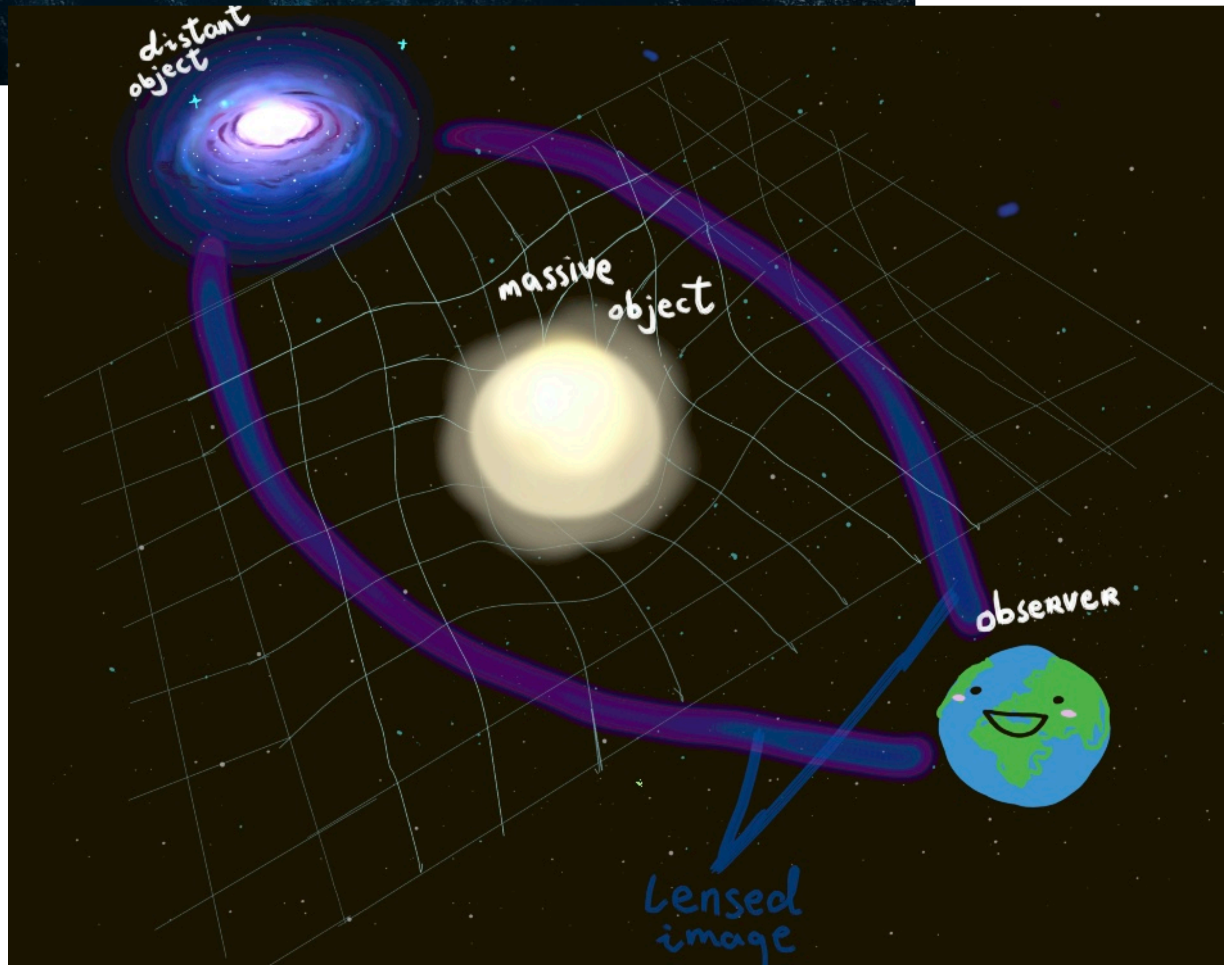
Propagation effects





Propagation effects carry cosmological and astrophysical information.

Research interests



Dark energy & modified gravity:
- How is the Universe expanding?
- Are the laws of gravity different on large scales?

- *AG et al. JCAP (2019),*
- *Tasinato, AG et al JCAP 2021,*
- *AG. et al arXiv:2110.14689*
- *AG et al. PRD (2021),*
- *Balaudo A., AG et al JCAP 2023,*
- *AG, et al, JCAP 2025*
- *S. Akama, AG et al arXiv:2603.03165*

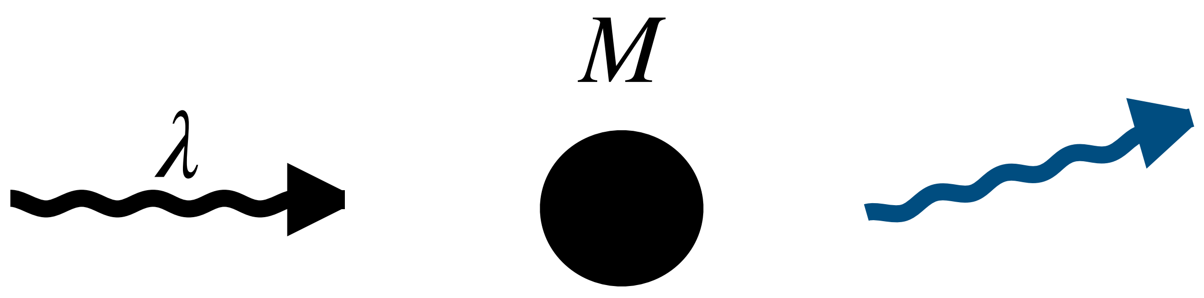
Dark matter & wave-optics effects:
- How is matter distributed in the Universe?
- Do small-scale halos have cores or cusps?

- *AG PDU 2024,*
- *Braga G., AG et al JCAP 2024,*
- *M. Carrillo-Gonzales, AG et al, PRD 2025*
- *R. Amoruso, AG et al arXiv:2604.15313*
- *AG, G. Tasinato arXiv:2605.21584*

Black holes:
- How do BH respond to perturbations?
- What can we learn about gravity in the strong-field regime?

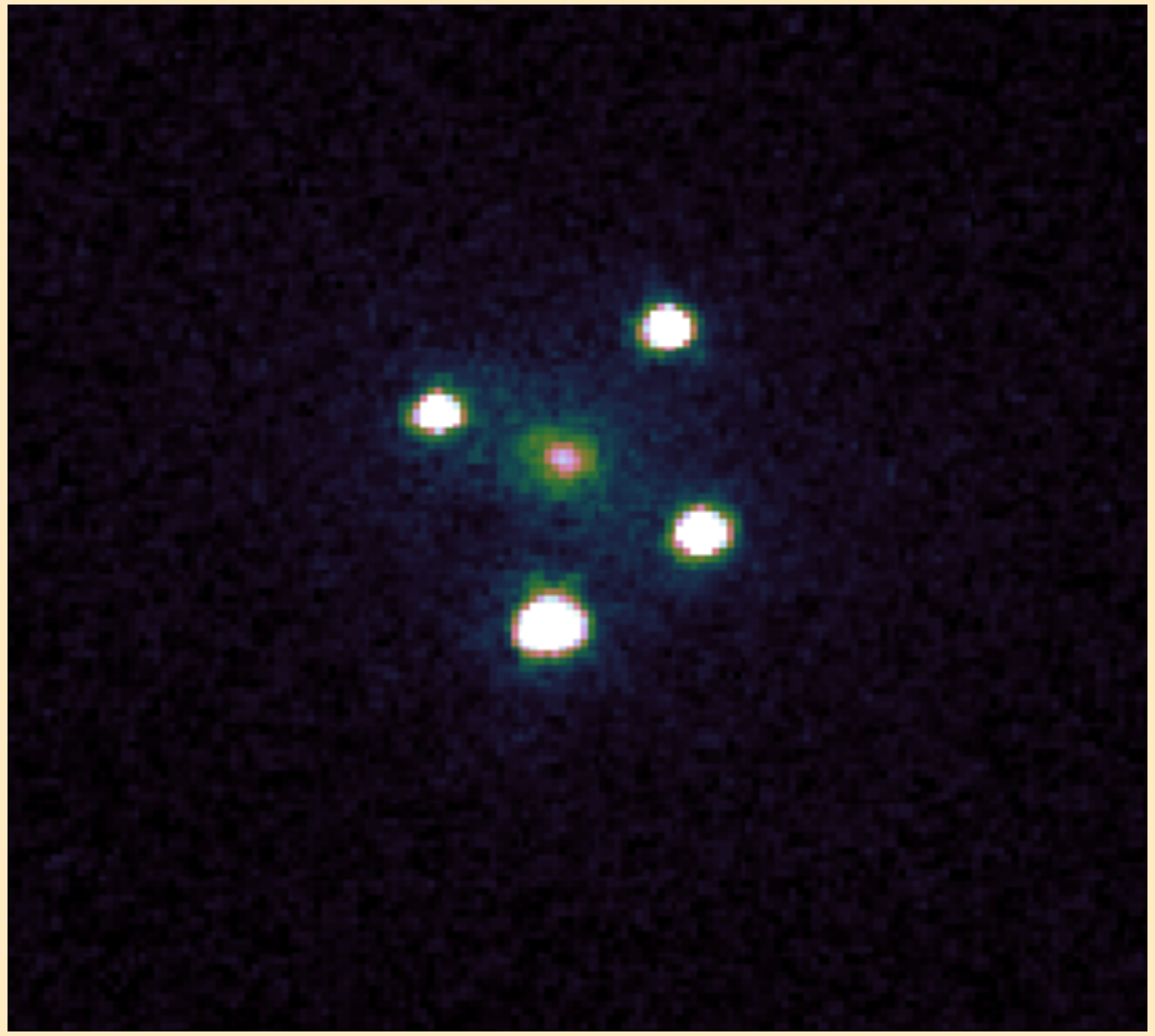
- De Luca V., *AG et al PRD (2024)*
- *M. Carrillo-Gonzales, AG et al, PRD 2025*

Gravitational lensing: Geometric & Wave optics



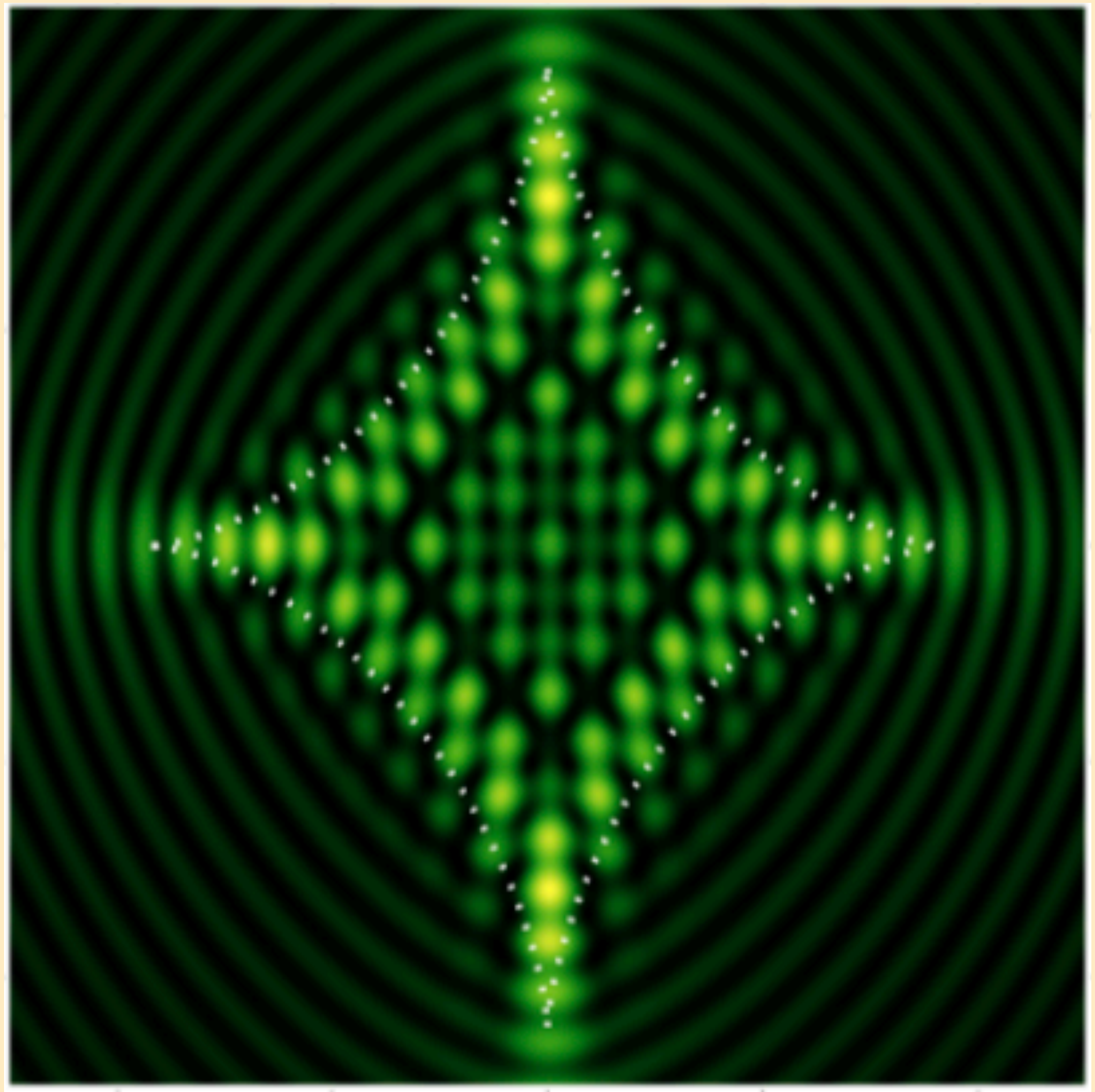
Geometric optics: $\lambda \ll M$

Weak lensing



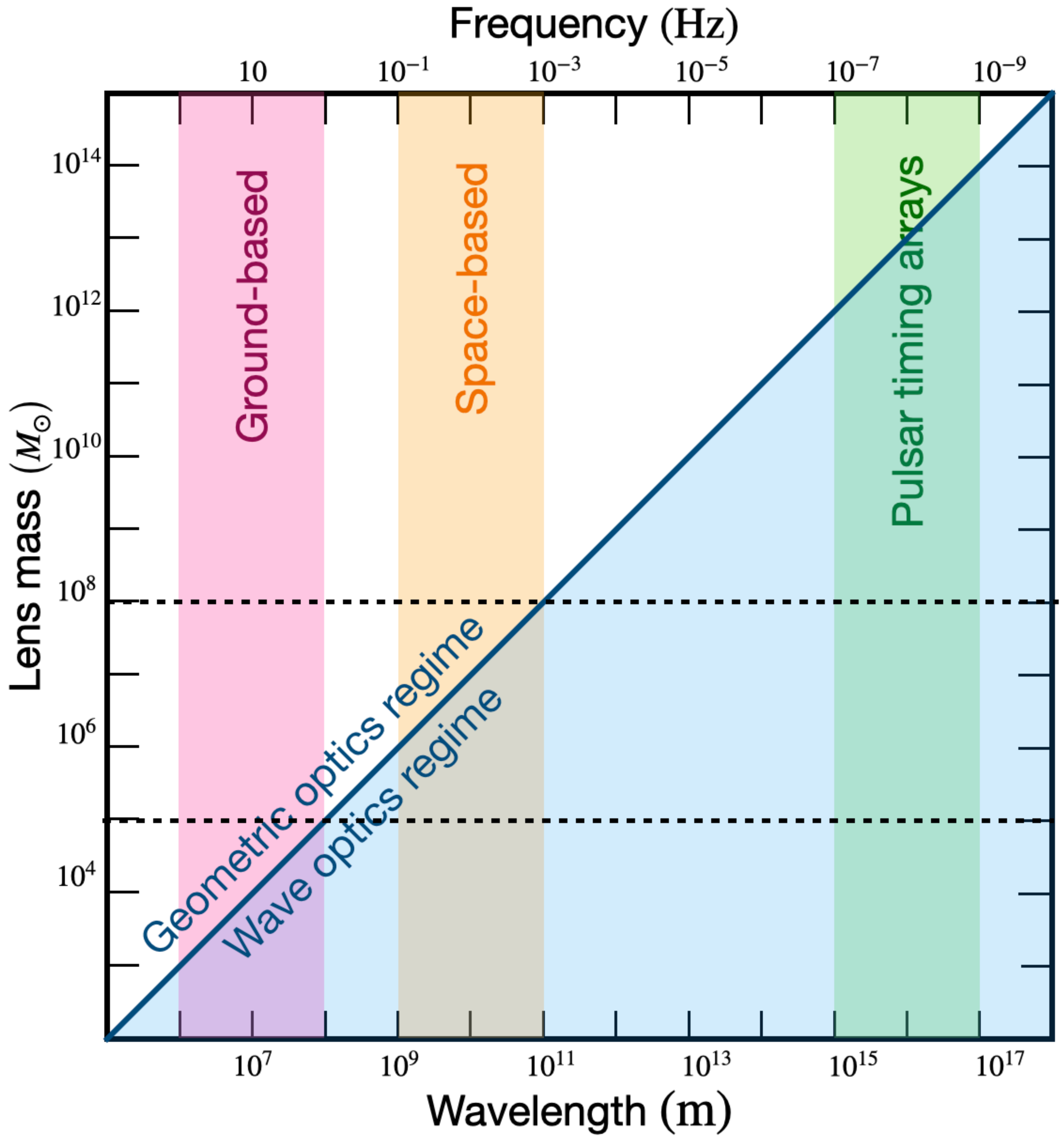
Strong lensing

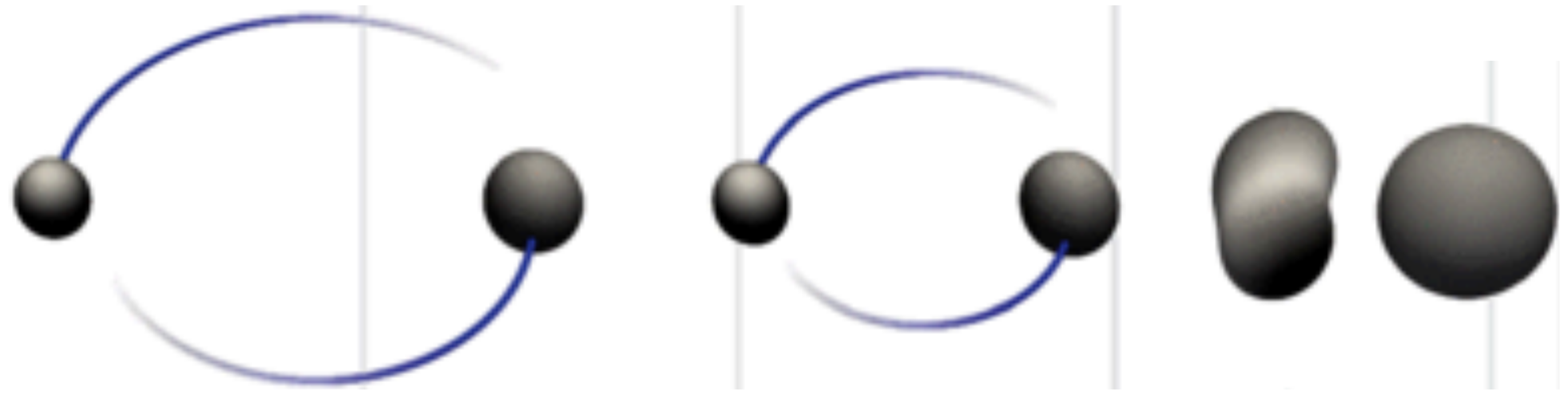
Wave optics: $\lambda \geq M$



Interference

Wave optics for GWs

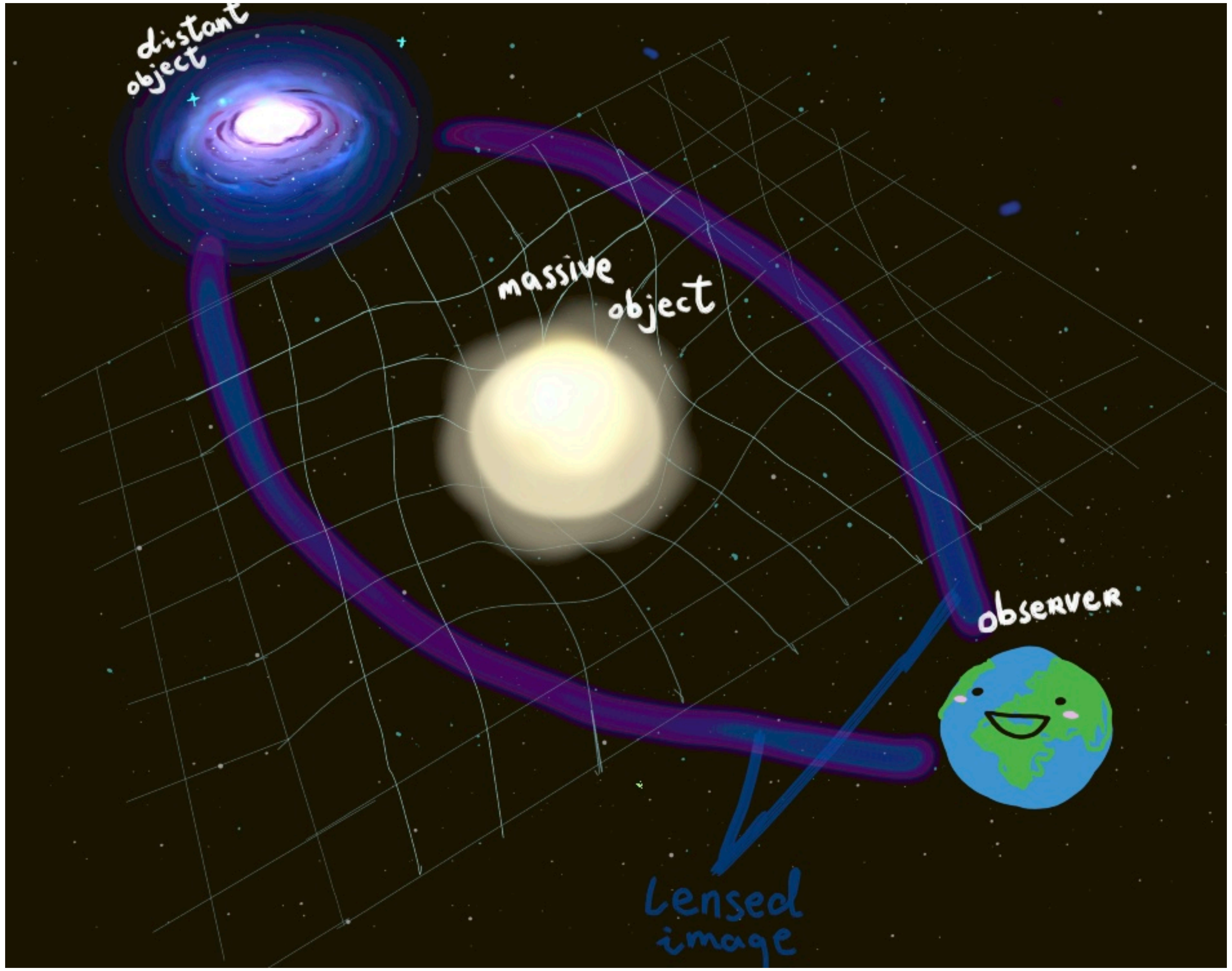




$$M < 10^8 M_\odot \left(\frac{f}{mHz}\right)^{-1}$$

- LVK: GW231123 → strong candidate for WO
  2507.08219, 2512.16916, 2512.17631, 2512.19077
- LISA: ~(0.1 - 1.6)% of MBHB with $(10^5 - 10^{6.5})M_\odot$ and $4 \leq z \leq 10$ 2102.10295

Rules of the game



Goal:

solve wave equation on the lens background

$$\Box h_{\dots} = 0$$

$\Box \rightarrow U(\mathbf{r})$

Talk's plan

1. Diffraction integral formalism for wave-optics effects
2. Scattering amplitudes for including PM effects → 2511.15797
3. Wave-optics in modified gravity → 2605.21584
4. Many lenses with open quantum systems → 2604.15313

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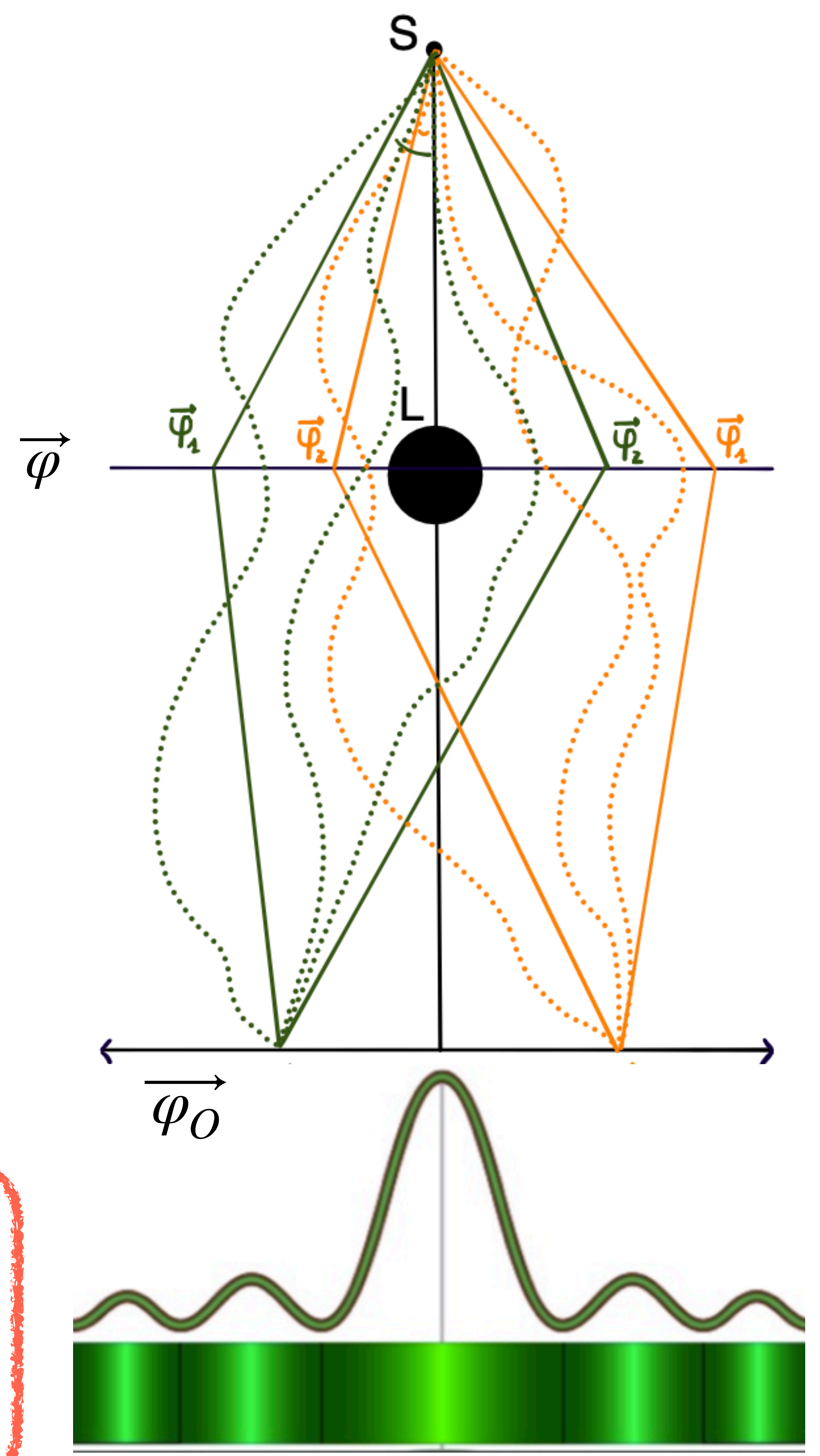
The diffraction integral

Nakamura&Deguchi 1999

- Consider a scalar wave $h = F(\mathbf{x})\ \left(e^{i\omega|\mathbf{x}-\mathbf{x}_s|}/|\mathbf{x}-\mathbf{x}_s|\right)$
- In $\Box h = 0$, assume:
  1. Eikonal: $\omega \gg |\partial_r^2 F|/|\partial_r F|$
  2. Paraxial: $\varphi \ll 1$
  3. Newtonian lens: $U \ll 1$
  4. Scalar fields
- KG equation becomes Schrödinger $i\omega\partial_r F = -\dfrac{\partial_\varphi^2 F}{2r^2} + 2\omega^2 UF$
- Solve with path integral:

$$F_{\rm diff}(r, \vec{\varphi}_O) = \int \mathcal{D}\vec{\varphi}(r) e^{i\omega\int_0^r \mathrm{d}r'\left[\frac{r^2}{2}|\dot{\vec{\varphi}}(r)|^2 - 2U(r,\vec{\varphi}(r))\right]}$$



The diffraction integral

Nakamura&Deguchi 1999

Considering that the lens extension along the optical axis is small (thin lens)

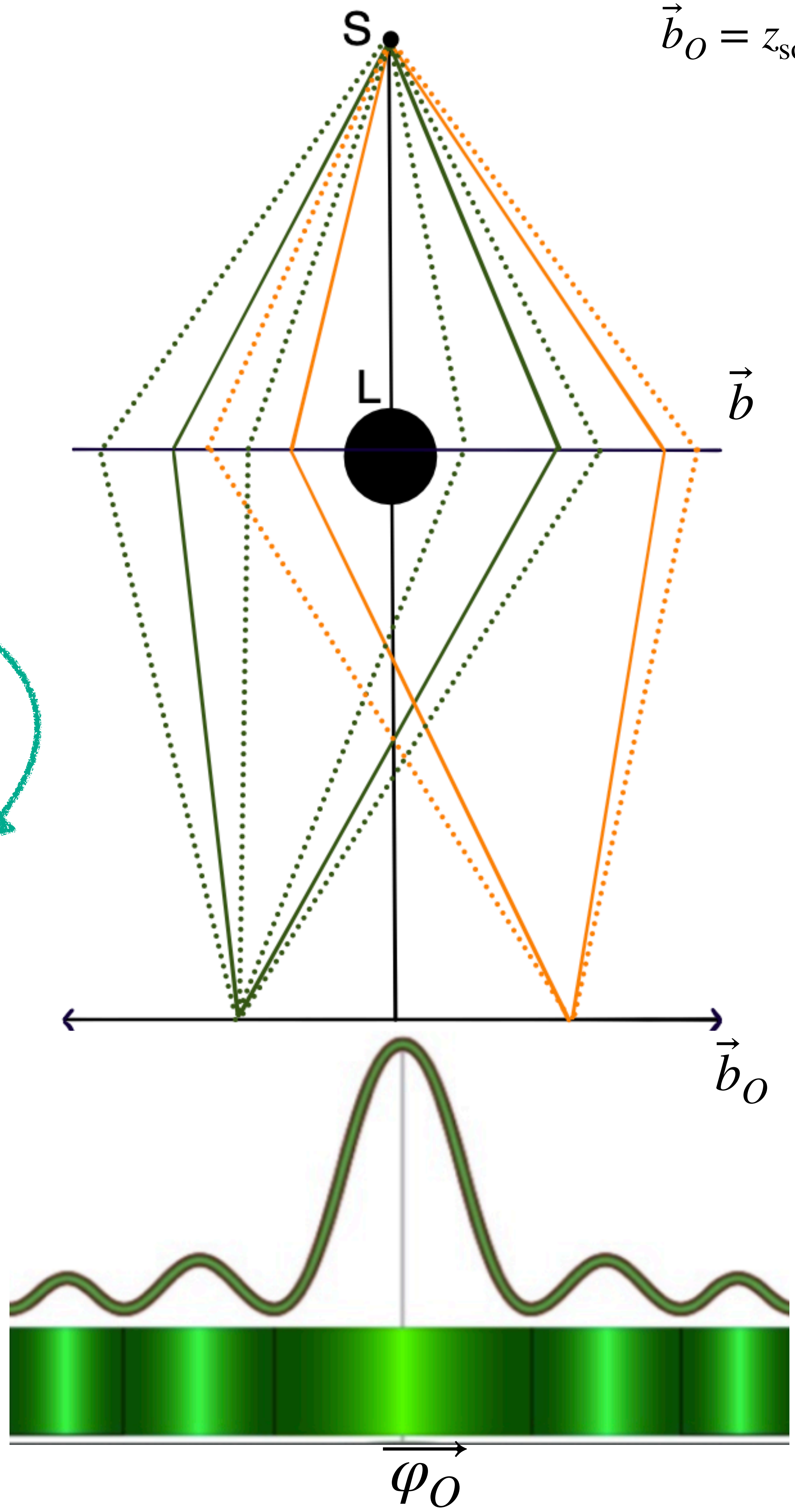
$\vec{b} = z_{\rm sl}\,\vec{\varphi}$

$\vec{b}_O = z_{\rm so}\,\vec{\varphi}_O$

$z_{\rm eff} = z_{\rm sl} z_{\rm lo} / z_{\rm so}$

$$F_{\rm diff}\left(\omega, \vec{b}_O\right) = \frac{\omega z_{\rm eff}}{2\pi i} \int \mathrm{d}^2\vec{b}\; e^{i\,\omega\,\Phi(\vec{b},\vec{b}_O)}$$

Fermat potential: $\Phi(\vec{b}, \vec{b}_O) = \frac{z_{\rm eff}}{2}\left(\vec{b} - \frac{z_{\rm sl}}{z_{\rm so}}\vec{b}_O\right)^2 - 2\int \mathrm{d}z\; U(z, \vec{b})$



Defining adimensional frequency parameter: $\nu = \omega L_{\rm lens}$

- If $\nu \gg 1$: sum of the saddle point $\nabla_{\vec{b}}\Phi(\vec{b}, \vec{b}_O) = 0$
- If $\nu \lesssim 1$: interference among all possible paths

Talk's plan

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2. Scattering amplitudes for including PM effects → 2511.15797
3. Wave-optics in modified gravity → 2605.21584
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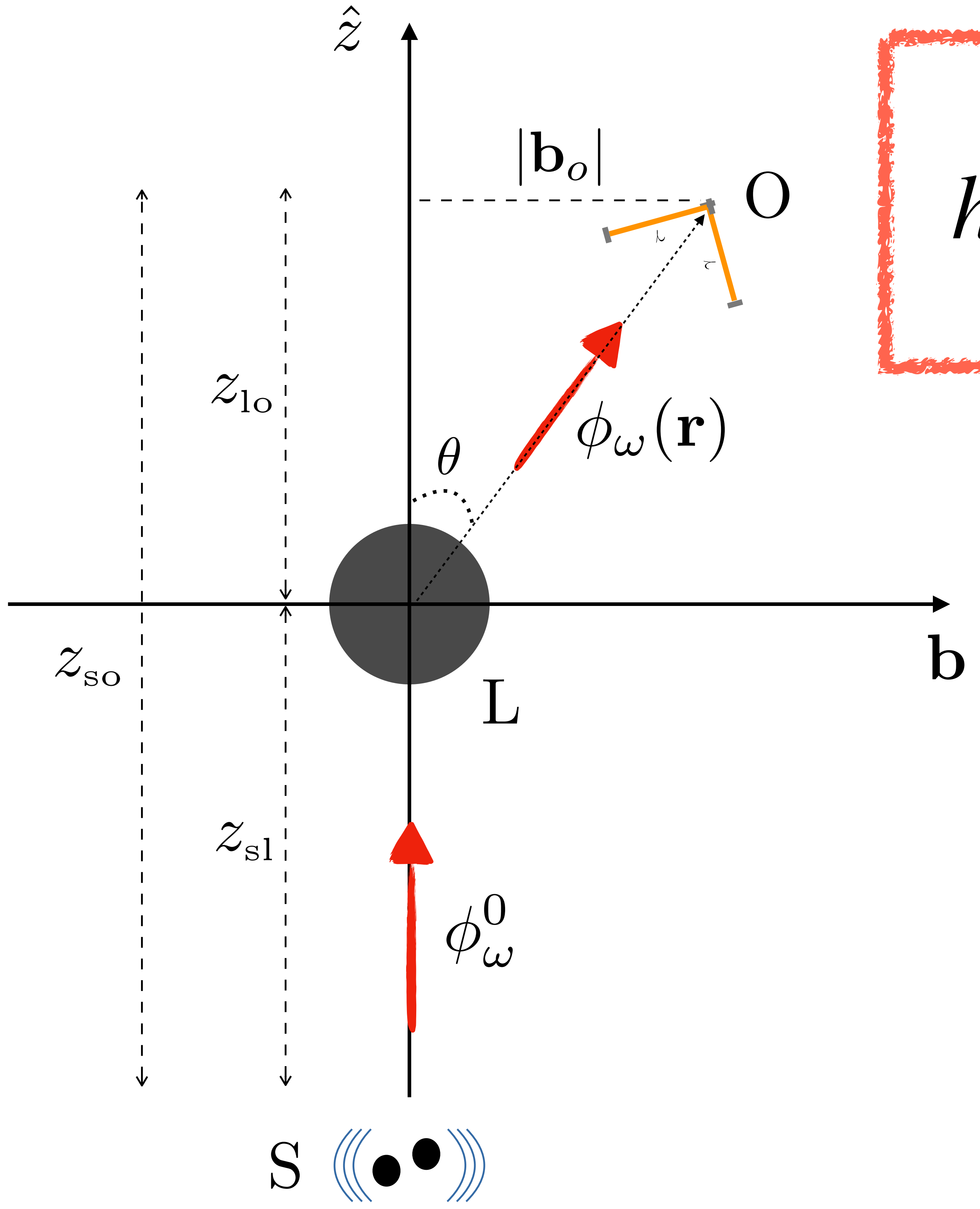
Scattering amplitudes

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797



$$h_{\rm lens.} = F(r_O, \theta_O)\,\frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|}$$

$$h_{\rm scatt.} = e^{i\omega z} + f(\theta_O)\,\frac{e^{i\omega r}}{r}$$

Scattering amplitudes: the Fresnel factor

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
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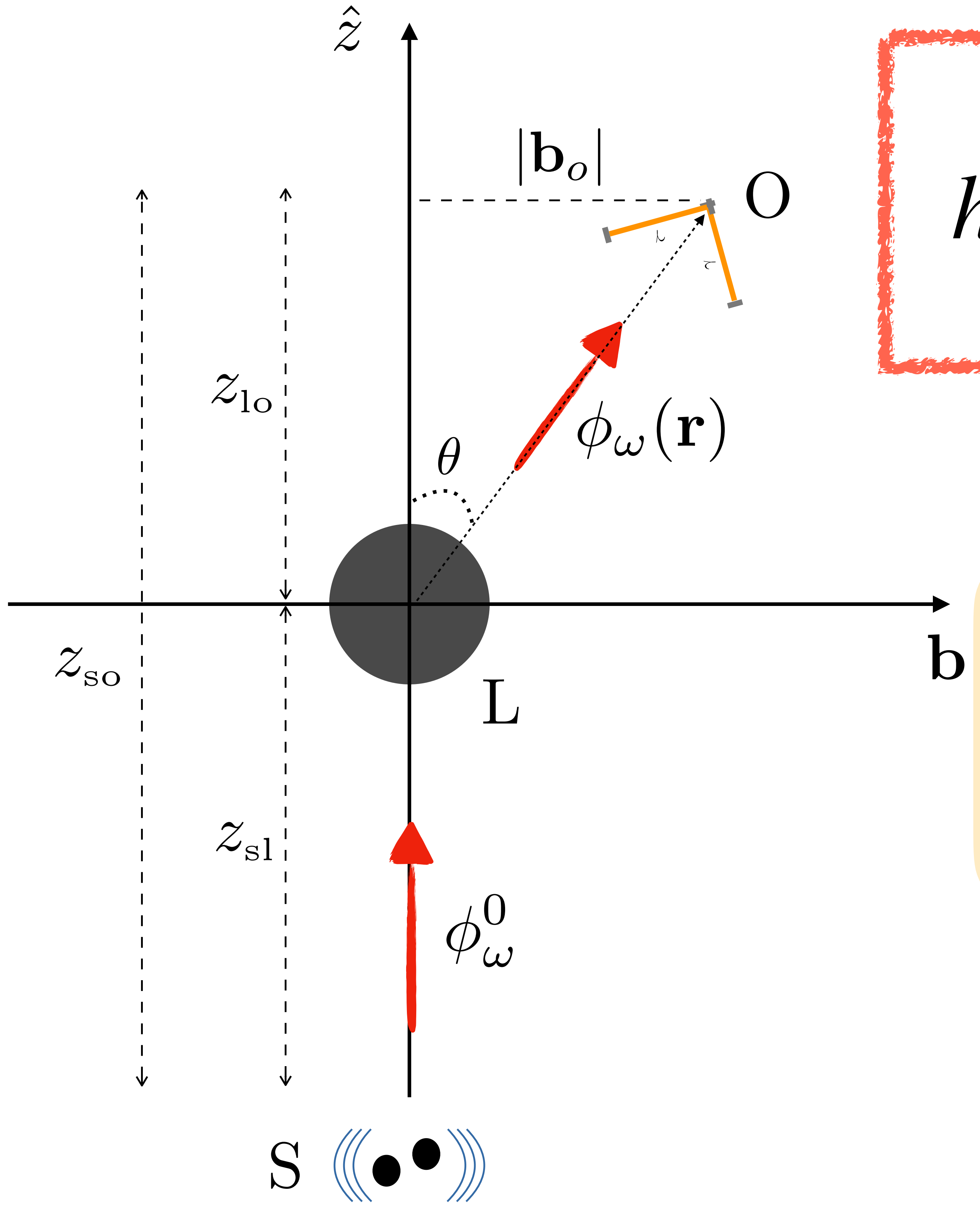


$$h_{\rm lens.} = F(r_O, \theta_O) \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|}$$

$$h_{\rm scatt.} = \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|} + f(r_O, \theta_O) \frac{e^{i\omega r}}{r}$$

Doing a partial wave expansion of $\Box h = 0$:

$$f(r_O, \theta_O) \frac{e^{i\omega r}}{r} = i\omega \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} (-1)^\ell \underbrace{\left(e^{i\delta_\ell} - 1\right)}_{\text{Phase shifts}} \left[\underbrace{h_\ell^{(1)}(\omega r_O) h_\ell^{(1)}(\omega r_s)}_{\text{Expand for large } \omega r \text{ including Fresnel factor}}\right] P_\ell(\cos\theta_O)$$

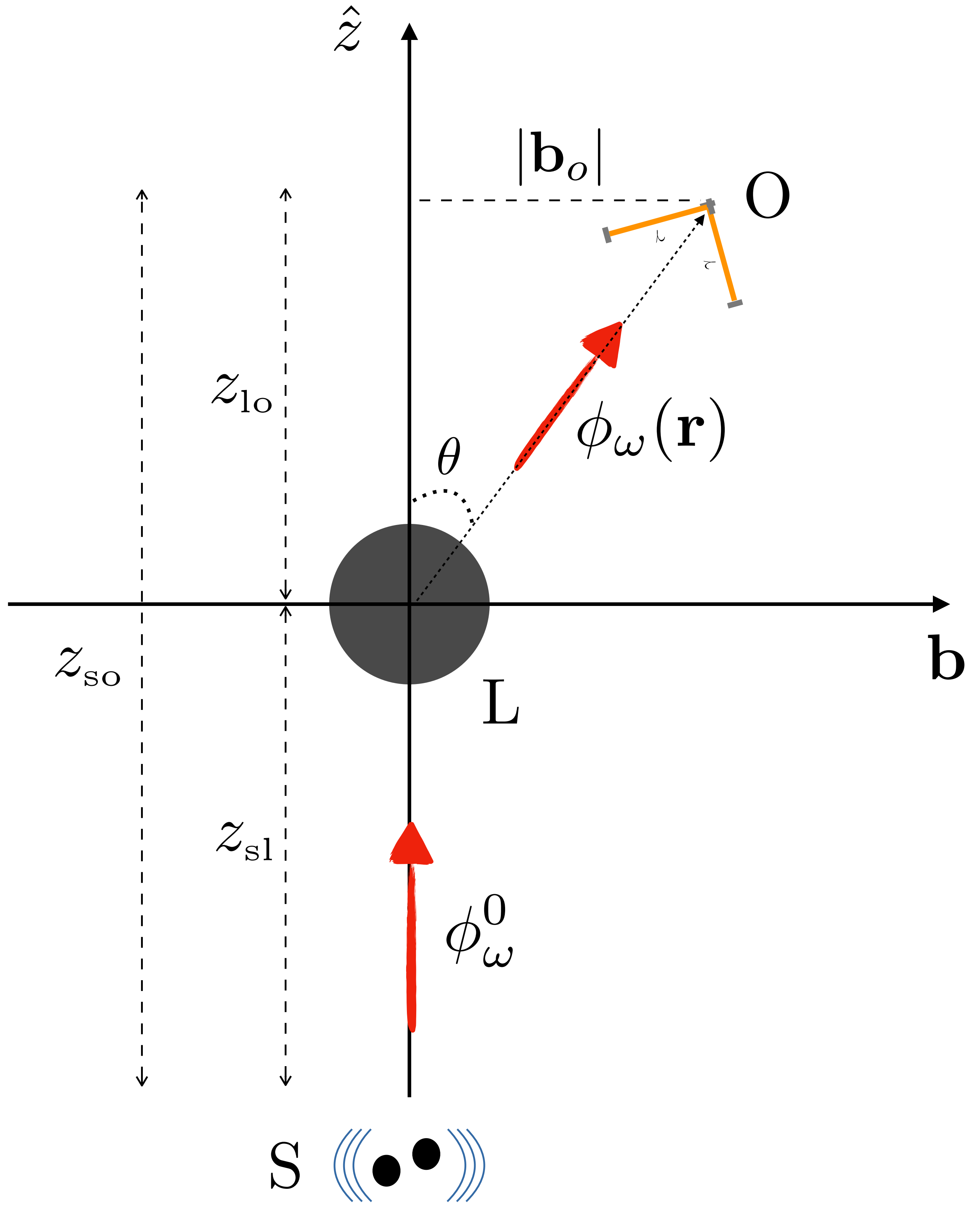
Phase shifts

Expand for large ωr including Fresnel factor

$$h_\ell^{(1)}(\omega r) h_\ell^{(1)}(\omega r_s) \approx (-1)^{\ell+1} \frac{e^{i\omega r}}{\omega r} \frac{e^{i\omega r_s}}{\omega r_s} e^{\frac{i\ell(\ell+1)}{2\omega}\left(\frac{1}{r}+\frac{1}{r_s}\right)}$$

Scattering amplitudes: the eikonal limit

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797



$$f(r_O,\theta_O) = \frac{e^{-\frac{i\omega z_{\rm eff}\vec{b}_O^2}{2z_{\rm lo}^2}}}{z_{\rm eff}} \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2i\omega}\left(e^{i\delta_\ell}-1\right) e^{\frac{i(\ell+1/2)^2}{2\omega z_{\rm eff}}} P_\ell(\cos\theta_O)$$

$\ell = \omega|\mathbf{b}| \gg 1$

Equivalent to $\omega \gg |\partial_r^2 F|/|\partial_r F|$

$$f^{\rm eik}(r_O,\theta_O) = \frac{e^{-\frac{i\omega z_{\rm eff}\vec{b}_O^2}{2z_{\rm lo}^2}}}{z_{\rm eff}} \left[\frac{\omega z_{\rm eff}}{2\pi i}\int d^2\vec{b}\, e^{\frac{z_{\rm eff}}{2}\left(\vec{b}-\frac{z_{\rm sl}}{z_{\rm so}}\vec{b}_O\right)^2}\left(e^{i\delta^{\rm eik}}-1\right)\right]$$

$$F_{\rm diff}\left(\omega,\vec{b}_O\right) = \frac{\omega z_{\rm eff}}{2\pi i}\int d^2\vec{b}\, e^{i\omega\Phi(\vec{b},\vec{b}_O)}$$

At $\mathcal{O}(U)$ one has $\delta^{\rm eik} = -2\omega\int dz\, U$

Scattering amplitudes: matching f & F

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797

Diffraction integral	$F_{\text{diff}}\left(\omega,\vec{b}_O\right)=\frac{\omega z_{\text{eff}}}{2\pi i}\int \mathrm{d}^2\vec{b}\; e^{\frac{z_{\text{eff}}}{2}\left(\vec{b}-\frac{z_{\text{sl}}}{z_{\text{so}}}\vec{b}_O\right)^2-2\int dz U}$
Scattering amplitude	$f^{\text{eik}}(r_O,\vec{b}_O)=\frac{e^{-\frac{i\omega z_{\text{eff}}\vec{b}_O^2}{2z_{\text{lo}}^2}}}{z_{\text{eff}}}\left[\frac{\omega z_{\text{eff}}}{2\pi i}\int \mathrm{d}^2\vec{b}\; e^{\frac{z_{\text{eff}}}{2}\left(\vec{b}-\frac{z_{\text{sl}}}{z_{\text{so}}}\vec{b}_O\right)^2-2\int dz U}-1\right]$

Scattering amplitudes: matching f & F

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797

Diffraction integral	$F_{\rm diff}\left(\omega,\vec{b}_O\right) = z_{\rm eff}\, e^{\frac{i\omega z_{\rm eff}\vec{b}_O^2}{2z_{\rm lo}^2}} f^{\rm eik}(r_O,\vec{b}_O) + 1$
Scattering amplitude	$f^{\rm eik}(r_O,\vec{b}_O) = \frac{e^{-\frac{i\omega z_{\rm eff}\vec{b}_O^2}{2z_{\rm lo}^2}}}{z_{\rm eff}}\left[\frac{\omega z_{\rm eff}}{2\pi i}\int \mathrm{d}^2\vec{b}\, e^{\frac{z_{\rm eff}}{2}\left(\vec{b}-\frac{z_{\rm sl}}{z_{\rm so}}\vec{b}_O\right)^2 - 2\int dz U} - 1\right]$

This came from $\delta^{\rm eik}$ at $\mathcal{O}(U)$: from scattering techniques and BHPT we can improve the computation of the phase shift

Scattering amplitudes: the phase shift

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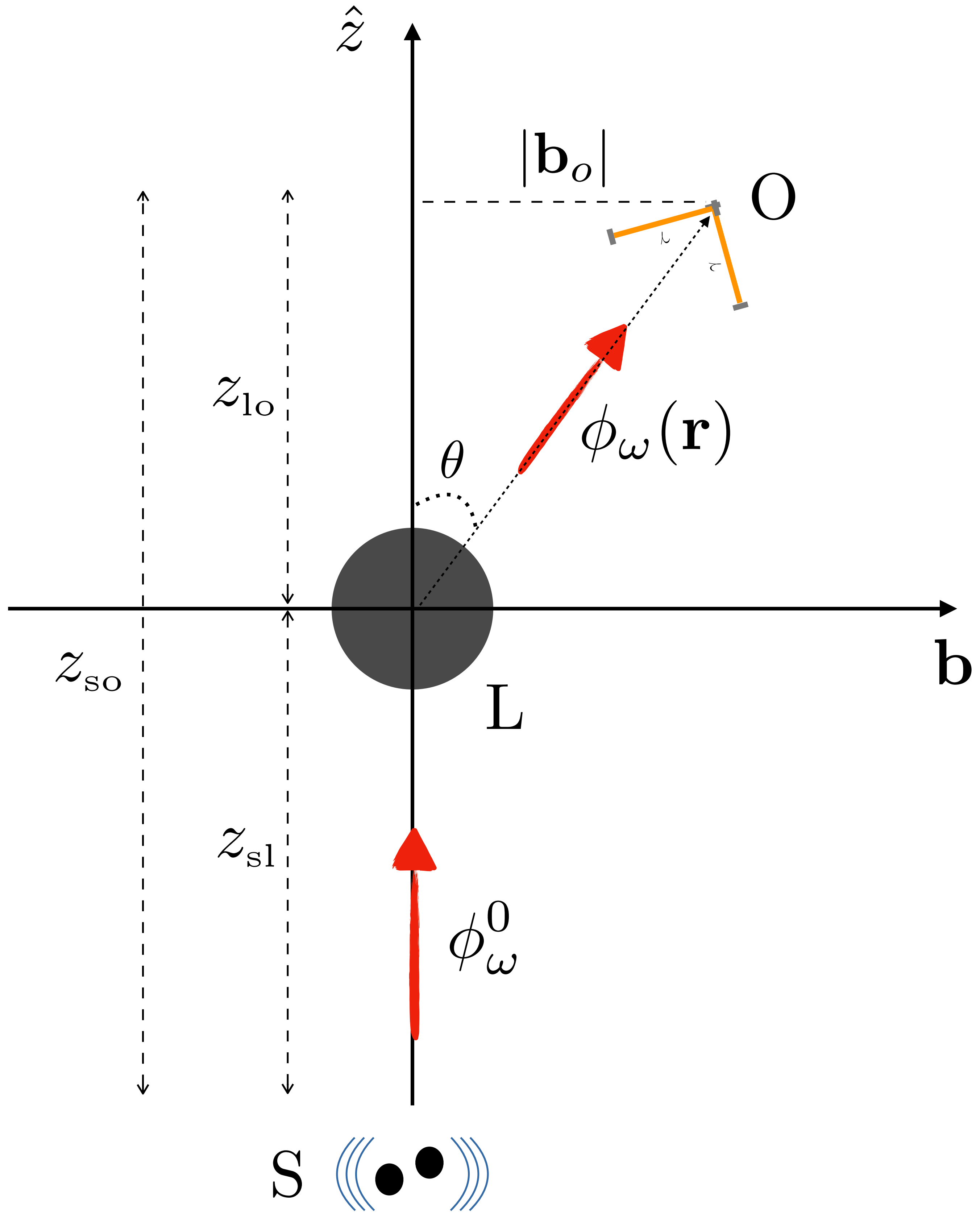


For a Schwarzschild black hole $\delta^{\rm eik}$ can be computed either via BHPT or resuming Feynman diagrams in the eikonal limit. Two small parameters:

- PM: $GM/|\mathbf{b}|$
- Eikonal ($\ell \gg 1$): $1/(\omega|\mathbf{b}|)$

2401.08752

$$\delta^{\rm eik} = \nu\left(-\log|\mathbf{b}| + \frac{15\pi}{16}\frac{GM}{|\mathbf{b}|} + \frac{(12s^2-1)}{24}\frac{1}{(\omega|\mathbf{b}|)^2}\right)$$

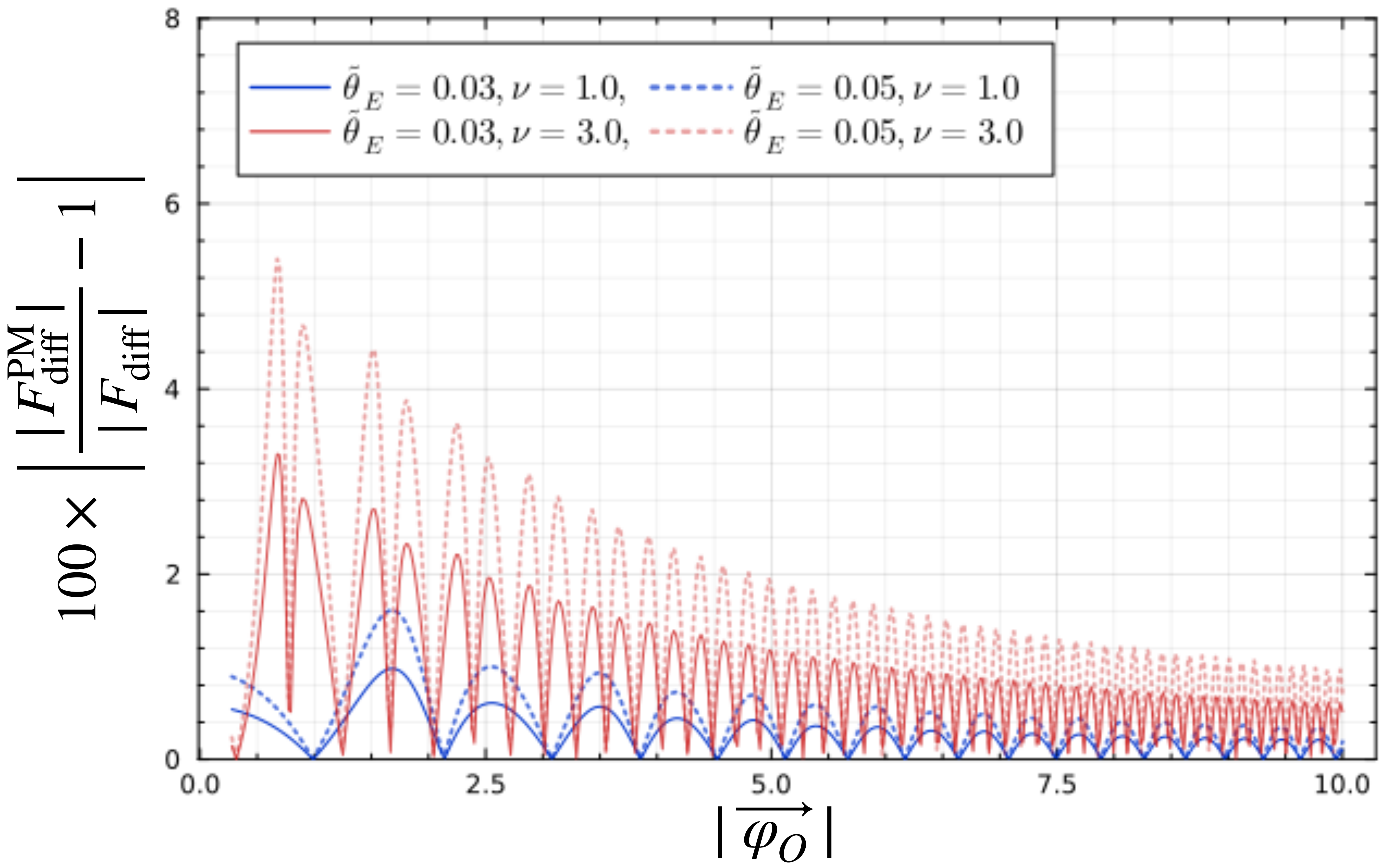
$\nu = 4MG\omega$

- First order BE vanishes for non-spinning lenses
- BE corrections depend on spin
- Use the matching $F_{\rm diff} = z_{\rm eff}\, e^{\frac{i\omega z_{\rm eff}\vec{b}_O^2}{2z_{\rm lo}^2}} f^{\rm eik} + 1$ to improve $F_{\rm diff}$

Post Minkowskian corrections: behavior in $|\vec{\varphi}_O|$

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797

$$F_{\rm diff}^{\rm PM}(\nu, \vec{\varphi}_O) = \frac{\nu}{2\pi i}\int d^2\vec{\varphi}\, e^{i\nu\left(\frac{1}{2}(\vec{\varphi}-\vec{\varphi}_O)^2 - \log|\vec{\varphi}| + \frac{15\pi}{64}\frac{\tilde{\theta}_E}{|\vec{\varphi}|}\right)} \qquad \tilde{\theta}_E = \sqrt{\frac{4MGz_{\rm so}}{z_{\rm sl}z_{\rm lo}}}$$



	$M(M_\odot)$	$z_{\rm so}$(pc)	$z_{\rm sl}$(pc)	$\frac{15\pi}{64}\tilde{\theta}_E$
Supermassive BH lens	10^8	10^8	10^8	10^{-7}
Hierarchical triples	$10^6 - 10^8$	10^8	$10^{-3} - 1$	$10^{-2} - 10^{-1}$

2407.00473, 2506.22519

Post Minkowskian corrections: behavior in $|\vec{\varphi}_O|$

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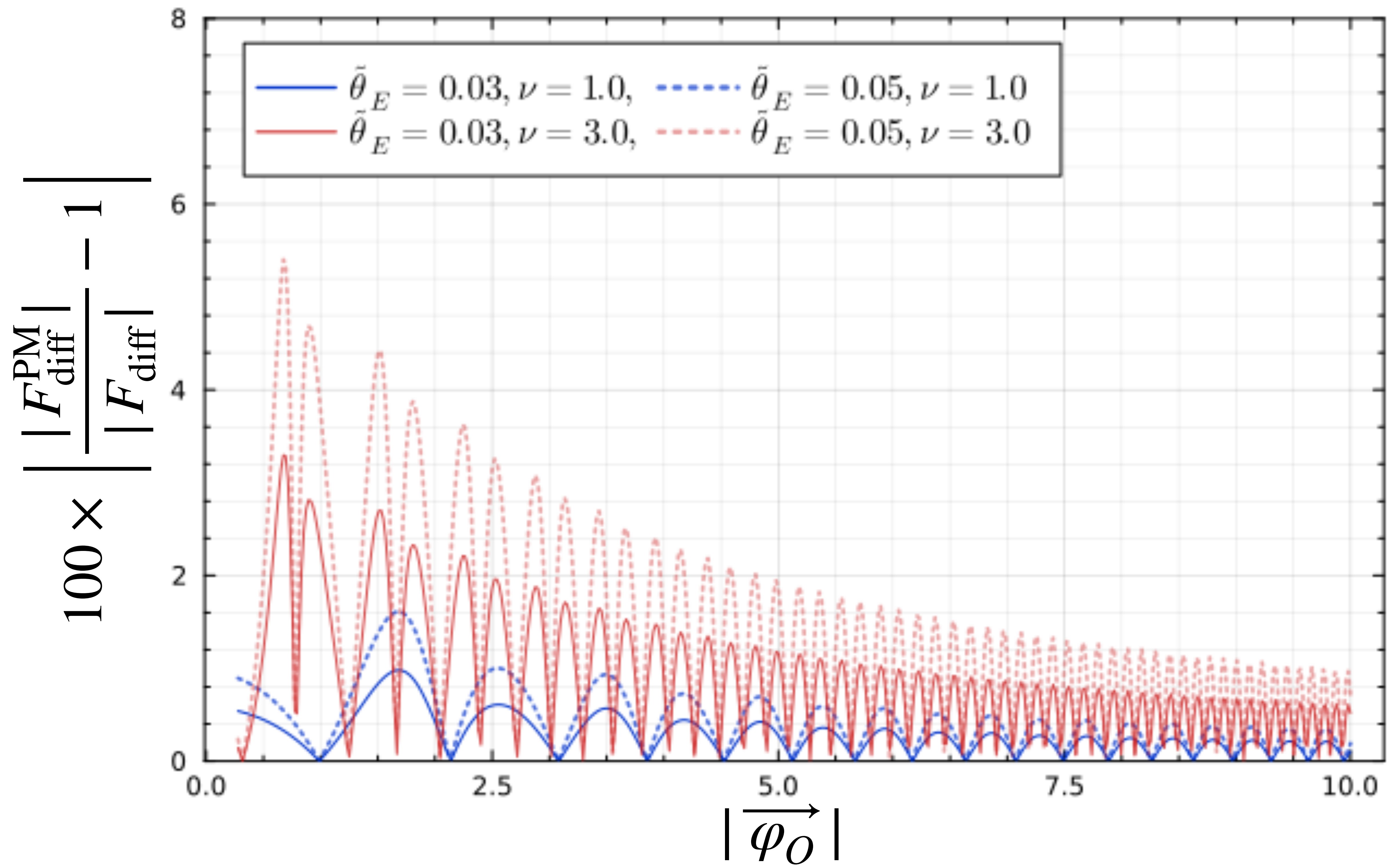
$$F^{\rm PM}_{\rm diff}(\nu, \vec{\varphi}_O) = \frac{\nu}{2\pi i}\int d^2\vec{\varphi}\; e^{i\nu\left(\frac{1}{2}(\vec{\varphi}-\vec{\varphi}_O)^2 - \log|\vec{\varphi}| + \frac{15\pi}{64}\frac{\tilde{\theta}_E}{|\vec{\varphi}|}\right)} \qquad \tilde{\theta}_E = \sqrt{\frac{4MGz_{\rm so}}{z_{\rm sl}z_{\rm lo}}}$$

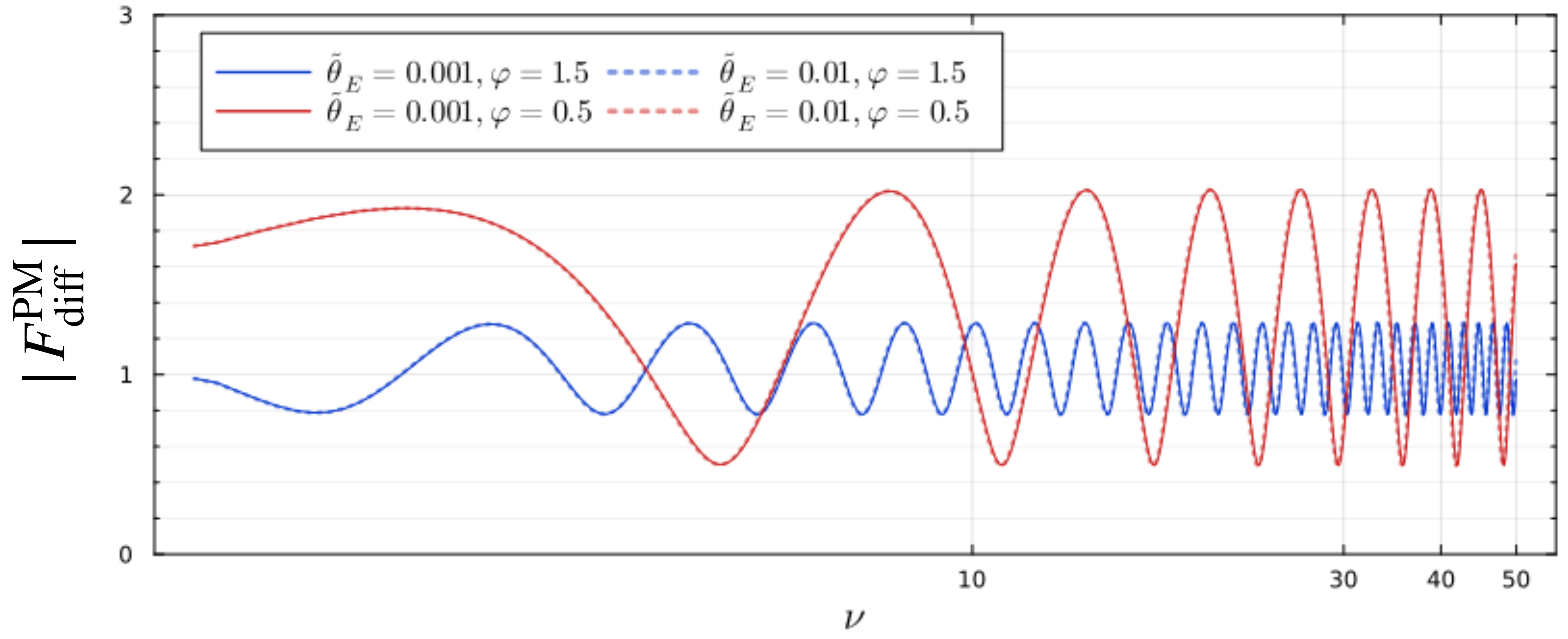


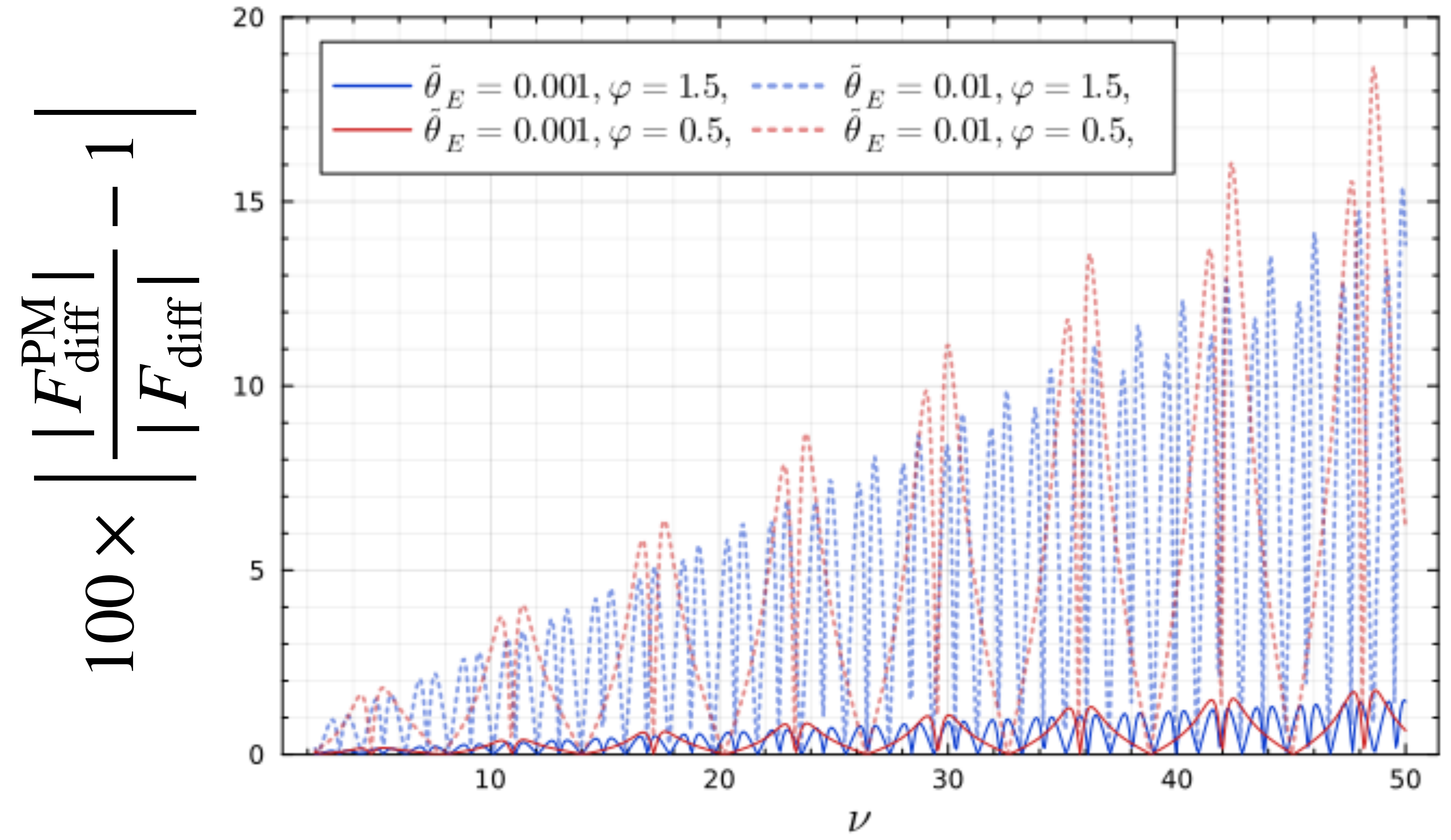
<u>*Decaying trend:*</u>

- For large $|\vec{\varphi}_O|$ paths passing close to the lens, accumulate large phase shifts.
- Dominant contribution comes from $|\vec{\varphi}| \sim |\vec{\varphi}_O|$, where PM corrections are small.

Post Minkowskian corrections: behavior in ν

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797





Nature of the oscillations:

- Interference btw Newtonian images: $\nabla_{\vec{\varphi}} \Phi^{\rm Newt}(\vec{\varphi}, \vec{\varphi}_O) = 0 \rightarrow \{\vec{\varphi}_1, \vec{\varphi}_2\}$
- Frequency: $\Delta\Phi^{\rm Newt} = \Phi^{\rm Newt}(\vec{\varphi}_1) - \Phi^{\rm Newt}(\vec{\varphi}_2)$.
- PM corrections: $\Delta\Phi^{\rm Newt} \rightarrow \Delta\Phi^{\rm Newt} + \Delta\psi^{\rm 2PM}$, misaligning the interference fringes wrt Newtonian case
- $\Delta\psi^{\rm 2PM} \propto \nu$, thus the phase difference increases (until it reaches 2π)

Post Minkowskian corrections: behavior in ν

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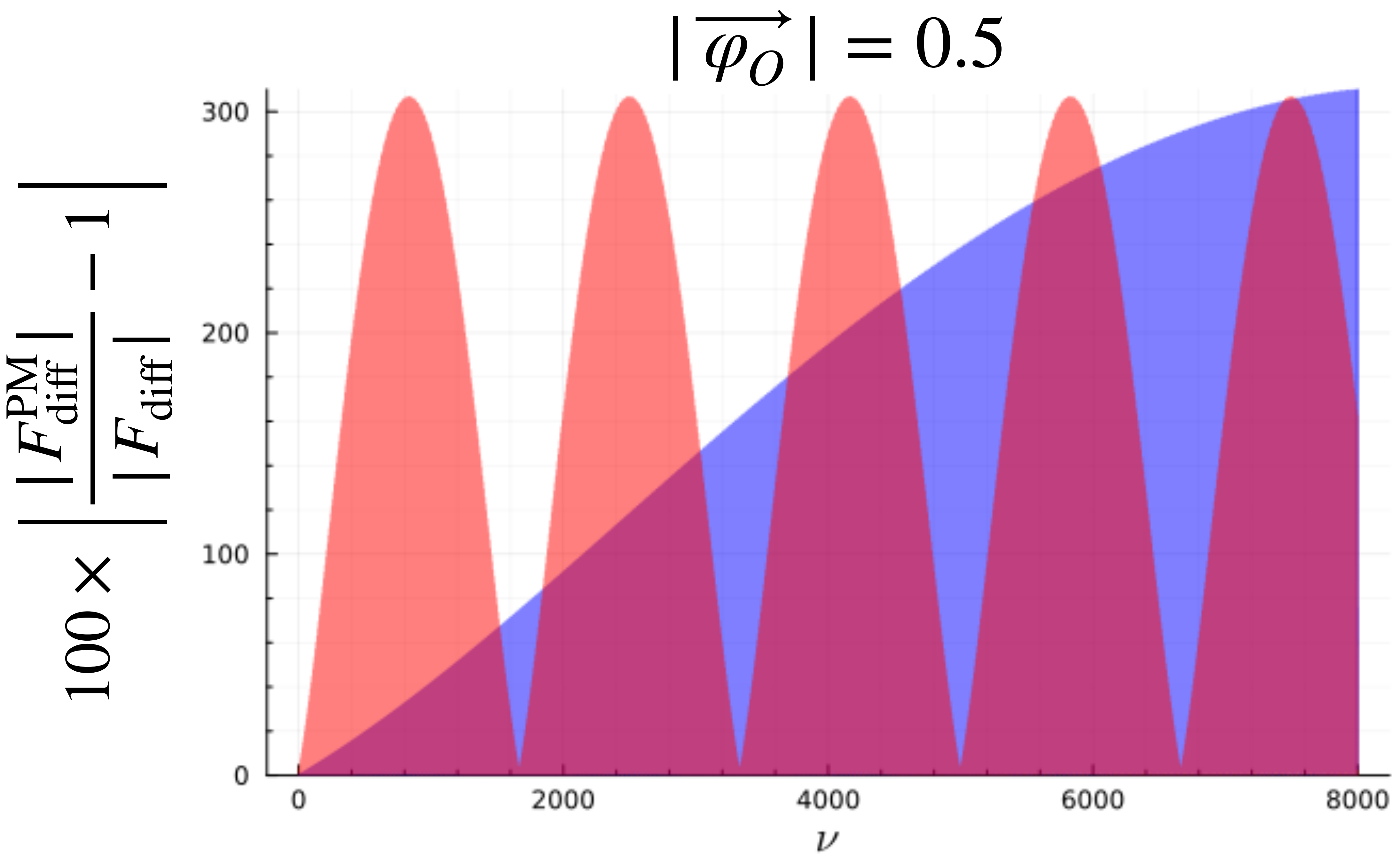

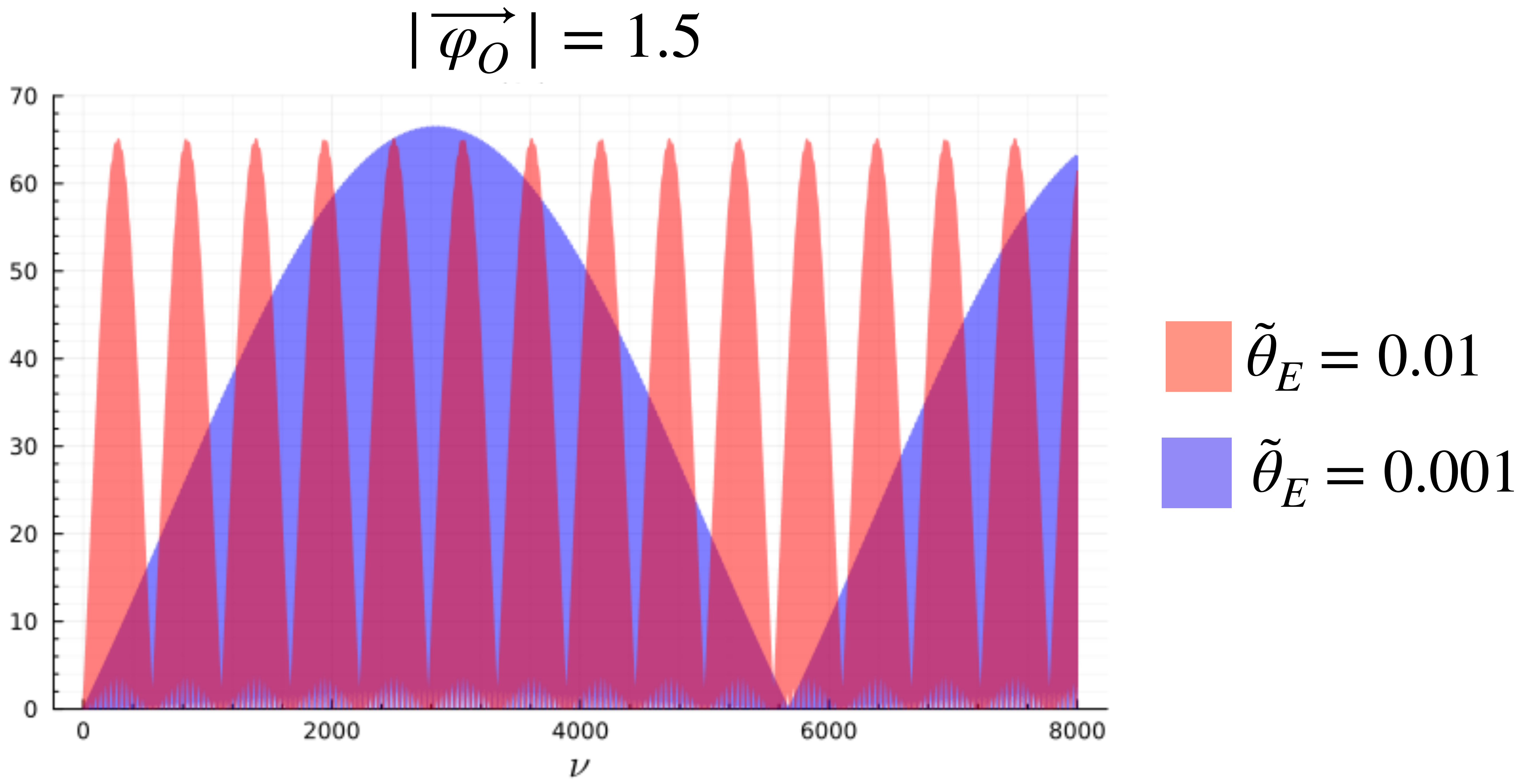

$$\Delta\psi^{2PM} = \frac{15\pi}{64}\left(\frac{1}{|\vec{\varphi_2}|} - \frac{1}{|\vec{\varphi_1}|}\right)\tilde{\theta}_E$$

$$T_{long} = \frac{2\pi}{\Delta\psi^{2PM}}$$

Observer position	$|\vec{\varphi}_O| = 0.5$	$|\vec{\varphi}_O| = 1.5$
Image position	$|\vec{\varphi}_1| = 1.28$ $|\vec{\varphi}_2| = 0.78$	$|\vec{\varphi}_1| = 2.0$ $|\vec{\varphi}_2| = 0.5$
Period	$T_{long} = 17\tilde{\theta}_E^{-1}$	$T_{long} = 5.7\tilde{\theta}_E^{-1}$

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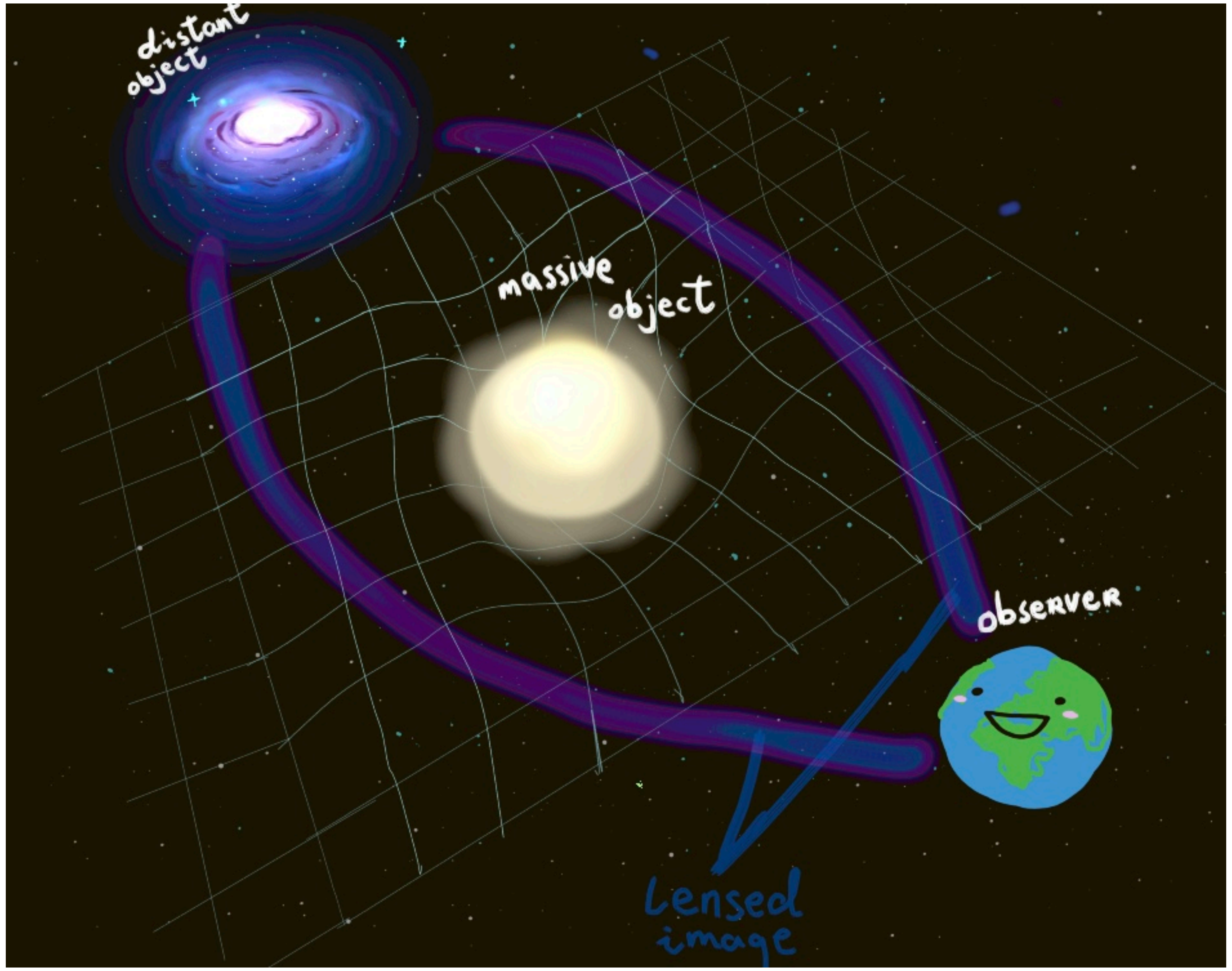
4. Many lenses with disorder systems → 2604.15313

Wave optics with non-minimally coupled scalar field

With: AG, G. Tasinato.
arXiv: 2605.21584



Goal:

solve modified wave equation on the lens background

$$\left(\Box - \xi R\right) h = 0$$

$$U(\mathbf{r})$$

These corrections are suppressed in GO: $\xi R/\omega^2 \sim \xi/(\omega L_{\rm lens})^2 \sim \xi/\nu^2$

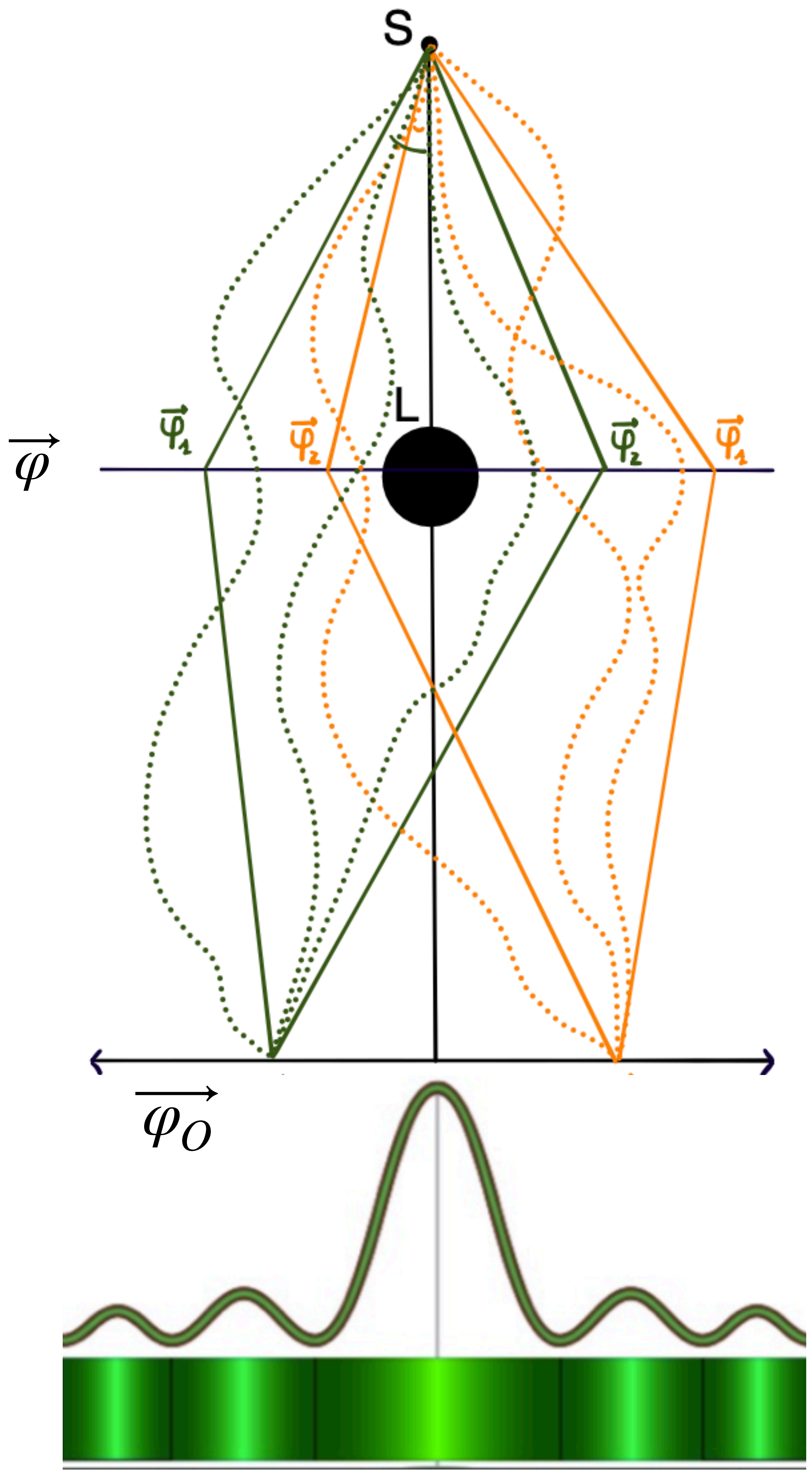
The diffraction integral in MG

With: AG, G. Tasinato
arXiv: 2605.21584

- Consider a scalar wave $h = F^{\rm MG}(\mathbf{x})\ \left(e^{i\omega|\mathbf{x}-\mathbf{x}_s|}/|\mathbf{x}-\mathbf{x}_s|\right)$
- In $\left(\Box - \xi R\right)\, h = 0$, assume:
  1. Eikonal: $\omega \gg |\partial_r^2 F^{\rm MG}|/|\partial_r F^{\rm MG}|$
  2. Paraxial: $\varphi \ll 1$
  3. Newtonian lens: $U \ll 1$
  4. Scalar fields

$$U_{\rm eff} = U + \frac{\xi \Delta U}{2\omega^2}$$

- KG equation becomes Schrödinger $i\omega\partial_r F^{\rm MG} = -\frac{\partial_\varphi^2 F^{\rm MG}}{2r^2} + 2\omega^2 U_{\rm eff} F^{\rm MG}$



Modified diffraction integral: Singular isothermal sphere

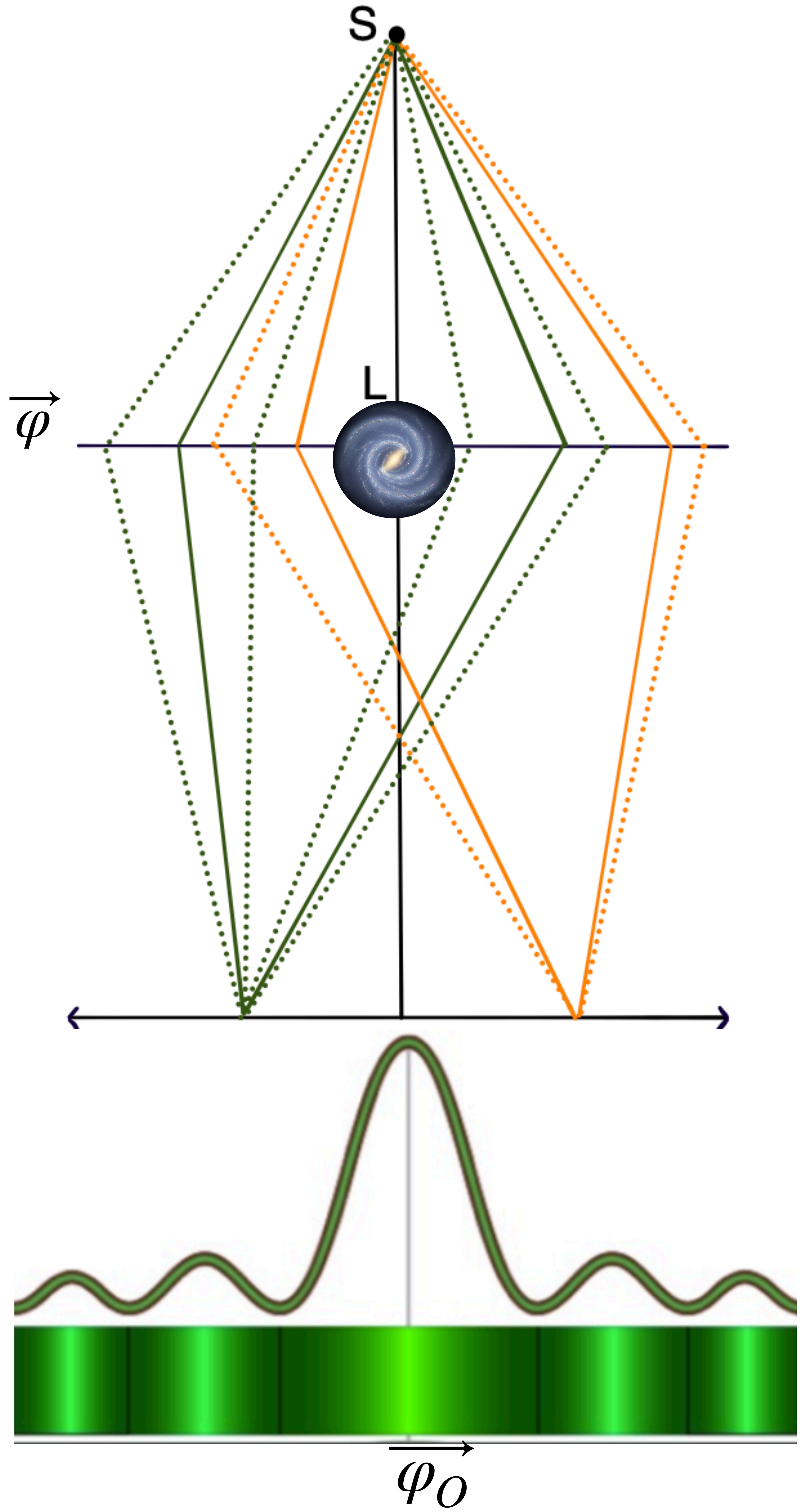
With: AG, G. Tasinato
arXiv: 2605.21584

- Consider a SIS galaxy lens: $\begin{cases} U &= 2\sigma_v^2 \log(r/r_0) \\ R &= 2\sigma_v^2/r^2 \end{cases}$

- dimensionless coordinates: $\begin{cases} \vec{x} &= \vec{\varphi}/\theta_E \\ \vec{y} &= \vec{\varphi}_O/\theta_E \end{cases}$

- Define key parameters: $\begin{cases} \nu &= 16\pi^2\sigma_v^4 z_{\rm eff}\,\omega \\ \epsilon &= 8\pi^2\sigma_v^4 \xi \end{cases}$

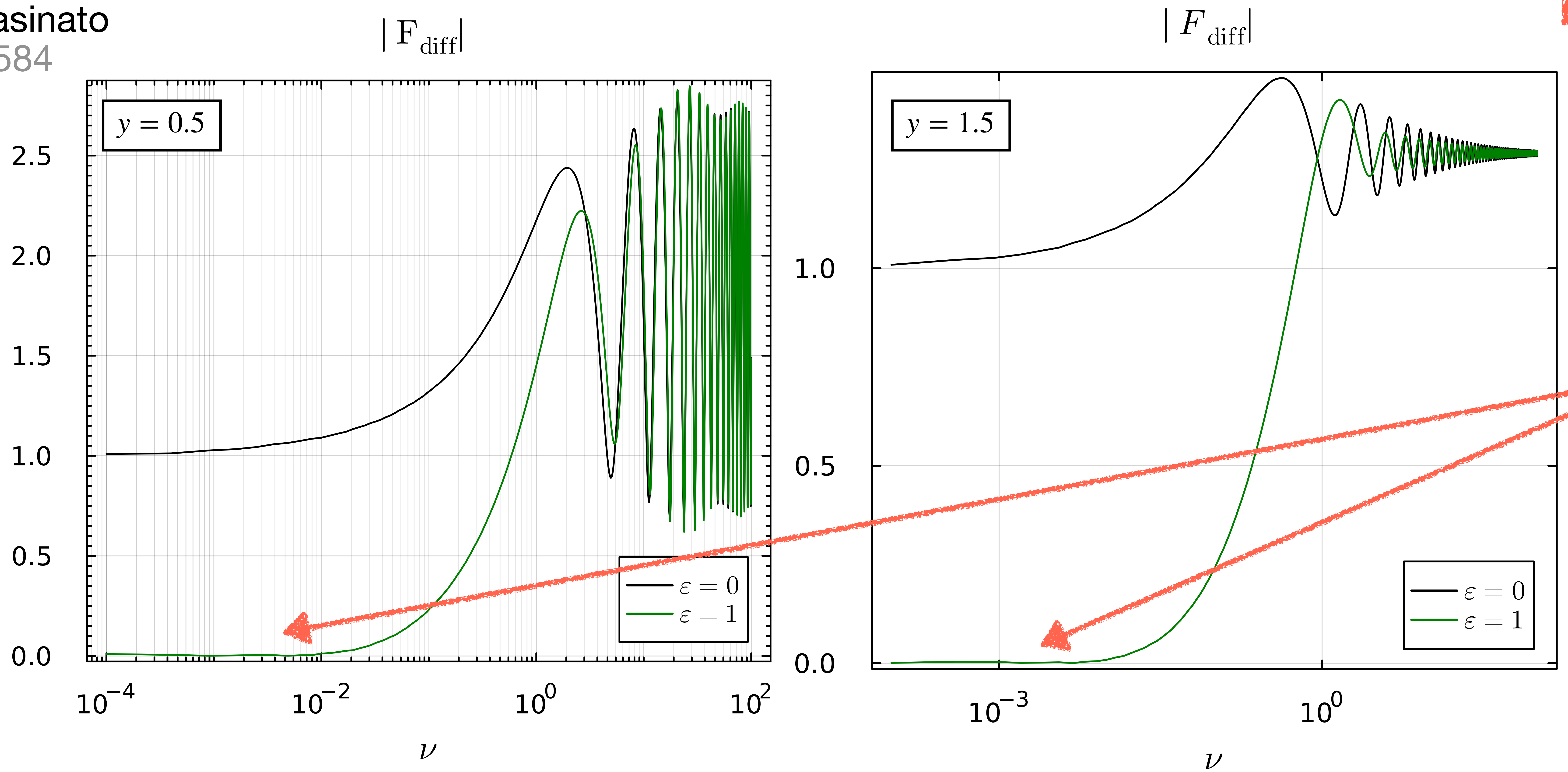
$$F_{\rm diff}^{\rm MG} = \frac{\nu}{2\pi i}\int {\rm d}^2\vec{x}\,\exp\left[i\nu\left(\frac{1}{2}|\vec{x}-\vec{y}|^2 - x - \frac{\epsilon}{x\nu^2}\right)\right]$$



Issue with IR behavior

With: AG, G. Tasinato
arXiv: 2605.21584

$$h = F_{\rm diff}^{\rm MG} \times \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|}$$



$F_{\rm diff}^{\rm MG} \to 0$, no wave?

Multiplying the Schrödinger equation by $F_{\rm diff}^{\rm MG\,*}$, and integrating over a sphere, one finds for $\omega \to 0$:

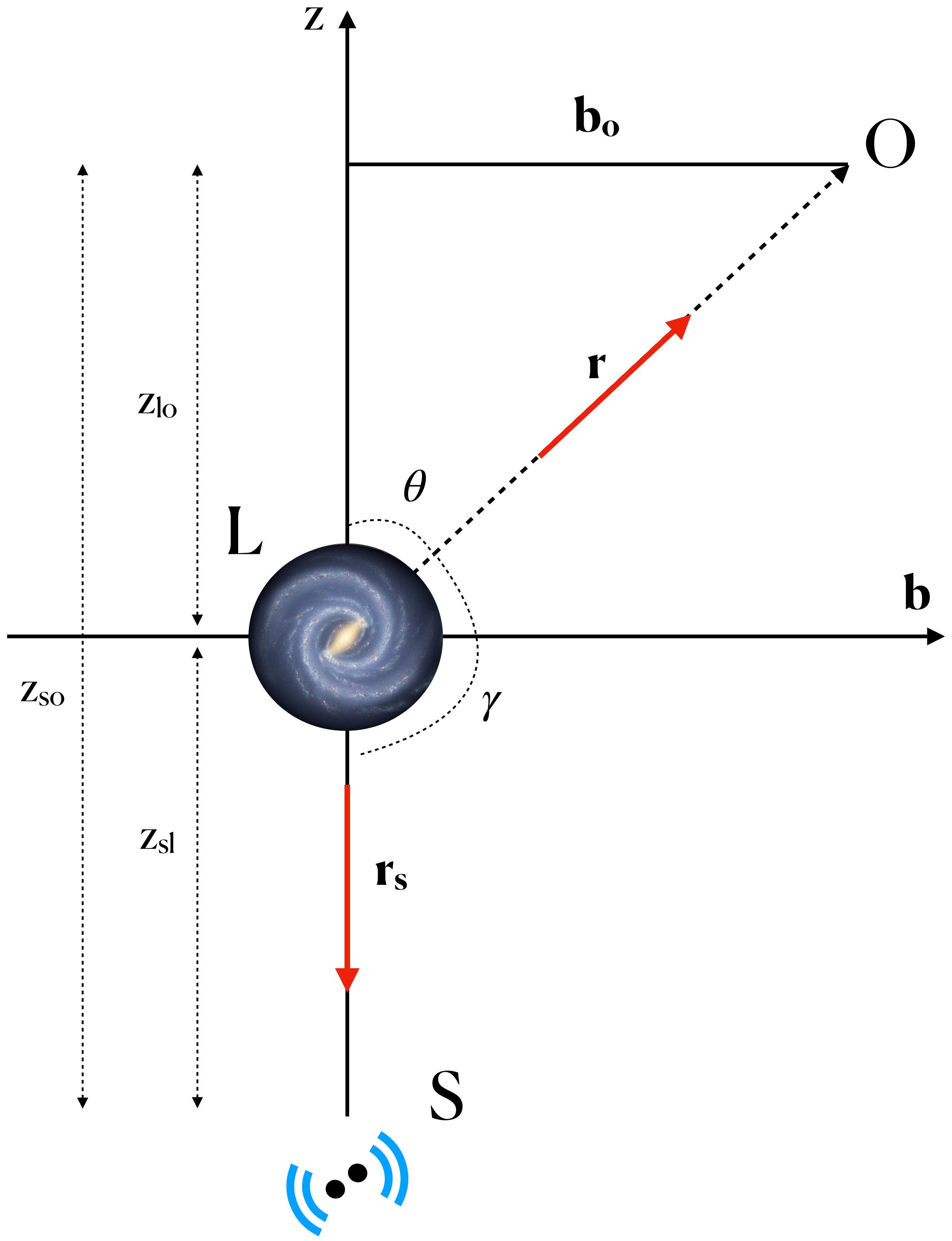
$$-\int d\Omega \,|\, \partial_\varphi F_{\rm diff}^{\rm MG} \,|^2 = 4\sigma_v^2 \xi \int d\Omega \,|\, F_{\rm diff}^{\rm MG} \,|^2$$

The eikonal approximation breaks down at small frequency: the diffraction integral is not a good solution.

Distorted base waveform

With: AG, G. Tasinato
arXiv: 2605.21584

In order to have a description that interpolate smoothly from $\nu \gg 1$ to $\nu \lesssim 1$, we change the reference function from to one with the correct static limit



$$h = F^{\rm MG} \times h_\omega^{\rm free}$$
$$\left[\nabla^2 + \omega^2\right] h_\omega^{\rm free} = 0$$

$\longrightarrow$

$$h = F^{\rm ref} \times h_\omega^{\rm ref}$$
$$\left[\nabla^2 + \omega^2 - 2\xi \Delta U(\mathbf{r})\right] h_\omega^{\rm ref} = 0$$

Under the modified eikonal & paraxial:

$$\left|\partial_r^2 F^{\rm ref}\right| \ll \left|\frac{2}{r}\partial_r F^{\rm ref} + 2\,\partial^i \ln h_\omega^{\rm ref}\,\partial_i F^{\rm ref}\right|$$

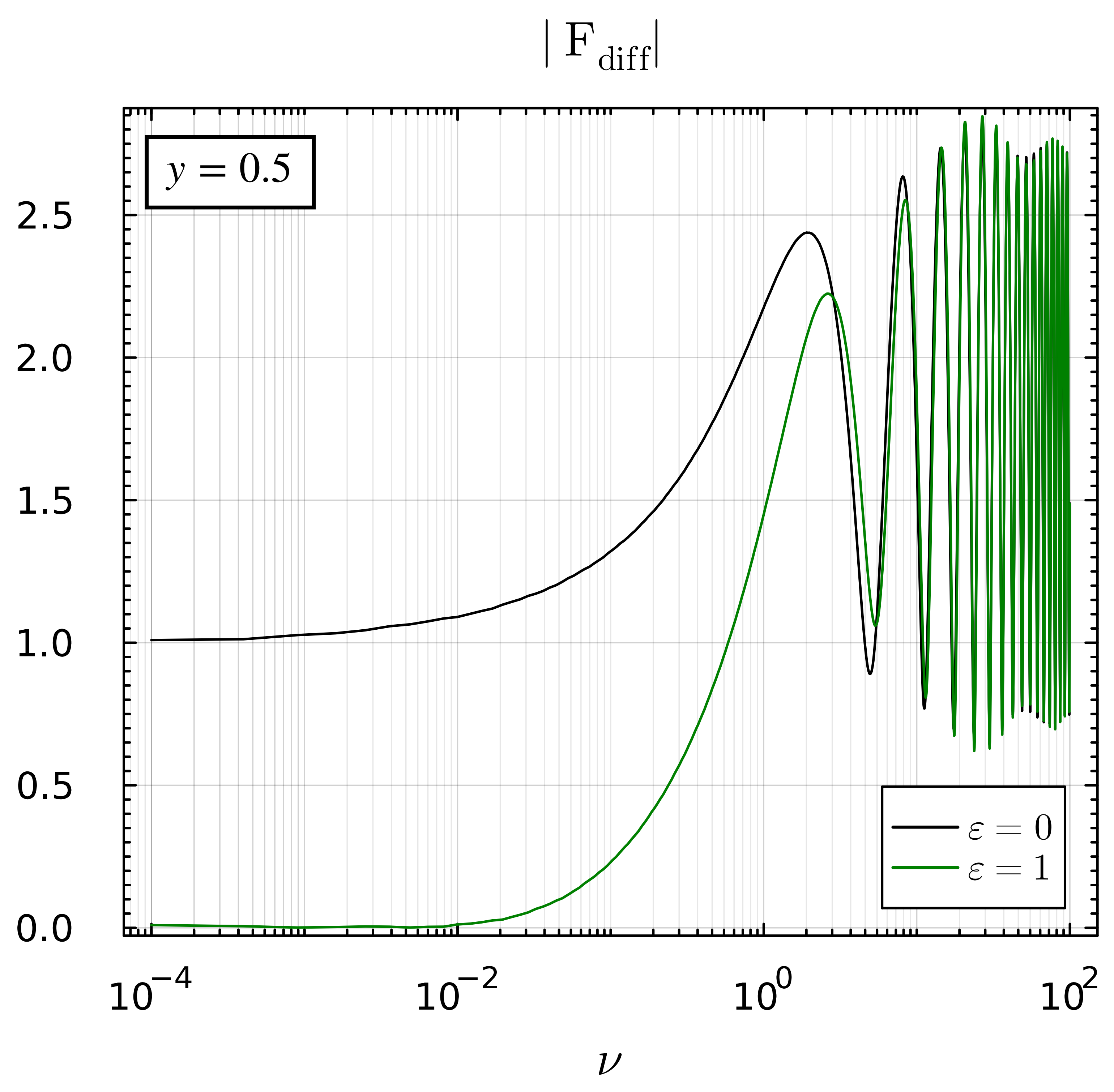
$F^{\rm ref}$ satisfies the same GR equation:

$$i\omega\partial_r F^{\rm ref} = -\frac{1}{2}\partial_\varphi^2 F^{\rm ref} + 2\omega^2 U F^{\rm ref}$$

Distorted base waveform

With: AG, G. Tasinato
arXiv: 2605.21584

In order to have a description that interpolate smoothly from $\nu \gg 1$ to $\nu \lesssim 1$, we change the reference function from to one with the correct static limit



$$h = F^{\mathrm{MG}} \times h_\omega^{\mathrm{free}}$$
$$\left[\nabla^2 + \omega^2\right] h_\omega^{\mathrm{free}} = 0$$

→

$$h = F^{\mathrm{GR}} \times h_\omega^{\mathrm{ref}}$$
$$\left[\nabla^2 + \omega^2 - 2\xi \Delta U(\mathbf{r})\right] h_\omega^{\mathrm{ref}} = 0$$

Under the modified eikonal & paraxial:

$$\left|\partial_r^2 F^{\mathrm{ref}}\right| \ll \left|\frac{2}{r}\partial_r F^{\mathrm{ref}} + 2\,\partial^i \ln h_\omega^{\mathrm{ref}}\,\partial_i F^{\mathrm{ref}}\right|$$

F^{ref} satisfies the same GR equation:

$$i\omega\partial_r F^{\mathrm{ref}} = -\frac{\partial_\varphi^2 F^{\mathrm{ref}}}{2r^2} + 2\omega^2 U F^{\mathrm{ref}}$$

Distorted base waveform

With: AG, G. Tasinato
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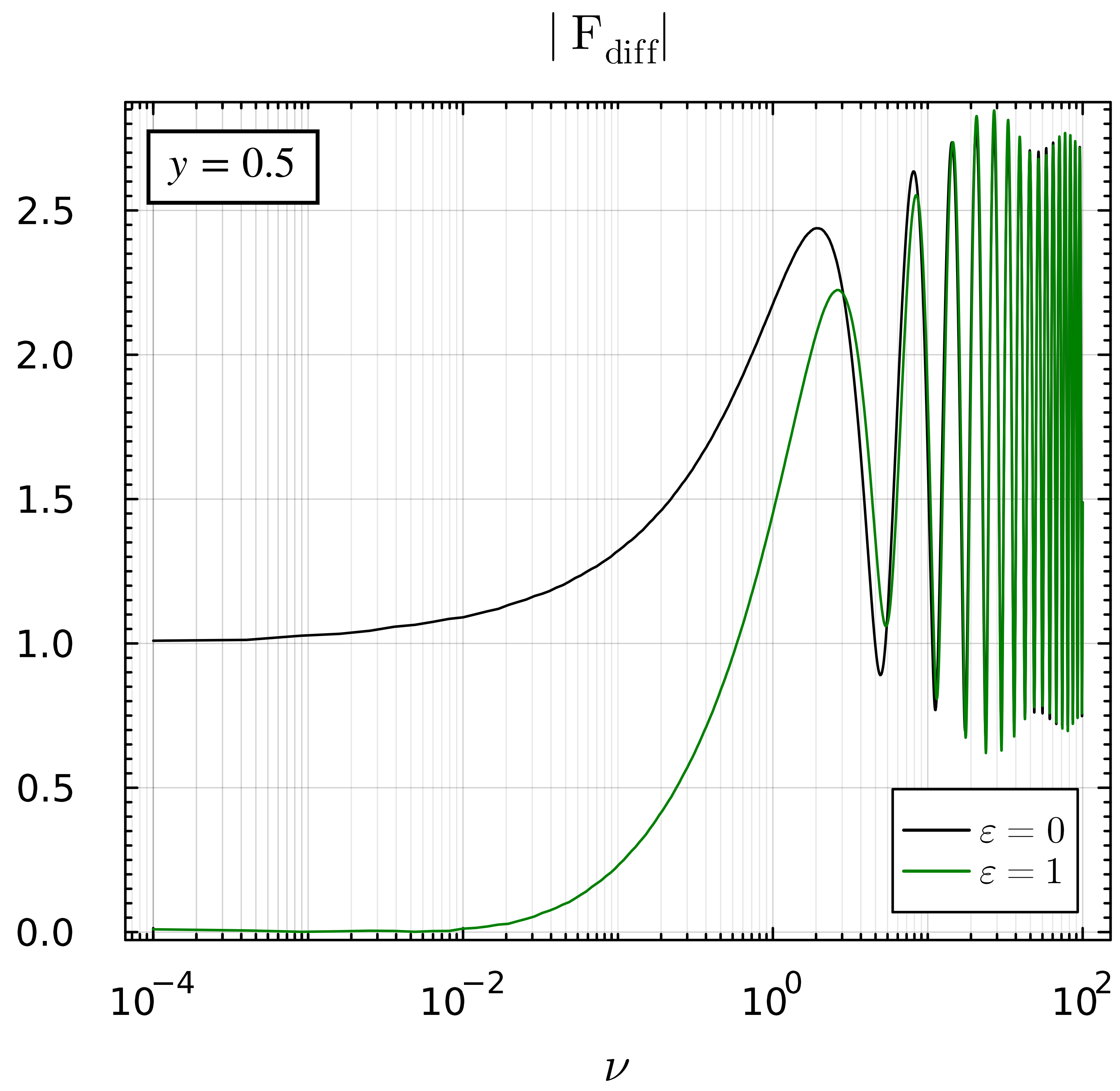
In order to have a description that interpolate smoothly from $\nu \gg 1$ to $\nu \lesssim 1$, we change the reference function from to one with the correct static limit



$$h = F^{\mathrm{MG}} \times h_\omega^{\mathrm{free}}$$
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$\longrightarrow$

$$h = F^{\mathrm{GR}} \times h_\omega^{\mathrm{ref}}$$
$$\left[\nabla^2 + \omega^2 - 2\xi \Delta U(\mathbf{r})\right] h_\omega^{\mathrm{ref}} = 0$$

The reference wave is a distorted spherical wave.

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \omega^2 - \frac{\ell(\ell+1) + 4\sigma_v^2 \xi}{r^2}\right] h_{\omega\ell}^{\mathrm{ref}}(r, r_s) = 0$$

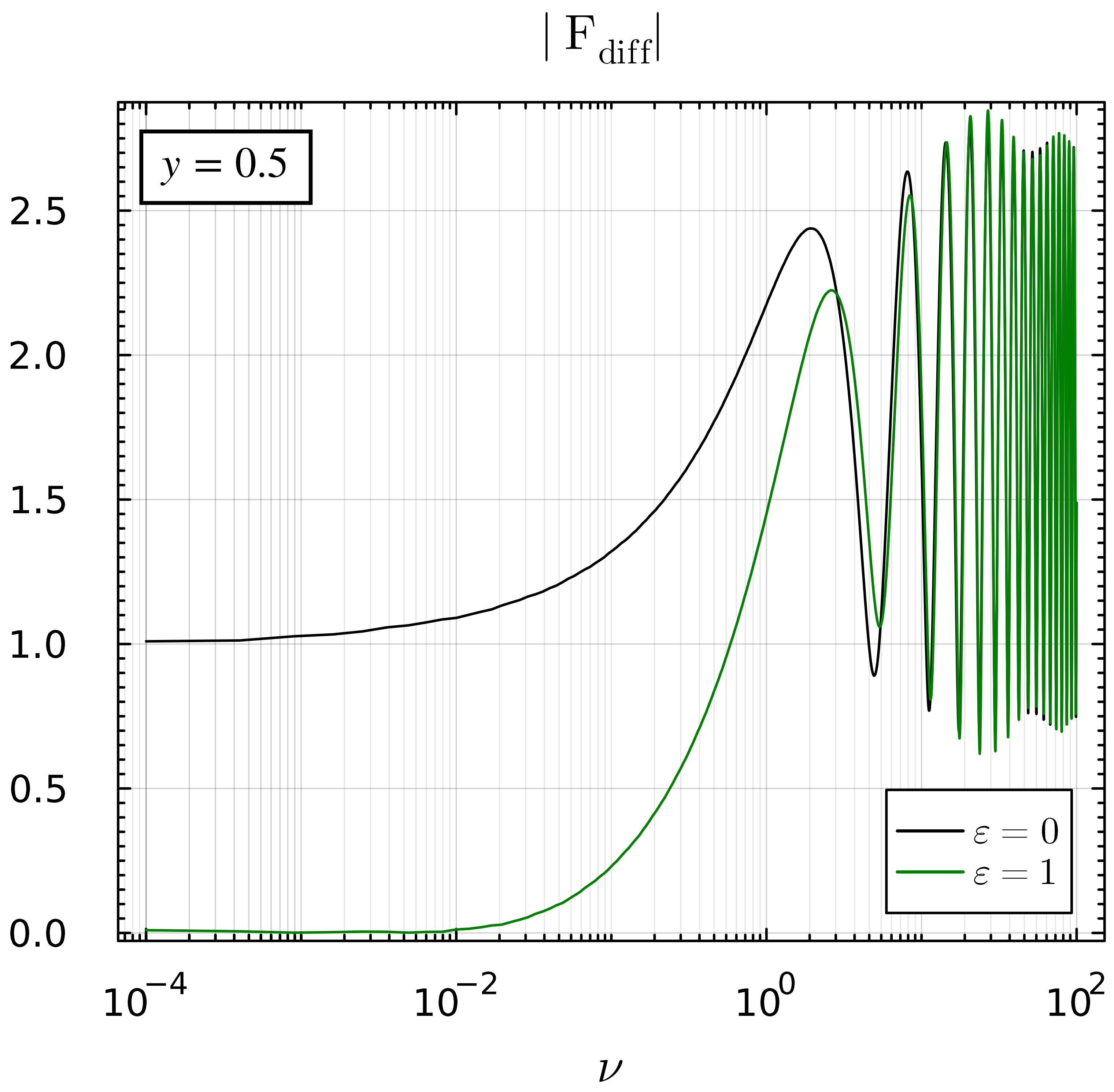
MG + SIS: shift of angular momentum

$$\lambda_\ell = -\frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 16\sigma_v^2 \xi}$$

Distorted base waveform

With: AG, G. Tasinato
arXiv: 2605.21584

In order to have a description that interpolate smoothly from $\nu \gg 1$ to $\nu \lesssim 1$, we change the reference function from to one with the correct static limit



$$h = F^{\rm MG} \times h^{\rm free}_\omega$$
$$\left[\nabla^2 + \omega^2\right] h^{\rm free}_\omega = 0$$

$\longrightarrow$

$$h = F^{\rm GR} \times h^{\rm ref}_\omega$$
$$\left[\nabla^2 + \omega^2 - 2\xi \Delta U(\mathbf{r})\right] h^{\rm ref}_\omega = 0$$

The reference wave is a distorted spherical wave.

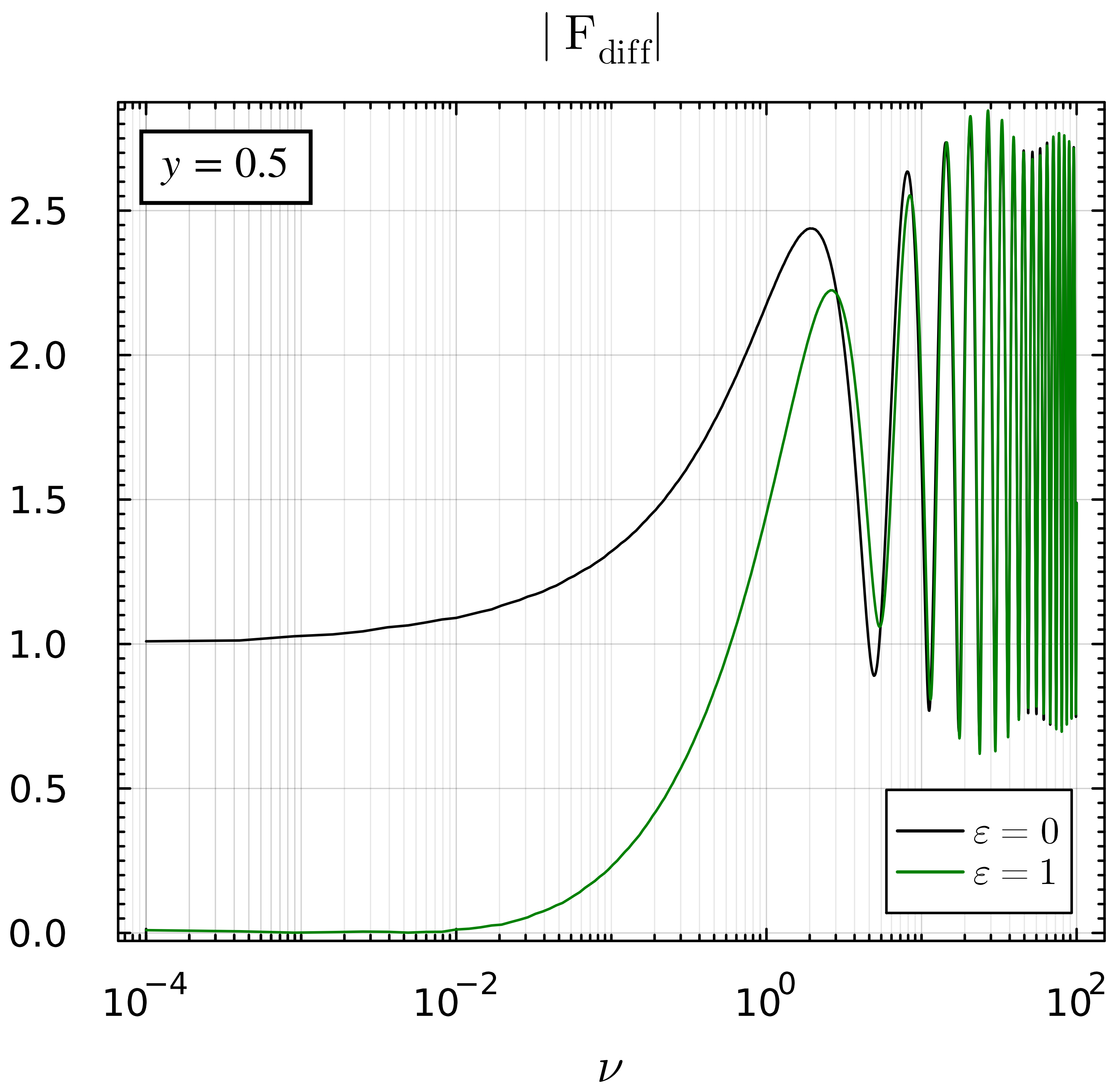
$$h^{\rm ref}_\omega = i\omega \sum_{\ell=0}^{\infty} (2\ell + 1) j_{\lambda_\ell}(\omega r_<) h^{(1)}_{\lambda_\ell}(\omega r_>) P_\ell(\cos\gamma)$$

$$\lambda_\ell = -\frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 16\sigma_v^2 \xi}$$

Distorted base waveform

With: AG, G. Tasinato
arXiv: 2605.21584

In order to have a description that interpolate smoothly from $\nu \gg 1$ to $\nu \lesssim 1$, we change the reference function from to one with the correct static limit



$$h = F^{\rm MG} \times h_\omega^{\rm free}$$
$$\left[\nabla^2 + \omega^2\right] h_\omega^{\rm free} = 0$$

→

$$h = F^{\rm GR} \times h_\omega^{\rm ref}$$
$$\left[\nabla^2 + \omega^2 - 2\xi \Delta U(\mathbf{r})\right] h_\omega^{\rm ref} = 0$$

The reference wave is a distorted spherical wave. When $\omega r_s \gg 1$

$$h_\omega^{\rm ref} \simeq \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|} \times e^{i\Delta\Phi(b)} \qquad \Delta\Phi = -\frac{\pi}{2}\left(\lambda_\ell - \ell\right)_{\ell+1/2=\omega b}$$

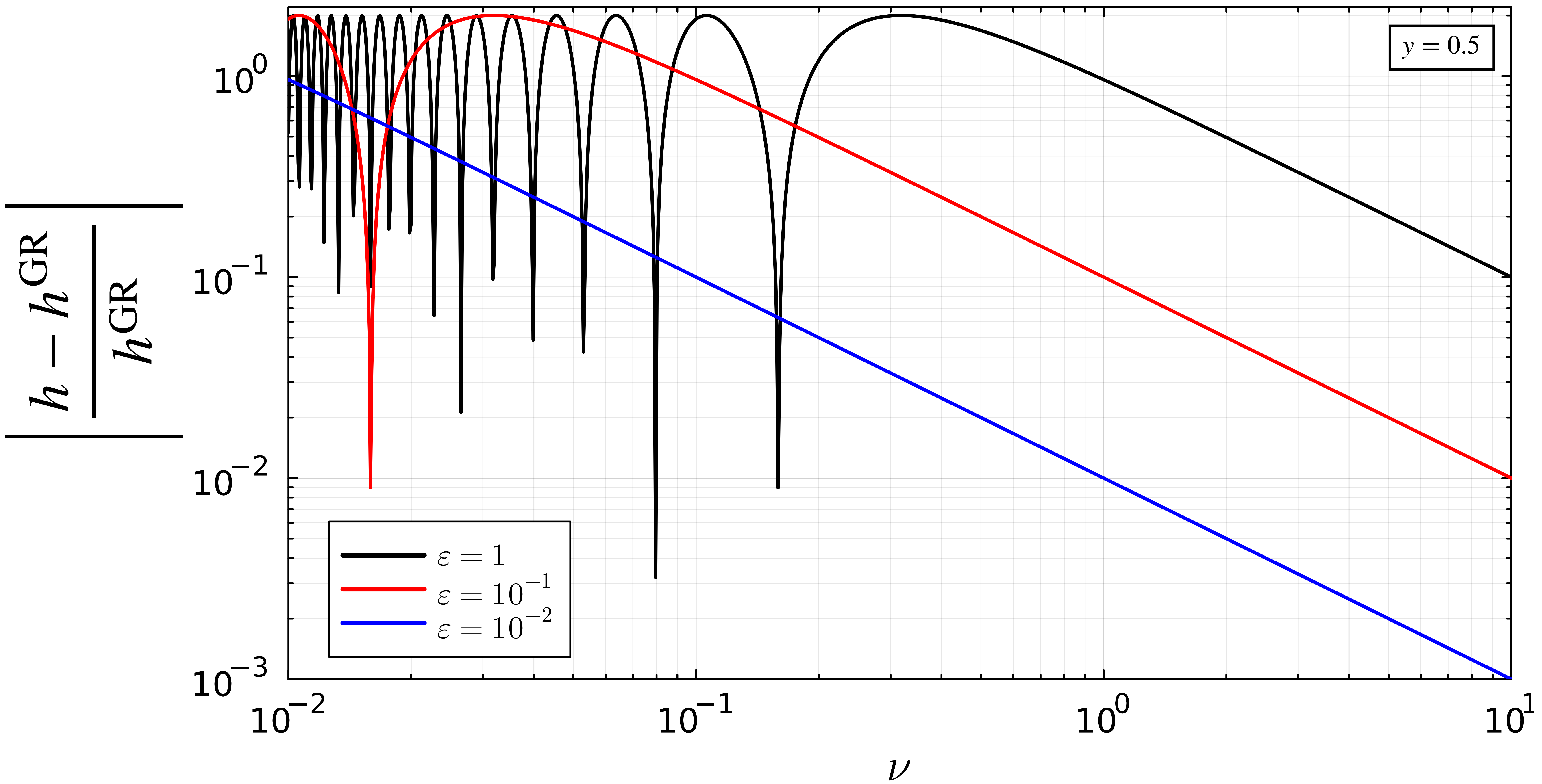
$$\lambda_\ell = -\frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 16\sigma_v^2 \xi}$$

Numerical results

With: AG, G. Tasinato
arXiv: 2605.21584

For $\omega r_s \gg 1 \rightarrow$

$$h_\omega \simeq F^{\rm GR} \times \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|} \times e^{i\Delta\Phi} \qquad \Delta\Phi = -\frac{\pi}{2}\left(\lambda_\ell - \ell\right)_{\ell+1/2=\omega b} = -\frac{\epsilon}{2y\nu}$$



Numerical results

With: AG, G. Tasinato
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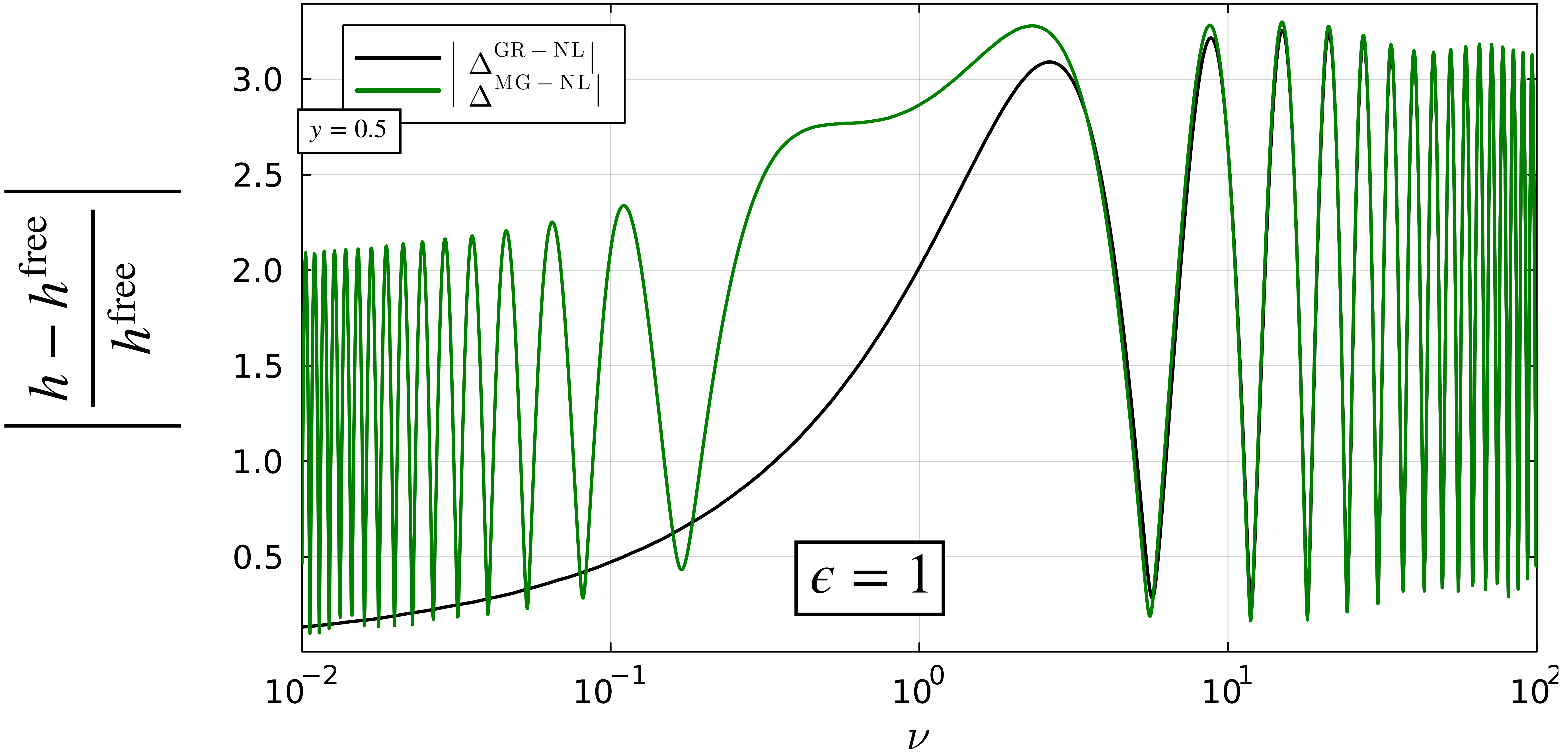
For $\omega r_s \gg 1 \rightarrow$ $h_\omega \simeq F^{\rm GR} \times \dfrac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|} \times e^{i\Delta\Phi}$ $\quad \Delta\Phi = -\dfrac{\pi}{2}\left(\lambda_\ell - \ell\right)_{\ell+1/2=\omega b} = -\dfrac{\epsilon}{2y\nu}$



Numerical results

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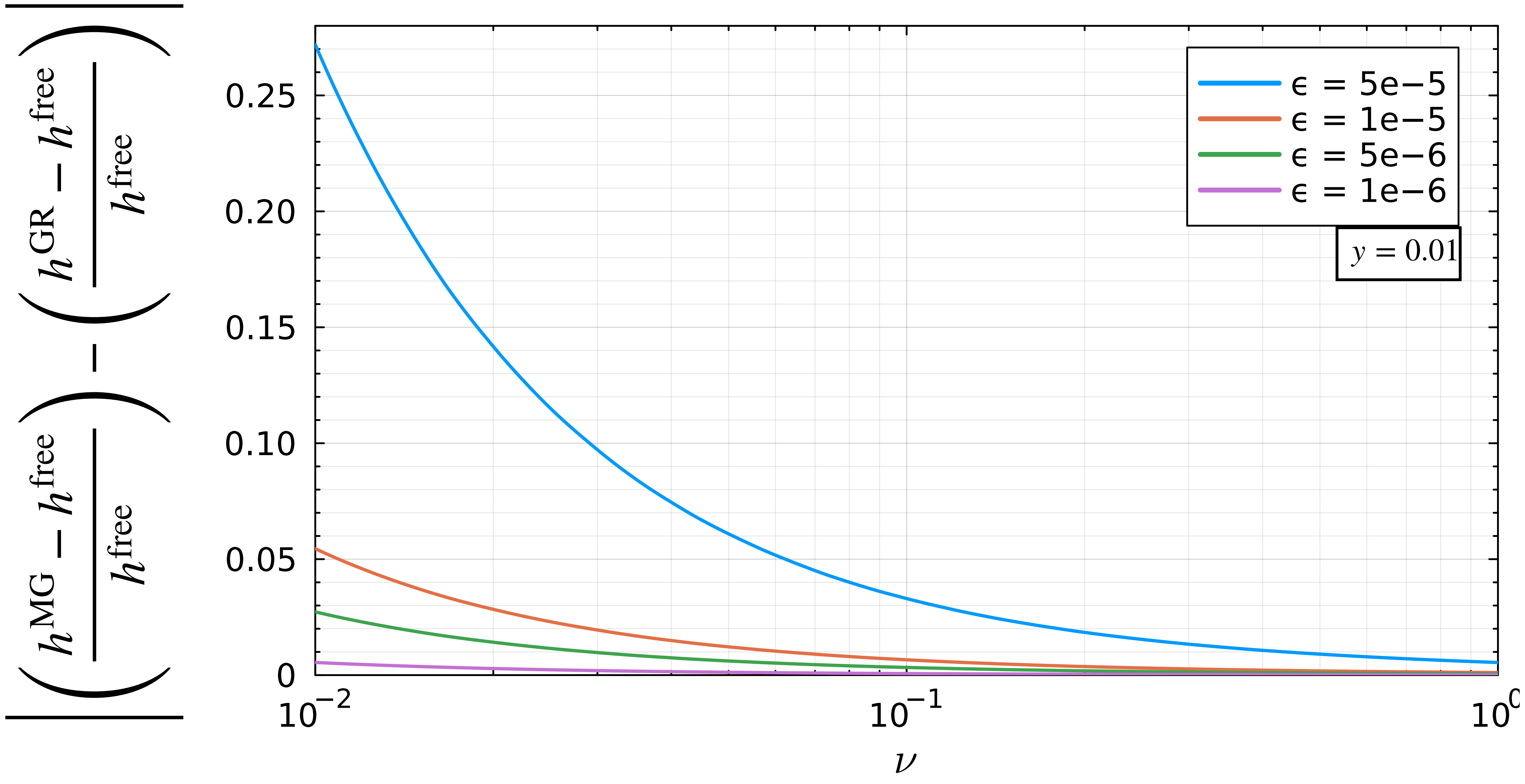
arXiv: 2605.21584

For $\omega r_s \gg 1 \rightarrow$ $$h_\omega \simeq F^{\rm GR} \times \frac{e^{i\omega|\mathbf{r}-\mathbf{r}_s|}}{|\mathbf{r}-\mathbf{r}_s|} \times e^{i\Delta\Phi} \qquad \Delta\Phi = -\frac{\pi}{2}\left(\lambda_\ell - \ell\right)_{\ell+1/2=\omega b} = -\frac{\epsilon}{2y\nu}$$



$$\xi \approx 10^{13}\epsilon\left(\frac{M_\odot}{M_{\rm SIS}}\right)\left(\frac{z_{\rm sl}}{1{\rm pc}}\right)\left(\frac{z_{\rm lo}}{z_{\rm so}}\right)$$

	$M_{\rm SIS}(M_\odot)$	$z_{\rm so}$(pc)	$z_{\rm sl}$(pc)	ξ
Supermassive BH lens	$10^6 - 10^8$	10^8	10^8	$\epsilon\left(10^{13} - 10^{15}\right)$
Hierarchical triples	$10^6 - 10^8$	10^8	$10^{-3} - 1$	$\epsilon\left(10^{2} - 10^{7}\right)$

2407.00473, 2506.22519

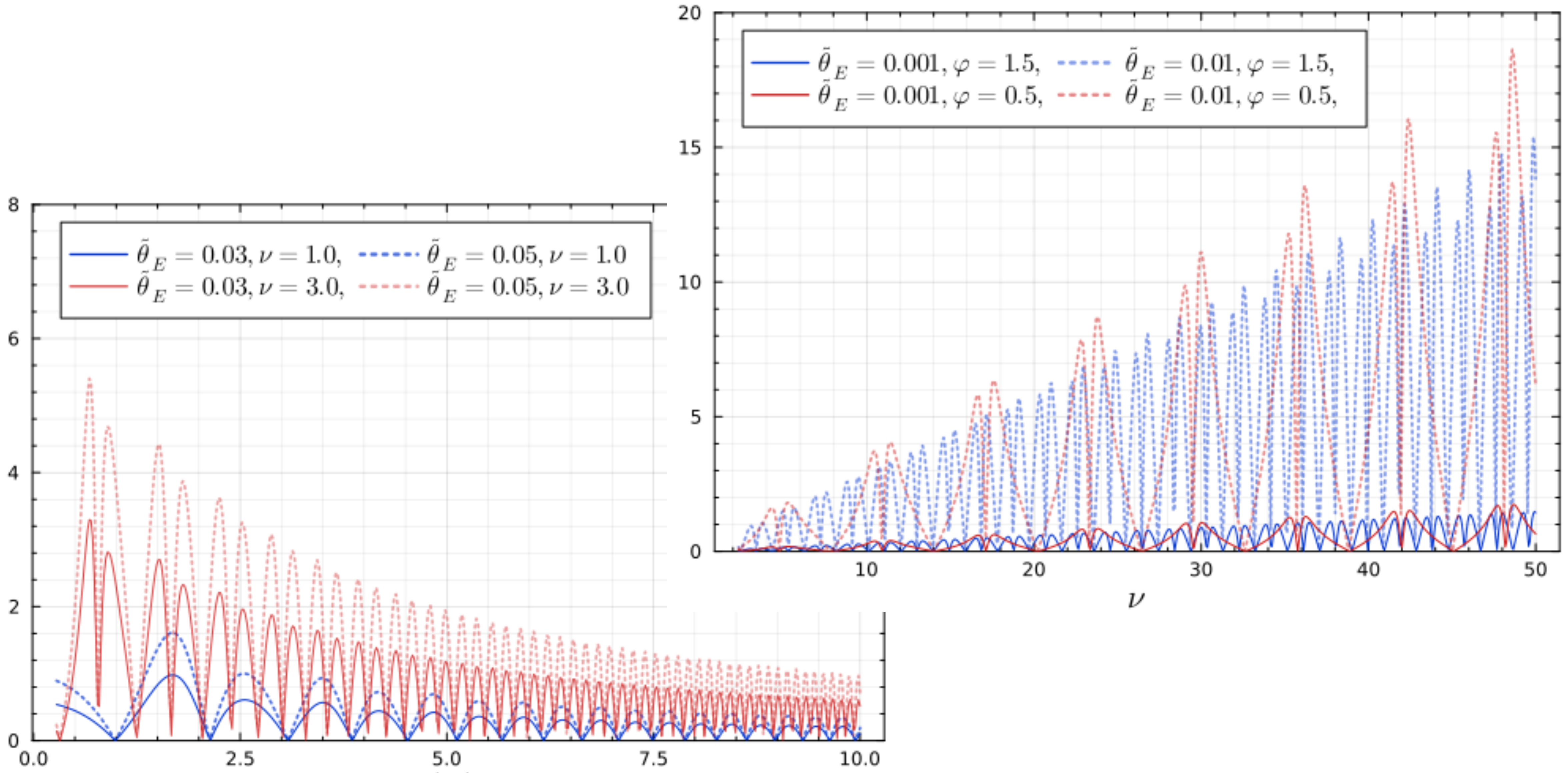
Conclusions

Lensing in wave-optics is an interesting testing ground for scattering approaches with observational prospects.

Scattering & Lensing

With: M. Carrillo Gonzales, V. De Luca, AG, J. Parra Martinez, M. Trodden.
arXiv: 2511.15797

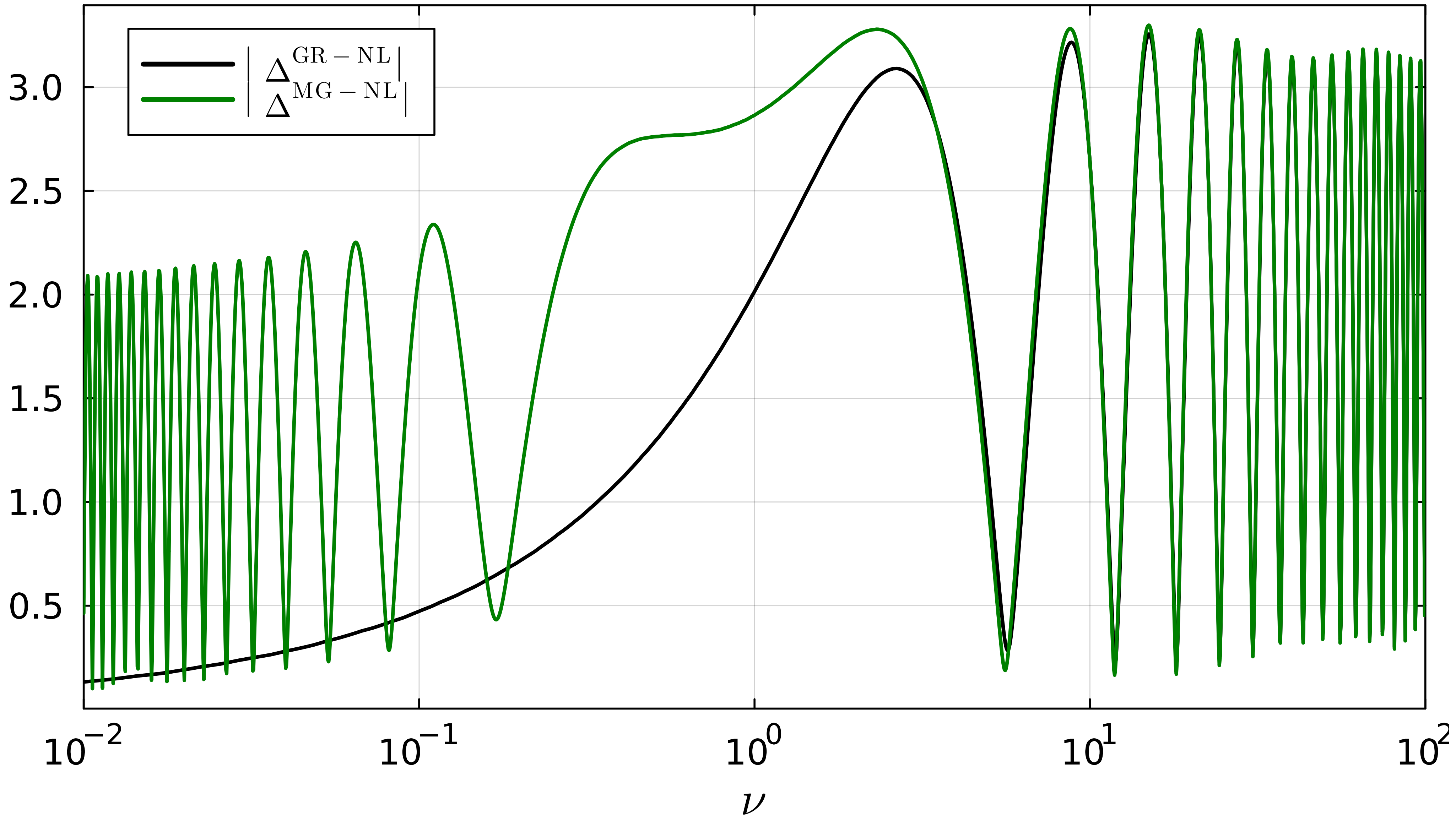
$$F_{\rm diff} \leftrightarrow f^{\rm eik}$$



WO in MG

With: AG, G. Tasinato
arXiv: 2605.21584

$$\left(\Box - \xi R\right) h = 0$$



Thank you!