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Dynamical Tidal Response of Compact Objects

Based on 2511.02372 with
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+ work in progress with
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A simple story of gravity

- Over the years, different approaches (scattering amplitudes, double copy, EFT, BHPT, hidden symmetries…) have revealed a striking simplicity of gravity, yet to be fully understood.

- Famously, BHs of GR display astonishing *beauty* and *simplicity*.
  Despite being assembled from unknown microscopic constituents, and behaving like maximally chaotic quantum systems with a rich entanglement structure, the classical features of BHs are surprisingly simple.
  [Maldacena, Shenker and Stanford '15; Almheiri, Hartman, Maldacena, Shaghoulian and Tajdini '20]

- Interestingly, not only do BH solutions are simple, but also their perturbations.

- Some aspects of this *simplicity* are well understood in terms of (hidden) symmetries of general relativity. Others remain mysterious.

A simple story of gravity

- One famous example: *isospectrality* – known to follow from Chandrasekhar's duality.

  [Chandrasekhar '75]

- True for massless (spin-0, 1, 2) fields on Schwarzschild/Kerr(-de Sitter) spacetime, as well as for partially-massless spin-2 fields.

  [Brito, Cardoso and Pani '13; Rosen and LS '20]

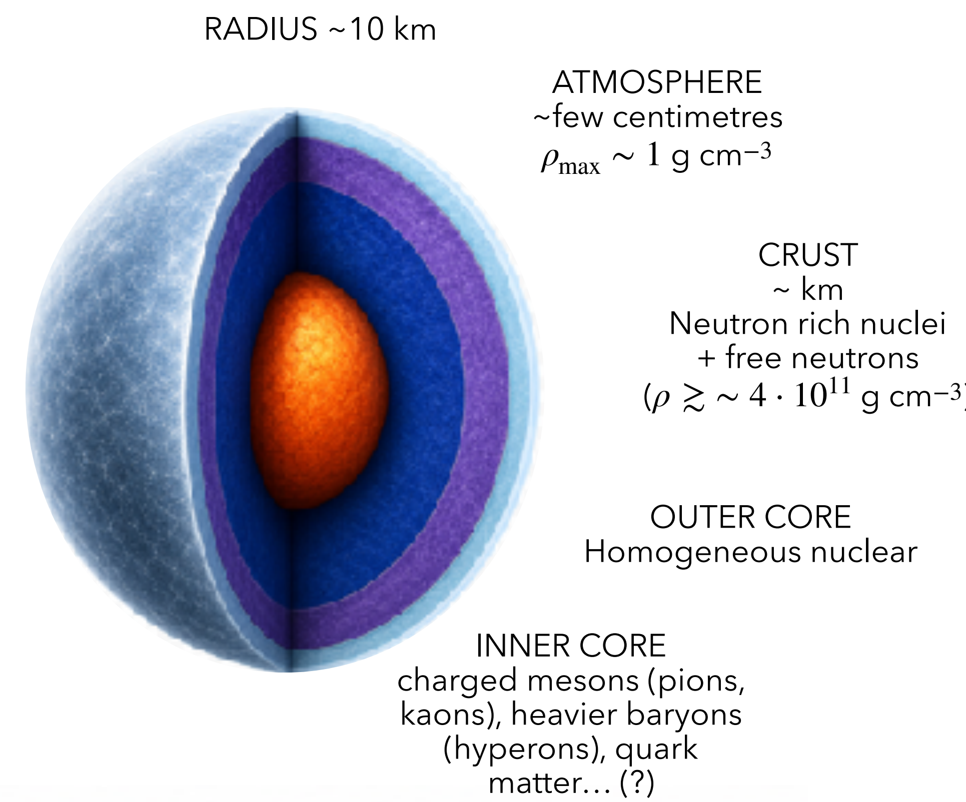
- Another intriguing example arises from the study of black holes' tidal responses: *black hole Love numbers vanish*.

- In the language of EFT, the remarkable property of black holes is that they behave like elementary point particles (in the static regime).

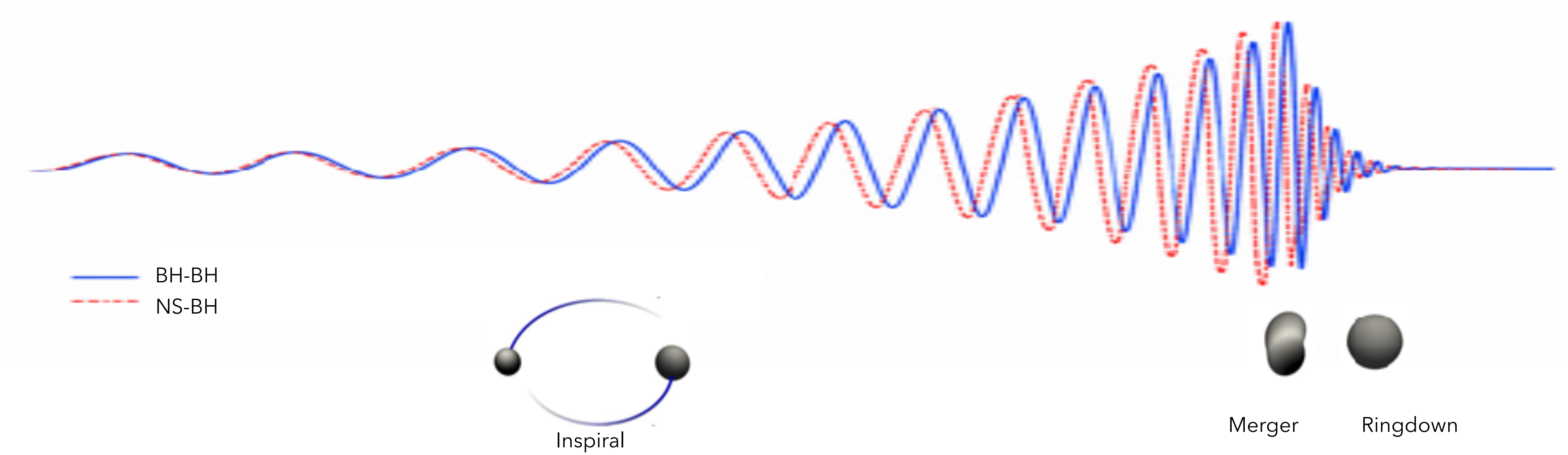
  [*See also Filippo's talk*]

New era of gravitational physics

- Gravitational-wave observations provide an unprecedented opportunity to gain insight into the internal structure of neutron stars through their tidal response.



- Tidal effects change the dynamics during the inspiral:



New era of gravitational physics

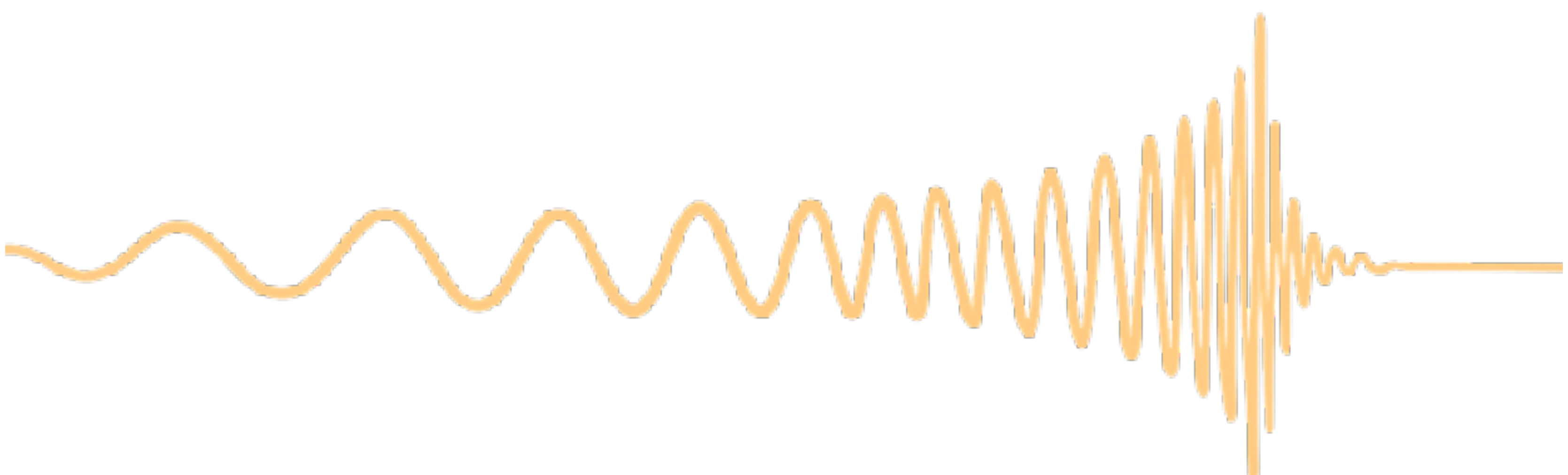
- As we go toward the era of precision physics with GWs, ever more accurate waveform templates are necessary.
  [Maggiore et al. '19; Sathyaprakash et al. '19; Kalogera et al. '21; Berti et al. '21; …]

- This requires to have a precise understanding of the conservative and dissipative dynamics of two-body systems.
  [Buonanno et al. '22; Flanagan and Hinderer '07; Henry, Faye and Blanchet '20; Kälin, Liu, and Porto '20; …]

- This includes in particular subdominant tidal effects.

Subleading tidal effects

- Various fundamental reasons (symmetries…) as well as practical ones (reduce systematics, improve waveform, break degeneracy…) to study subdominant effects.



- Two types of subleading corrections to Love Numbers:
  - $\omega \neq 0$ (*dynamical* LNs)
  - nonlinearities

[*See Filippo's talk*]

Subdominant tidal effects

- Intense activity on dynamical tidal response:

Dynamical tides	References
Foundational / EFT	Goldberger & Rothstein *et al.* [10, 11]; Chakrabarti *et al.* [20]†; Porto [13]; Endlich & Penco [168]; Goldberger *et al.* [29]
Waveform modelling	Hinderer *et al.* [386]†; Steinhoff *et al.* [387]†; Schmidt & Hinderer [388]†; Pratten *et al.* [86]†; Saketh *et al.* [57]; Pitre & Poisson [97, 109]†; Jakobsen *et al.* [389]†; Mandal *et al.* [390, 391]†; Hegade *et al.* [392]†; Yu *et al.* [393]†; Chakraborty *et al.* [394]
BH perturbation theory / PN framework	Poisson [64]; Pitre & Poisson [97]†; Katagiri *et al.* [347, 395]; Chakraborty *et al.* [394, 396]; Bhatt *et al.* [397]; Andersson *et al.* [398]†; Hegade *et al.* [392, 399–401]†
Scattering amplitudes / Raman scattering	Saketh *et al.* [58]; Ivanov *et al.* [218, 223, 225, 229]; Saketh *et al.* [224]†; Caron-Huot *et al.* [227]; Correia & Isabella [402]; Correia *et al.* [403]; Jarequi *et al.* [404]†
EFT off-shell matching	Chakrabarti *et al.* [20]†; Combaluzier–Szteinsznaider *et al.* [179]
Dissipation, tidal heating ($\omega \neq 0$)	Goldberger *et al.* [11, 29]; Chia [28]; Starobinsky & Churilov [405]; Page [406]; Chia *et al.* [188]; Saketh *et al.* [57]; Bhatt *et al.* [407]; Hegade *et al.* [399, 401]†; Kobayashi *et al.* [408]
Dynamical Love numbers (RG running)	Chakrabarti *et al.* [20]*; Charalambous *et al.* [30]; Saketh *et al.* [58]; Perry & Rodriguez [385]; Ivanov *et al.* [218, 225]*; Katagiri *et al.* [395]; De Luca *et al.* [409]; Chakraborty *et al.* [394]; Caron-Huot *et al.* [227]*; Combaluzier–Szteinsznaider *et al.* [179]*; Kobayashi *et al.* [222]*
Spinning objects	Landry & Poisson [410]†; Steinhoff *et al.* [85]†; Gupta *et al.* [90, 91]†
Shell EFT	Kosmopoulos *et al.* [411]*; Bhattacharyya *et al.* [412]*
Neutron star dynamical tides (normal-mode description)	Hinderer *et al.* [386]†; Steinhoff *et al.* [387]†; Andersson & Pnigouras [83]†; Passamonti *et al.* [84, 87]†; Steinhoff *et al.* [85]†; Yu & Lau [413]†; Ghosh *et al.* [414]†; Pereira *et al.* [415]†; Pnigouras *et al.* [416]†

Adapted from [Rodriguez, LS and Solomon '26]

Outline and summary

Most of the existing results in the literature are incomplete for gravitational perturbations.

- I will show how to fully compute dynamical Love numbers for Schwarzschild black holes and neutron stars, including scheme-dependent finite terms in addition to their universal RG running.
- For BHs, I will discuss some aspects of hidden symmetry structure.

For a complete list of refs., see recent reviews: [Rodriguez, LS and Solomon ’26; Chakraborty and Pani ’26]

Beyond the static linear response

- A (post-)Newtonian-inspired ansatz for the relativistic analogue of the tidal potential is:

$$
\begin{aligned}
g^{\rm PN}_{\mu\nu}(r \gg r_s) \sim & \left(1+\frac{r_s}{r}+\frac{r_s^2}{r^2}+\ldots\right) + \sum_{\ell}\sum_{m} \mathcal{E}_{\ell m}\left[r^{\ell}\left(1+\frac{r_s}{r}+\ldots\right) + k_{\ell}\left(\frac{r_s}{r}\right)^{\ell+1}\left(1+\frac{r_s}{r}+\ldots\right)\right] \\
& + \sum_{\ell}\sum_{m} \ddot{\mathcal{E}}_{\ell m}\left[r^{\ell+2}\left(1+\frac{r_s}{r}+\ldots\right) + k_{\ell}\left(\frac{r_s}{r}\right)^{\ell-1}\left(1+\frac{r_s}{r}+\ldots\right) + \ddot{k}_{\ell}\left(\frac{r_s}{r}\right)^{\ell+1}\left(1+\frac{r_s}{r}+\ldots\right)\right] \\
& + \sum_{\ell\ell_1\ell_2}\sum_{mm_1m_2} \mathcal{G}^{\ell\ell_1\ell_2}_{mm_1m_2}\mathcal{E}_{\ell_1 m_1}\mathcal{E}_{\ell_2 m_2}\left[r^{\ell_1+\ell_2}\left(1+\frac{r_s}{r}+\ldots\right) + p_{\ell\ell_1\ell_2}\left(\frac{r_s}{r}\right)^{\ell+1}\left(1+\frac{r_s}{r}+\ldots\right)\right] \\
& + \mathcal{O}(\dddot{\mathcal{E}}, \mathcal{E}^3)
\end{aligned}
$$

- Are there non-analytic contributions in r beyond the static/linear regime?
- Ambiguities in the definition of the response coefficients for integer ℓ?
- How to describe dissipation?

Beyond the static linear response

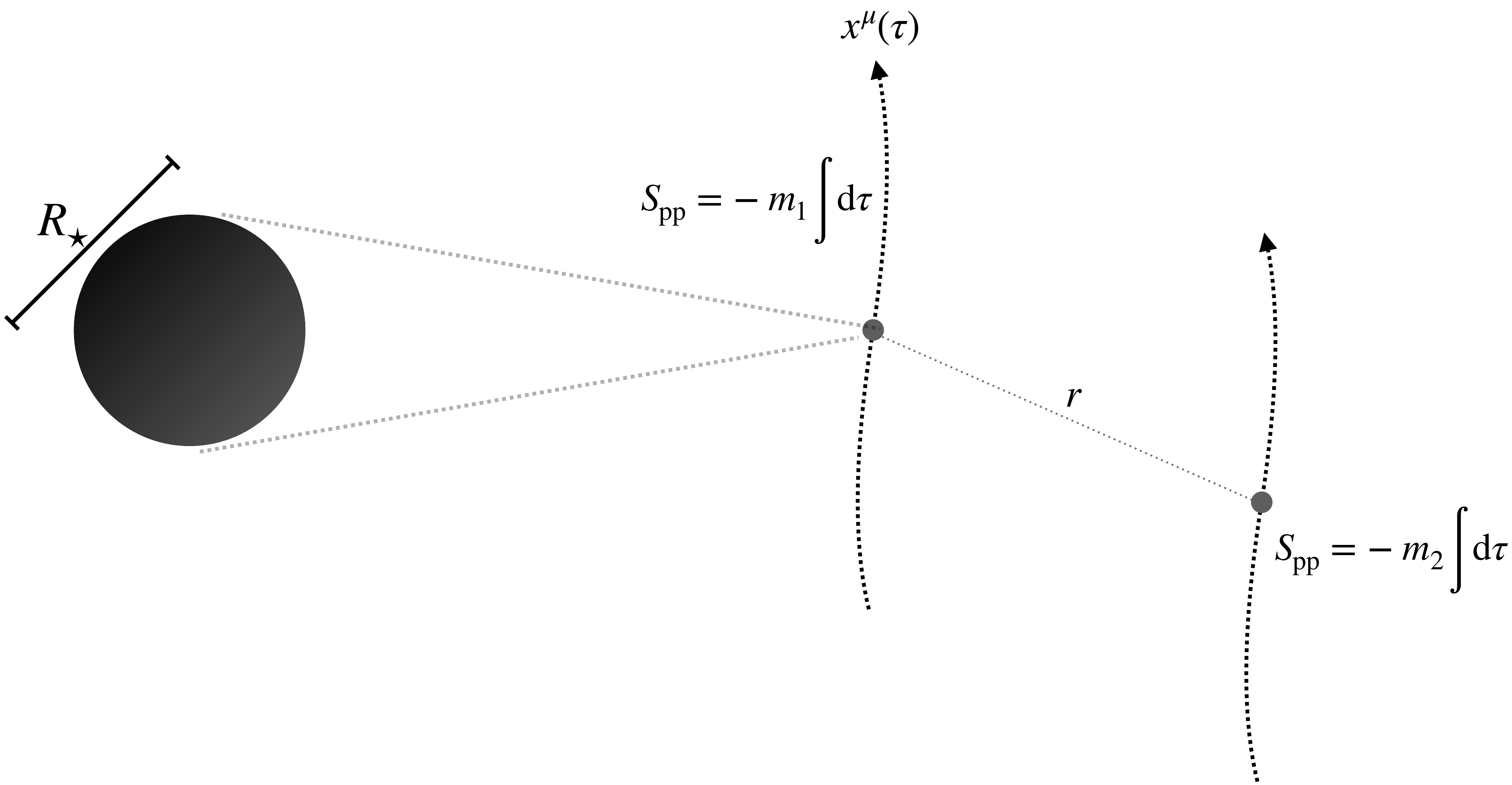
- For static nonlinear response, no non-analytic contributions in r (no running) in δg for generic compact objects.

- Non-analyticities and running appear at finite ω.

- More in general, one may worry whether the definition of Love numbers is reparametrization invariant, or that the source/response split is not well-defined because of nonlinearities.

- Ambiguities can indeed arise in the calculation of induced response in general relativity, as opposed to electromagnetism and Newtonian gravity.

EFT approach to compact objects and binary dynamics



[Goldberger and Rothstein ’04, ’05, ’20;
Porto ’05, ’16; Kol and Smolkin ’07, ’11; …]

The point-particle EFT

- At distances $r \gg R_\star \gtrsim r_s$:

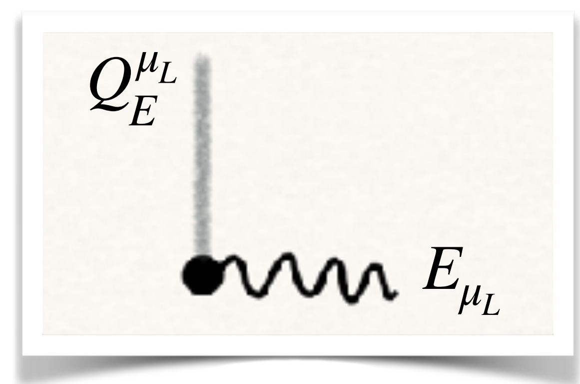
$$S = S_{\rm EH} + S_{\rm pp} + S_{\rm int}$$

$$S_{\rm EH} = \frac{M_{\rm Pl}^2}{2}\int {\rm d}^4x\sqrt{-g}\,R\,, \quad S_{\rm pp} = -\,M\int {\rm d}\tau\,, \quad S_{\rm int} = \sum_{\ell=2}^{\infty}\int {\rm d}\tau\left(Q_E^{\mu_L}(X)E_{\mu_L} + Q_B^{\mu_L}(X)B_{\mu_L}\right) + \ldots\,,$$

where $\mu_L \equiv \mu_1\cdots\mu_\ell$ and, schematically,

$$E_{\mu_1\cdots\mu_\ell} = \nabla_{\langle\mu_1}\cdots\nabla_{\mu_{\ell-2}}E_{\mu_{\ell-1}\mu_\ell\rangle}\,, \qquad B_{\mu_1\cdots\mu_\ell} = \nabla_{\langle\mu_1}\cdots\nabla_{\mu_{\ell-2}}B_{\mu_{\ell-1}\mu_\ell\rangle}\,,$$

$$E_{\mu\nu} \equiv C_{\mu\rho\nu\sigma}u^\rho u^\sigma, \qquad B_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\gamma\mu}{}^{\alpha\beta}C_{\nu\delta\alpha\beta}u^\delta u^\gamma\,.$$



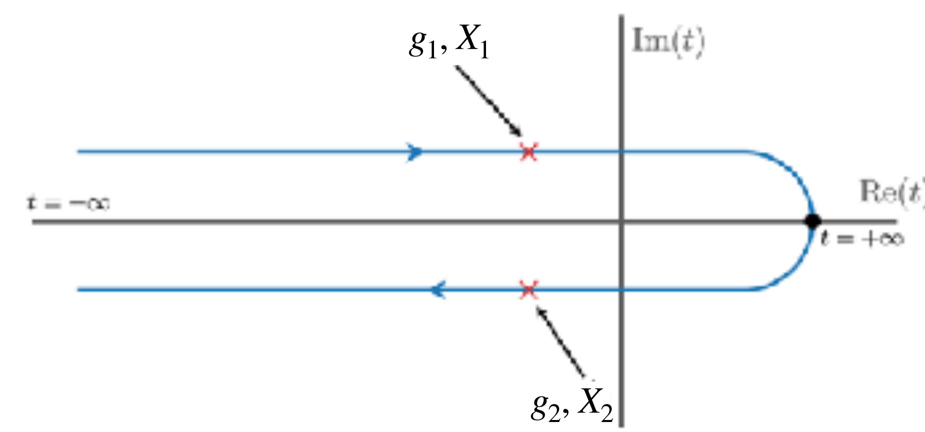
- $Q_{E,B}^{\mu_L}$ carry information about the object's finite-size properties and microscopic physics: tidal response, dissipation (absorption across the horizon, viscosity)…
  → composite operators built from worldline degrees of freedom X.

Tidal effects in worldline EFT

- One shall integrate out Q by solving its equation of motion. For instance, in linear response theory:

$$\left\langle Q^{i_1\cdots i_\ell}(\tau)\right\rangle = \sum_{\ell'} \int \mathrm{d}\tau' G^{(Q)i_1\cdots i_\ell\, j_1\cdots j_{\ell'}}(\tau-\tau') E_{j_1\cdots j_{\ell'}}(\tau') \,, \qquad \text{(similarly for } B\text{)}$$

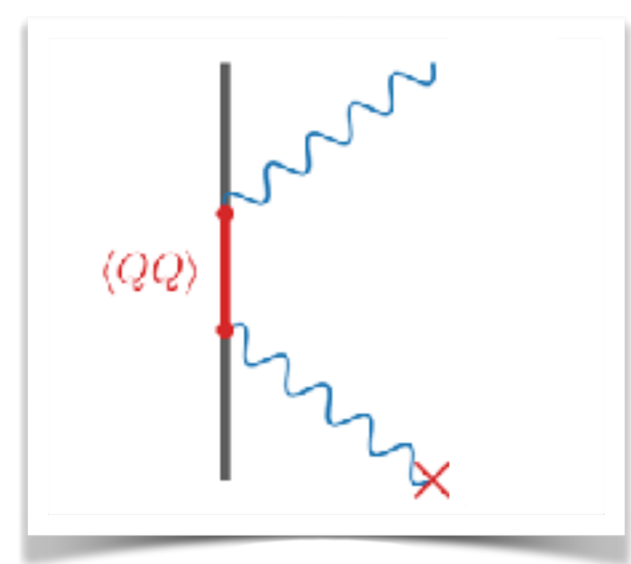
with $G^{(Q)}$ the Green's function.



- More precisely, one shall perform an in-in path integral

$$\mathrm{e}^{i\Gamma^{\text{in-in}}} = \int \mathscr{D}X_1 \mathscr{D}X_2 \, \mathrm{e}^{iS[X_1,\ldots]-iS[X_2,\ldots]}$$

leading to an effective interaction in the in-in action:

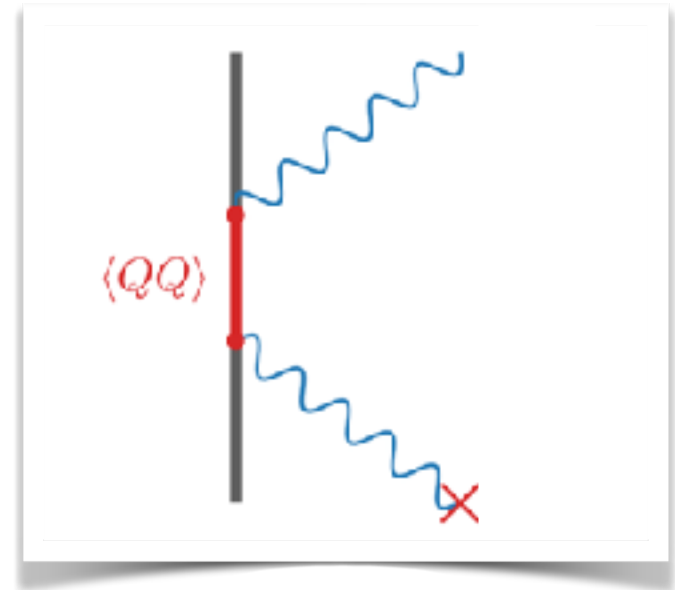


$$i\Gamma^{\text{in-in}}_{\text{int}} = \sum_{I,J=1,2} \sum_{\ell,\ell'} \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \langle Q_I^{i_1\cdots i_\ell}(\tau_1) Q_J^{j_1\cdots j_{\ell'}}(\tau_2)\rangle \, E^I_{i_1\cdots i_\ell}(\tau_1) E^J_{j_1\cdots j_{\ell'}}(\tau_2) \ .$$

Tidal effects in worldline EFT

$$\langle Q_I^{i_1\cdots i_\ell}(\tau_2) Q_{J,j_1\cdots j_{\ell'}}(\tau_1)\rangle = \delta^{i_1}_{j_1}\cdots\delta^{i_\ell}_{j_\ell}\delta_{\ell\ell'}F_{IJ}(\tau_2-\tau_1) + \text{perms.}$$

$$i\Gamma^{\text{in-in}}_{\text{int}} = \sum_{I,J=1,2}\sum_{\ell}\int \mathrm{d}\tau_1\mathrm{d}\tau_2 F_{IJ}(\tau_2-\tau_1)\, E^I_{i_1\cdots i_\ell}(\tau_2)E^{J\,i_1\cdots i_\ell}(\tau_1)$$



Tidal field

- Let's write $g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + h_{\mu\nu}$; $\quad h_+ \equiv \frac{1}{2}\left(h_1 + h_2\right)$, $\quad h_- \equiv h_1 - h_2$.

Response

- We can use $\Gamma^{\text{in-in}}_{\text{int}}$ to compute the one-point function of the (classical) response field $h^+_{\mu\nu}$:

$$\langle h^+_{\mu\nu}(t,\vec{x})\rangle_{\text{in-in}} = \int \mathscr{D}h_+\mathscr{D}h_- h^+_{\mu\nu}(t,\vec{x})\mathrm{e}^{i\Gamma^{\text{in-in}}_{\text{int}}} = i\sum_{\ell}\int \mathrm{d}\tau_1\mathrm{d}\tau_2 F_{+-}(\tau_2-\tau_1)\, E^+_{i_1\cdots i_\ell}(\tau_2)\bar{E}^{-i_1\cdots i_\ell}(\tau_1),$$

$$F_{+-}(\tau_2-\tau_1) = i\langle[Q_+(\tau_2), Q_-(\tau_1)]\rangle\Theta(\tau_2-\tau_1) \qquad \text{(retarded Green's function of } Q\text{)}$$

- In the adiabatic limit (hierarchy between probe timescale and X's timescale):

static Love numbers

dynamical Love numbers

$$F_{+-}(\omega) = \lambda^{(0)} + \lambda^{(1)}i\omega + \lambda^{(2)}\omega^2 + \ldots$$

dissipative response

Nonlinear tidal Love numbers

$$F_{+-}(\omega) = \lambda^{(0)} + \lambda^{(1)} i\omega + \lambda^{(2)}\omega^2 + \dots$$

- The EFT provides a systematic, unambiguous and gauge-invariant definition of the Love numbers.

- The tidal response coefficients λ's can be computed by matching with a general-relativistic calculation.

- To perform the matching, we compare the one-point function for the response $\langle h^+_{\mu\nu}\rangle$ in the presence of a tidal field $H_{\mu\nu}$ in the EFT, with an analogous calculation of $\delta g_{\mu\nu}$ in BHPT.

$$\delta g^{\rm EFT}_{\mu\nu} = H_{\mu\nu} + \langle h^+_{\mu\nu}\rangle_H \qquad \Longleftrightarrow \qquad \delta g^{\rm GR}_{\mu\nu}(r \gg r_s)$$

- To avoid gauge issues, we will actually perform the matching at the level of the Weyl tensor one-point function:

$$\langle W^+_{\mu\nu\rho\sigma}\rangle \qquad \Longleftrightarrow \qquad W^{\rm GR}_{\mu\nu\rho\sigma}(r \gg r_s)$$

Dynamical Love numbers: general-relativistic solution via small-frequency expansion

Luca Santoni

General-relativistic solution via small-frequency expansion

- For convenience, let's consider a D-dimensional spacetime.
  (We will eventually use dim-reg in the EFT.)

- From $SO(D-1)$, metric perturbations can be decomposed into scalar, vector and tensor.

- Let's start from the vector sector (parity-odd sector in $D=4$):

$$h_{\mu\nu} = \sum_{\ell,m} \begin{pmatrix} 0 & 0 & h_0(t,r)Y_i^{(T)m}{}_{\ell} \\ * & 0 & h_1(t,r)Y_i^{(T)m}{}_{\ell} \\ * & * & r^2 h_2(t,r)\,\nabla_{\langle i}Y_{j\rangle}^{(T)m}{}_{\ell} \end{pmatrix} .$$

- In the Regge-Wheeler gauge ($h_2 = 0$), after solving the constraint:

$$\frac{\mathrm{d}^2\Psi_{\mathrm{RW}}}{\mathrm{d}r_\star^2} + \Big(\omega^2 - V_{\mathrm{RW}}(r)\Big)\Psi_{\mathrm{RW}} = 0\,, \qquad \frac{\mathrm{d}r_\star}{\mathrm{d}r} = \frac{1}{f(r)}$$

$$V_{\mathrm{RW}}(r) = f\,\frac{(\ell+1)(\ell+D-4)}{r^2} + f^2\frac{(D-4)(D-6)}{4r^2} - ff'\frac{(D+2)}{2r}\,, \qquad f(r) \equiv 1 - \left(\frac{r_s}{r}\right)^{D-3} .$$

General-relativistic solution via small-frequency expansion

$$\frac{\mathrm{d}^2\Psi_{\mathrm{RW}}}{\mathrm{d}r_\star^2} + \left(\omega^2 - V_{\mathrm{RW}}(r)\right)\Psi_{\mathrm{RW}} = 0$$

- We solve this using boundary-layer theory.
  [White, "*Asymptotic Analysis of Differential Equations*"; Unruh '76; Bucciotti, Trincherini '23; …]

- We introduce three different zones: [Unruh '76]

  - **Near Zone** (NZ): $r - r_s \ll r_s$
  - **Intermediate Zone** (IZ): $r_s \lesssim r \ll \omega^{-1}$
  - **Far Zone** (FZ): $r \gg r_s\,,\ \omega r \gtrsim 1$

- We first solve the equation in each zone, and then match across the boundary.
  [Katagiri, Yagi, Cardoso '24; Chakraborty, De Luca, Gualtieri, Pani '25; …]

General-relativistic solution: Near Zone

- In $D = 4$:

$$\partial_r\big(f(r)\partial_r\Psi_{\mathrm{RW}}(r)\big) + \left(\frac{\omega^2}{f(r)} - \frac{\ell(\ell+1)}{r^2} + \frac{3r_s}{r^3}\right)\Psi_{\mathrm{RW}}(r) = 0.$$

- In the near-zone limit, $r \to r_s$ ($f \to 0$), the potential is dominated by the ω^2-term, leading to the usual wave equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r_\star^2} + \omega^2\right)\Psi_{\mathrm{RW}} = 0$$

with solutions $\mathrm{e}^{\pm i\omega r_\star}$.

- Imposing standard in-falling boundary conditions at the horizon selects

$$\Psi_{\mathrm{RW,NZ}}(r) = B\,\mathrm{e}^{-i\omega\left(r + r_s\log\left(\frac{r}{r_s} - 1\right)\right)}$$

General-relativistic solution: Intermediate Zone

- In the IZ ($r_s \lesssim r \ll \omega^{-1}$), the ω^2-term is genuinely small and is treated perturbatively:

$$\Psi_{\rm RW} = \Psi^{(0)} + \omega \Psi^{(1)}(r) + \omega^2 \Psi^{(2)}(r) + O(\omega^3)\,.$$

- Free integration constants fixed by matching IZ solution with the small-ω limit of NZ solution:

$$\lim_{\omega \to 0} \Psi_{\rm RW,NZ}(r) = \lim_{r \to r_s} \Psi_{\rm RW,IZ}(r)\,.$$

- At large r:

$$\begin{aligned} r^{-1}\Psi^{\ell=2}_{\rm RW,IZ}(r) \underset{\frac{r}{r_s}\to\infty}{=} & B\frac{r^2}{r_s^3} + iB\omega r_s\left(\frac{13r^2}{12r_s^3} + \frac{r_s^2}{5r^3}\right) \\ & + B\omega^2 r_s^2\left[\frac{r^4}{14r_s^5} + \frac{13r^3}{42r_s^4} + \frac{r^2}{r_s^3}\left(\frac{95+8\pi^2}{48} - \frac{107}{210}\log\left(\frac{r_s}{r}\right)\right) - \frac{319r}{420r_s^2} - \frac{153}{280r_s}\right. \\ & \left. + \frac{223}{420r} + \frac{363r_s}{560r^2} + \frac{r_s^2}{25r^3}\left(6 - 5\log\left(\frac{r_s}{r}\right)\right)\right] + O\left(\frac{r_s^3}{r^3}\right) \end{aligned}$$

to be contrasted with $g_{\mu\nu}^{\rm PN}(r \gg r_s)$ we wrote above.

General-relativistic solution: Intermediate Zone

$$r^{-1}\Psi^{\ell=2}_{\mathrm{RW,IZ}}(r) \underset{\frac{r}{r_s}\to\infty}{=} B\frac{r^2}{r_s^3} + iB\omega r_s\left(\frac{13r^2}{12r_s^3} + \frac{r_s^2}{5r^3}\right)$$
$$+B\omega^2 r_s^2\left[\frac{r^4}{14r_s^5} + \frac{13r^3}{42r_s^4} + \frac{r^2}{r_s^3}\left(\frac{95+8\pi^2}{48} - \frac{107}{210}\log\left(\frac{r_s}{r}\right)\right) - \frac{319r}{420r_s^2} - \frac{153}{280r_s}\right.$$
$$\left.+\frac{223}{420r} + \frac{363r_s}{560r^2} + \frac{r_s^2}{25r^3}\left(6 - 5\log\left(\frac{r_s}{r}\right)\right)\right] + O\left(\frac{r_s^3}{r^3}\right)$$

- Logarithmic scaling in r from order-ω^2: $r^\ell\omega^2 r_s^2\log(r/r_s)$, $r^{-\ell-1}\omega^2 r_s^2\log(r/r_s)$.
- This translates into a (classical) RG running of the dynamical Love numbers in the EFT.
- By construction, no non-analyticities in ω in the intermediate zone.

General-relativistic solution: Far Zone

- We can similarly solve the Regge-Wheeler equation in the far zone and obtain a solution valid from $\omega r \sim O(1)$ up to $\omega r = \infty$.

- Integration constants can be fixed as before by comparing the FZ and IZ solutions across their boundary:

$$\lim_{r\to\infty} \Psi_{\mathrm{RW,IZ}}(r) = \lim_{\omega r\to 0} \Psi_{\mathrm{RW,FZ}}(r) \,.$$

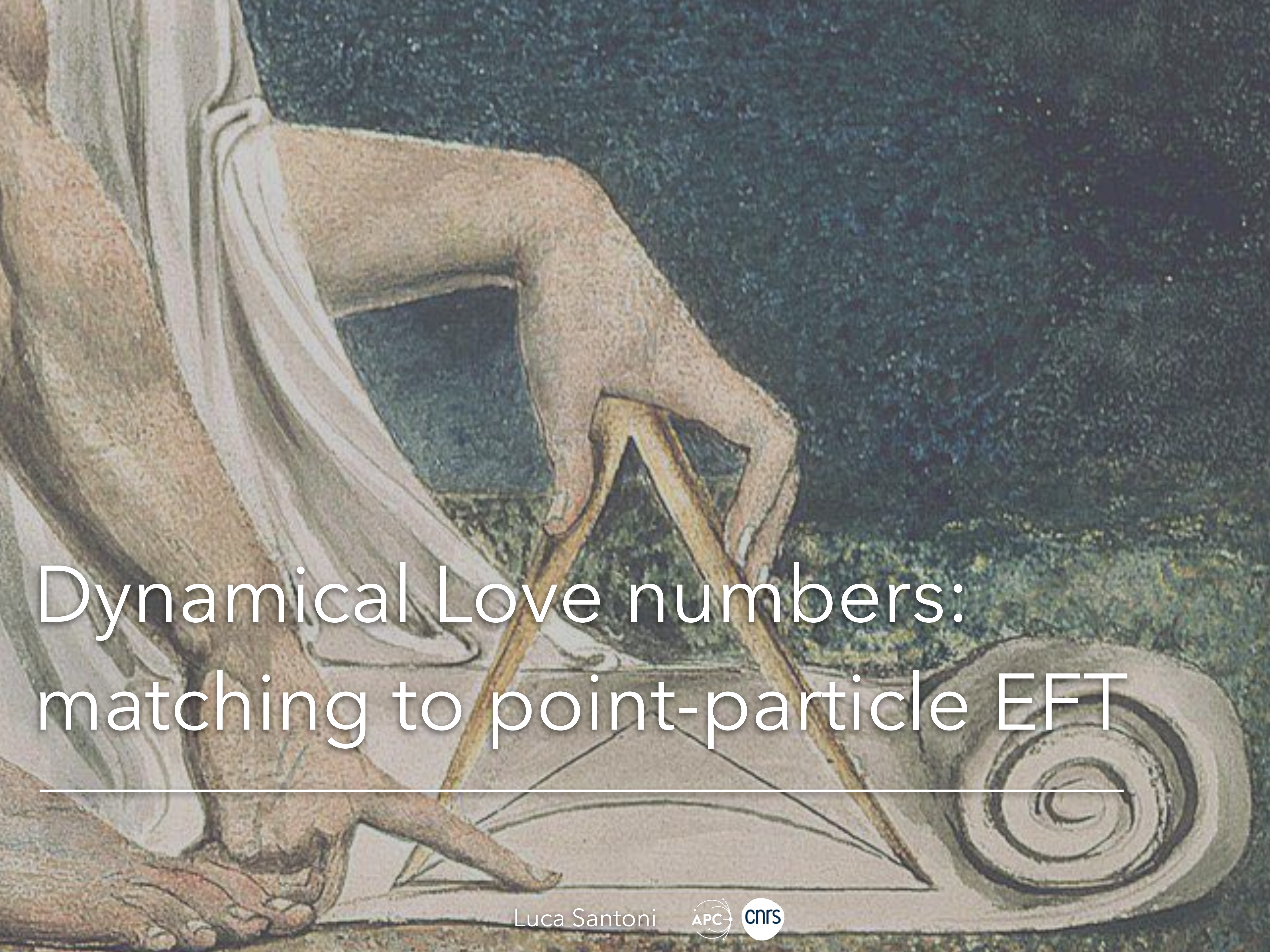
- The FZ solution can be used to compute scattering amplitudes and matched to an analogous calculation in the EFT.

  [Ivanov, Li, Parra-Martinez, Zhou '25, '26; Caron-Huot, Correia, Isabella, Solon '25]

- We showed that the far-zone solution reproduces the MST result in a scalar-field example.

- For the rest, we will not need the FZ: matching can be done directly in the IZ at the level of the Weyl tensor one-point function.

  [Combaluzier-Szteinsznaider, Glazer, Joyce, LS, Rodriguez '25]



Dynamical Love numbers: matching to point-particle EFT

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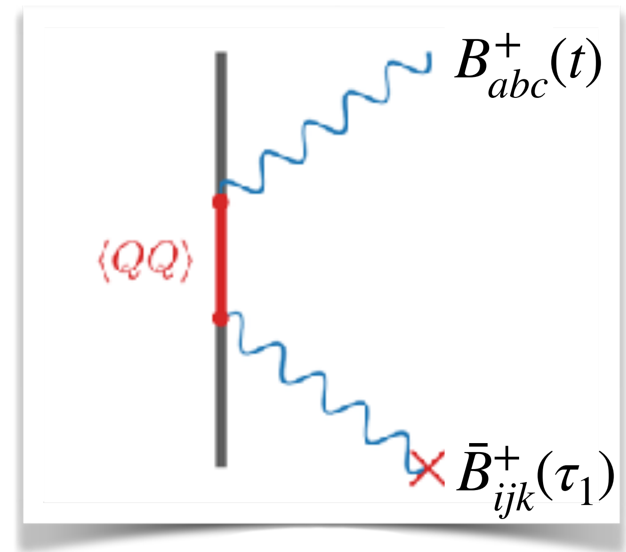
Gravitational dynamical Love numbers: vector sector

- From the in-in effective action, we can compute the one-point function of the magnetic component of the Weyl tensor: $B_{i_1 i_2 j} \equiv C_{0 i_1 i_2 j}$.

- For instance, for an external $\ell = 2$ tidal field:

$$\langle B^{+}_{abc}(t, \vec{x}) \rangle_{\text{in-in}} = i \int \mathrm{d}\tau_1 \mathrm{d}\tau_2 \, F_{\ell=2}(\tau_2 - \tau_1) \langle B^{+}_{abc}(t, \vec{x}) B^{-}_{ijk}(\tau_2) \rangle \bar{B}^{+ijk}(\tau_1)$$



- After some algebra, we find (for generic D and ℓ):

$$\bar{B}_{rij} + \langle B^{+}_{rij} \rangle_{\text{in-in}} = 2 \mathrm{e}^{-i\omega t} \sum_{\ell=2}^{\infty} (\ell - 1) c_{\text{ext}} \nabla_{[i} Y^{(T)m}_{j]\,\ell}$$

$$\times \left(r^{\ell} + \mu^{2\varepsilon} F_{\ell}(\omega) \frac{(\ell+1)!(D+\ell-2)}{\ell(\ell-1)} \frac{2^{\ell-2}\Gamma\left(\ell + \frac{D-3}{2}\right)}{\pi^{\frac{D-1}{2}}} r^{-\ell+3-D} \right)$$

with $\varepsilon = 2 - \dfrac{D}{2}$ and μ an arbitrary scale.

Gravitational dynamical Love numbers: vector sector

- This result represents the leading response of the point particle in the $G_N \to 0$ limit. However, in order to make contact with the UV solution at subleading order in G_N we need to include gravitational effects.

- Gravitational effects introduce UV divergences, which we must regulate via suitable counterterms in the EFT.

- To account for gravitational (background) nonlinearities, we use a Born-series approach.
[Correia, Isabella '24; Caron-Huot, Correia, Isabella, Solon '25]

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{(\ell-\varepsilon)(\ell-\varepsilon+1)}{r^2}\right)\Psi_{\mathrm{RW}}(r) = V_{\Psi_{\mathrm{RW}}}(r)\Psi_{\mathrm{RW}}(r)$$

$$V_{\Psi_{\mathrm{RW}}} = \sum_{n=1}^{\infty}\left(\frac{2G_N M n_D \mu^{2\varepsilon}}{r^{1-2\varepsilon}}\right)^n\left[\frac{2\varepsilon-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\ell^2+\ell-3-\varepsilon(2\ell-5)-2\varepsilon^2}{r^2} - (n+1)\omega^2\right] - \omega^2$$

With $n_D \equiv \dfrac{4\pi^{\frac{3-D}{2}}\Gamma\left(\frac{D-1}{2}\right)}{D-2}$.

Born series approach: homogeneous solution

- We first solve for the homogeneous solution:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{(\ell - \varepsilon)(\ell - \varepsilon + 1)}{r^2}\right)\Psi_{\mathrm{RW}}(r) = 0$$

$$\Psi_{\mathrm{RW,h}}(r) = \mu^{-\varepsilon} B_{\mathrm{reg}} r^{\ell+1-\varepsilon} + \frac{\mu^{\varepsilon} B_{\mathrm{irr}}}{2\ell + 1 - 2\varepsilon} r^{-\ell+\varepsilon}$$

- Comparing the Weyl tensor computed from $\Psi_{\mathrm{RW,h}}$ with the one-point function $\bar{B}_{rij} + \langle B^{+}_{rij}\rangle_{\mathrm{in-in}}$:

$$\frac{(\ell+1)!(D+\ell-2)}{\ell(\ell-1)} \frac{2^{\ell-1}\Gamma\left(\ell + \frac{D-1}{2}\right)}{\pi^{\frac{D-1}{2}}} F_{\ell}(\omega) = \frac{B_{\mathrm{irr}}}{B_{\mathrm{reg}}}$$

- This expression relates the dynamical Love numbers in $F_{\ell}(\omega)$ to the coefficients B_{irr} and B_{reg}, which we will fix by matching to the UV solution.

Born series approach: particular solution

- We now solve for the particular solution:

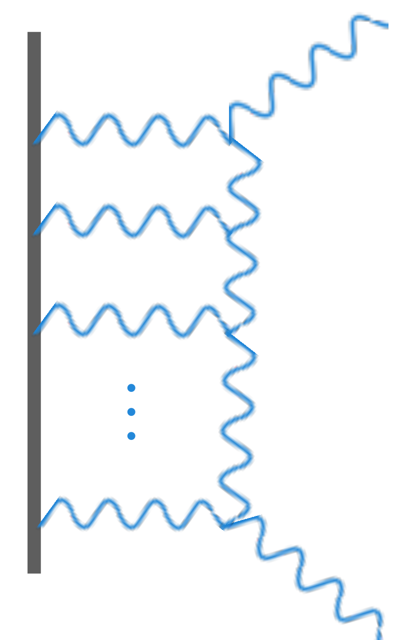
$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{(\ell-\varepsilon)(\ell-\varepsilon+1)}{r^2}\right)\Psi_{\rm RW}(r)=V_{\Psi_{\rm RW}}(r)\Psi_{\rm RW}(r)$$

using the Born series, up to the order $O(\omega^2 G_N^2 G_N^{2\ell+1})$:

$$\Psi_{\rm RW}(r)=\Psi_{\rm RW,h}(r)+\int^r \mathrm{d}r' G(r,r')V_\Psi(r')\Psi_{\rm RW,h}(r')+\int^r \mathrm{d}r' G(r,r')V_\Psi(r')\int^{r'}\mathrm{d}r'' G(r',r'')V_\Psi(r'')\Psi_{\rm RW,h}(r'')+\dots$$

- Taking the $\varepsilon\to 0$ limit (for $\ell=2$, with $\bar G\equiv G_N M n_D$):

$$\begin{aligned}\Psi_{\rm RW}^{\ell=2}(r)=\;& r^3 B_{\rm reg}\left(1+\frac{468\bar G^2\omega^2}{1225}-\frac{107\bar G^2\omega^2}{210\varepsilon}-\frac{107\bar G^2\omega^2}{70}\log(\mu r)\right)\\ &+\frac{B_{\rm reg}}{r^2}\left(\frac{17704\bar G^7\omega^2}{315}+\frac{32\bar G^7\omega^2}{15\varepsilon}+\frac{416\bar G^7\omega^2}{15}\log(\mu r)\right)\\ &+\frac{B_{\rm irr}}{r^2}\left(\frac{1}{5}+\frac{26998\bar G^2\omega^2}{55125}+\frac{107\bar G^2\omega^2}{1050\varepsilon}+\frac{107\bar G^2\omega^2}{210}\log(\mu r)\right).\end{aligned}$$



Born series approach: renormalized solution

- To subtract the infinities, we introduce the renormalized coefficients $\bar{B}_{\rm reg}$ and $\bar{B}_{\rm irr}$:

$$B_{\rm reg} = \bar{B}_{\rm reg}(1+\omega^2\delta_{11}) + \bar{B}_{\rm irr}\omega^2\delta_{12}$$
$$B_{\rm irr} = \bar{B}_{\rm irr}(1+\omega^2\delta_{22}) + \bar{B}_{\rm reg}\omega^2\delta_{21}$$

$$\delta^{\ell=2} = \begin{pmatrix} \frac{107\bar{G}^2}{210\varepsilon} & 0 \\ -\frac{32\bar{G}^7}{3\varepsilon} & -\frac{107\bar{G}^2}{210\varepsilon} \end{pmatrix}.$$

- Plugging these back into the expression for $\Psi_{\rm RW}$ yields the renormalized solution:

$$\Psi^{\mathsf{R},\ell=2}_{\rm RW} = \bar{B}_{\rm reg}\left(r^3 + r^3\bar{G}^2\omega^2\left(\frac{468}{1225} - \frac{214}{105}\log(\mu r)\right) + \frac{\bar{G}^7\omega^2}{r^2}\left(\frac{79472}{1575} + \frac{128}{5}\log(\mu r)\right)\right)$$
$$+\bar{B}_{\rm irr}\left(\frac{1}{5r^2} + \frac{\bar{G}^2\omega^2}{r^2}\left(\frac{24751}{55125} + \frac{214}{525}\log(\mu r)\right)\right).$$

- Comparing $\Psi^{\mathsf{R},\ell=2}_{\rm RW}$ with the UV solution $\Psi^{\ell=2}_{\rm RW,IZ}(r)$, we obtain the matching for the renormalized constants $B_{\rm irr}$ and $B_{\rm reg}$ as functions of ω, r_s, μ and the tidal field amplitude B:

$$\bar{B}^{\ell=2}_{\rm reg} = B\left[1 + i\omega r_s\frac{13}{12} - \omega^2 r_s^2\left(\frac{121991}{58800} + \frac{\pi^2}{6} - \frac{107}{210}\log(\mu r_s)\right)\right]$$
$$\bar{B}^{\ell=2}_{\rm irr} = B\left[i\omega r_s^6 - \omega^2 r_s^7\left(\frac{1943}{2520} + \log(\mu r_s)\right)\right]$$

Born series approach: renormalized solution

- Finally, we obtain for the response function and dynamical Love numbers:

$$
\begin{aligned}
\frac{45}{\pi}F_{\ell=2}(\omega) &= i\omega r_s^6 - \omega^2 r_s^7\left(\log(\mu r_s) - \frac{787}{2520}\right) \\
\frac{525}{\pi}F_{\ell=3}(\omega) &= \frac{i\omega r_s^8}{36} - \frac{\omega^2 r_s^9}{36}\left(\log(\mu r_s) - \frac{1\,727}{2\,520}\right) \\
\frac{14\,175}{\pi}F_{\ell=4}(\omega) &= \frac{i\omega r_s^{10}}{784} - \frac{\omega^2 r_s^{11}}{784}\left(\log(\mu r_s) - \frac{237\,529}{277\,200}\right) \\
&\vdots
\end{aligned}
$$

- Similarly one can obtain the dynamical Love numbers in the parity-even sector.

- In all the examples, we find (up to order ω^2):

$$
\mathrm{Im}(\omega F_\ell(\omega)) = -\,\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F_\ell(\omega)\,.
$$

- $\mathrm{Im}(\omega F_\ell(\omega))$ and $\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F_\ell(\omega)$ are identical for even and odd sectors, as expected from the Chandrasekhar duality (not the finite terms though – the duality is broken by dim-reg).

Neutron star dynamical Love numbers

Luca Santoni

Dynamical Love numbers of neutron stars

- Let's focus on the even sector.
- The EFT part is unchanged:

$$\Psi_{\rm Z}^{{\rm R},\ell=2}(r) = \bar{B}_{\rm reg}\left[r^3\left(1-\bar{G}^2\omega^2\left(\frac{214}{105}\log(\mu r)+\frac{2731}{9800}\right)\right)+\frac{\bar{G}^7\omega^2}{r^2}\left(\frac{3011}{80}\log(\mu r)+\frac{1545209}{57600}\right)-\frac{189\bar{G}^5}{32r^2}\right]$$
$$+\frac{\bar{B}_{\rm irr}}{r^2}\left(\bar{G}^2\omega^2\left(\frac{214}{525}\log(\mu r)+\frac{111383}{441000}\right)+\frac{1}{5}\right)$$

- What is different is the UV part (no near zone):

$$\Psi_{\rm Z}^{\ell=2}(r) \xrightarrow{r\to\infty} a_0\frac{r^3}{r_s^3}-\left(\underbrace{\frac{189}{1024}a_0}_{\text{Tidal field tail}}\underbrace{-\frac{1}{5}b_0}_{\text{Response}}\right)\frac{r_s^2}{r^2}$$
$$+\omega r_s\left[a_1\frac{r^3}{r_s^3}-\left(\frac{189}{1024}a_1-\frac{1}{5}b_1\right)\frac{r_s^2}{r^2}\right]$$
$$+\omega^2 r_s^2\left[\frac{r^3}{r_s^3}\left(a_2+\frac{107}{210}a_0\log\left(\frac{r_s}{r}\right)\right)+\left(b_2-\left(\frac{107}{1050}b_0+\frac{3011}{10240}a_0\right)\log\left(\frac{r_s}{r}\right)\right)\frac{r_s^2}{r^2}\right]+\cdots$$

Dynamical Love numbers of neutron stars

$$S_{\rm int} \sim \int {\rm d}\omega\, F^{(E)}_{\ell=2}(\omega) E_{ij} E^{ij}$$

$$F^{(E)}_{\ell=2}(\omega) = \frac{R_\star^5}{G} c_E + i\omega M R_\star^5 \nu_E + \omega^2 \frac{R_\star^8}{G^2 M} c_{\dot E} + \cdots$$

- Matching to bulk relativistic solution yields:

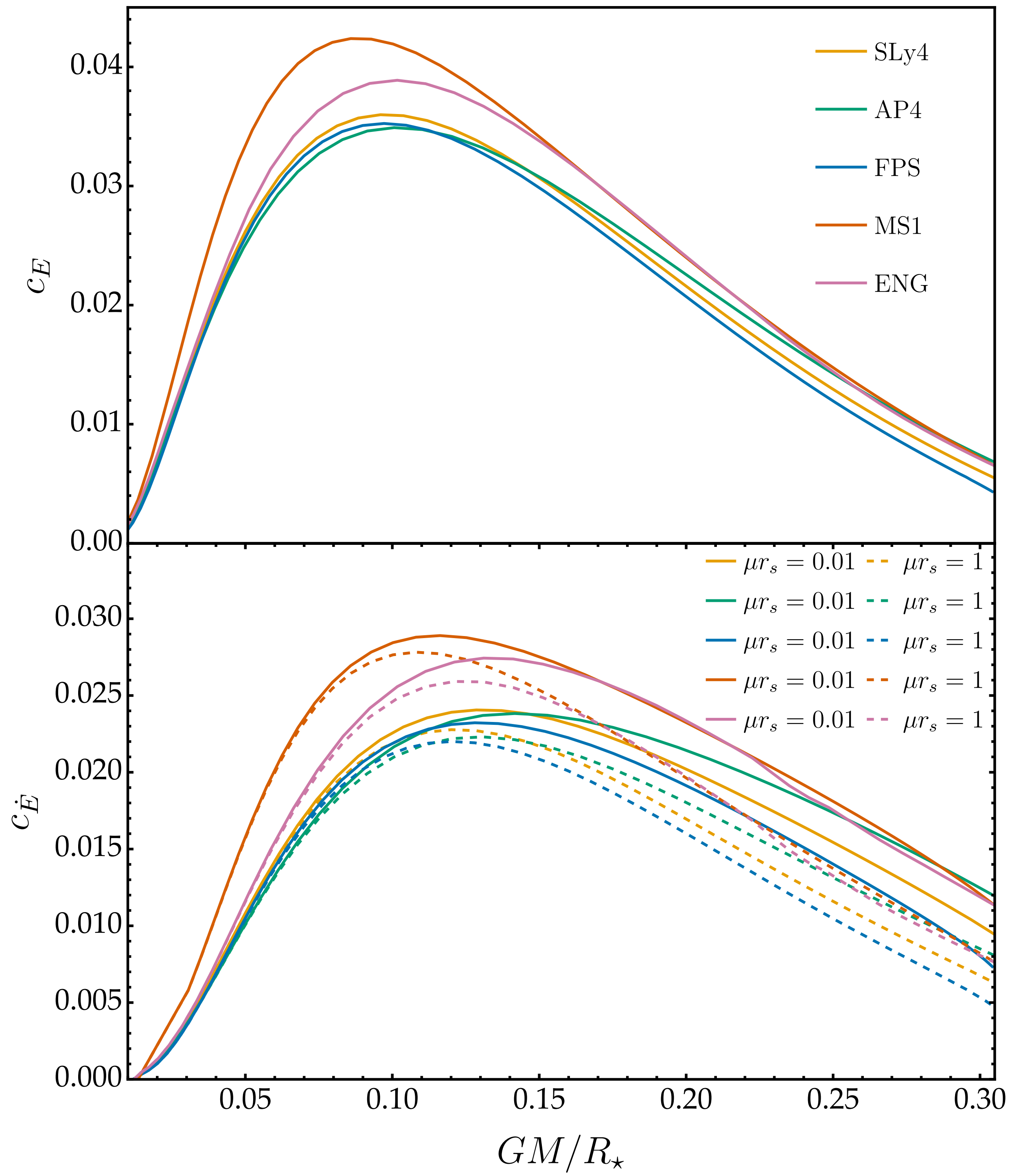
$$c_E = \frac{8G^5M^5}{45R_\star^5}\frac{b_0}{a_0},$$

$$i\nu_E = \frac{16G^5M^5}{45R_\star^5}\left(\frac{b_1}{a_0} - \frac{b_0 a_1}{a_0^2}\right),$$

$$c_{\dot E} = \frac{32G^8M^8}{45R_\star^8}\left[-\left(1 + \frac{107}{105}\frac{b_0}{a_0}\right)\log(\mu r_s) - \frac{67981}{176400}\frac{b_0}{a_0} + \frac{a_1^2 b_0}{a_0^3} - \frac{a_1 b_1}{a_0^2} - \frac{b_0 a_2}{a_0^2} + \frac{945 a_2}{1024 a_0} + \frac{5b_2}{a_0} - \frac{1015283}{1032192}\right].$$

Dynamical Love numbers of neutron stars

- In the case of a perfect-fluid neutron star:



[Apostolidis, De Luca, Gualtieri, Katagiri, Pani, LS, *in preparation*]


Broader view,
observational prospects,
and future horizons

Conclusions and open questions

- We computed gravitational dynamical Love numbers of black holes and neutron stars.
- As opposed to static response coefficients, the dynamical Love numbers exhibit RG running, in line with standard EFT expectations.
- The coefficient of the log was known before.
  [Chakrabarti, Delsate, Steinhoff '13; Charalambous, Dubovsky, Ivanov '21; Saketh, Zhou, Ivanov '23; Ivanov, Li, Parra-Martinez, Zhou '25, '26; Katagiri, Yagi, Cardoso '24; Chakraborty, De Luca, Gualtieri, Pani '25; …]
- For the first time, we computed the scheme-dependent finite terms of the gravitational dynamical Love numbers.

A simple story of gravity (to be continued)

- Generalization to Kerr black holes: [Combaluzier-Szteinsznaider, Glazer, Joyce, LS, Rodriguez, *in preparation*]

- How general/fundamental is the relation $\mathrm{Im}(\omega F_\ell(\omega)) = -\mu \frac{\mathrm{d}}{\mathrm{d}\mu} F_\ell(\omega)$?

  Note that $\mathrm{Im}(\omega F_\ell(\omega))$ is dictated by ladder symmetries. [Hui, Joyce, Penco, LS and Solomon '22]

  This is suggestive that there is some symmetry structure at order ω^2 as well.

- A similar structure relating the discontinuity of the conservative response to the dissipative one persists at subleading orders in ω (at least for a scalar field).

- This is suggestive that there is still further structure to be mined in black hole response coefficients…

Subdominant tidal effects

- Dynamical Love numbers:
  _ *Black holes:* unlikely to be measured.
  _ *Neutron stars:* within reach of future detectors.

- Nonlinear Love numbers: *see Filippo's talk*.

- The systematic incorporation of subdominant tidal effects into high-precision GW models will be crucial to maximize the scientific return of future GW detectors.
  [Apostolidis, De Luca, Gualtieri, Katagiri, Pani, LS, *in preparation*]